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On the existence of Eshelby's equivalent ellipsoidal inclusion solution

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Abstract
The existence of Eshelby's equivalent inclusion solution is proved for a non-degenerate 'transformed' ellipsoidal inhomogeneity in an infinite anisotropic linear elastic matrix. We prove the invertibility of the fourth rank tensor expression, $\mathbf{C}'\boldsymbol{\mathcal{S}}^E + \mathbf{C}(\mathbf{I} - \boldsymbol{\mathcal{S}}^E)$, where $\mathbf{C}$ is the stiffness tensor of the matrix, $\mathbf{C}'$ is the stiffness tensor of the inhomogeneity, $\boldsymbol{\mathcal{S}}^E$ is the Eshelby tensor, and $\mathbf{I}$ is the symmetric identity tensor. Taking advantage of the positive definiteness of certain tensor expressions, a proof-by-contradiction using energy arguments is posited that eliminates the possibility that the above expression is singular. Because the tensor expression is non-singular, it can always be inverted and Eshelby's equivalent ellipsoidal inclusion method can be used to find the stress and strain fields in both the matrix and inhomogeneity.

Keywords
inhomogeneity, invertibility, Eshelby's inclusion

1. Introduction

The problem of finding the stress fields resulting from an inhomogeneity has been investigated over the decades [1–4]. More than 50 years ago, Eshelby formulated a strategy for finding the stress field of a 'transformed' ellipsoidal inhomogeneity in an infinite matrix [5]. He used the concept of an 'equivalent inclusion', a method for replacing an inhomogeneity with elastic constants (C'_{ijkl}) different from those of the matrix with an inclusion. This inclusion has the same elastic constants (C_{ijkl}) as the matrix (Figure 1), but with an eigenstrain (denoted with e^*_{ij}), which is unknown and different from the actual transformation strain ($e^{*\prime}_{ij}$) prescribed for the transformed inhomogeneity. For Eshelby's equivalent inclusion method to work, the fictitious eigenstrain e^*_{ij} must be determined from

$$[(C'_{ijkl} - C_{ijkl})\mathcal{S}^E_{klmn} + C_{ijmn}]e^*_{mn} = C'_{ijkl}e^{*\prime}_{kl}, \tag{1}$$

where $e^{*\prime}$ is the transformation strain actually prescribed for the inhomogeneity and $\mathcal{S}^E$ is the Eshelby tensor. $\mathcal{S}^E$ relates the eigenstrain e^* to the constrained strain e^c (i.e. the actual total (elastic and transformation) strain of the inclusion when embedded in the matrix), associated with the equivalent transformed inclusion, through the following equation:

$$e^c_{ij} = \mathcal{S}^E_{ijkl}e^*_{kl}. \tag{2}$$

If one uses a similar scheme employed in the 'reduced' Voigt notation for linear elastic stiffnesses, Equation (1) may be written as

$$[(C'_{IK} - C_{IK})\mathcal{S}^E_{KM} + C_{IM}]e^*_M = C'_{IM}e^{*\prime}_M, \tag{3}$$

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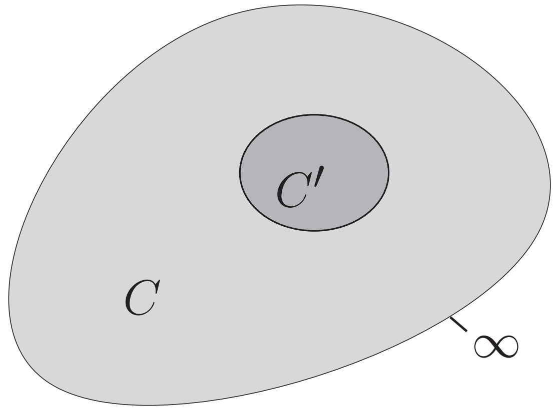

Figure 1. Inclusion inside an infinite matrix with different elastic constants.

where all uppercase indices range from 1 to 6. (For more information on the Voigt notation, refer to Appendix A). Solutions of Equation (3) require that the 6×6 matrix $[(\mathbf{C}' - \mathbf{C})\mathcal{S}^E + \mathbf{C}]$ be non-singular. Eshelby did not discuss whether this matrix will always be non-singular (and the problem would thus be solvable), or whether there exists some set of conditions that preclude obtaining such a solution. On physical grounds, one would expect Eshelby's equivalent transformed inclusion method to always work, but for completeness a proof is desirable. Here we show that $[(C'_{IK} - C_{IK})\mathcal{S}^E_{KM} + C_{IM}]$ is always invertible so that the equivalent eigenstrain e^*_{ij} can always be found. The proof takes advantage of the positive definiteness of certain tensors appearing in energy arguments.

To simplify the notation, we shall define a 6×6 matrix

$$A_{IM} \equiv (C'_{IK} - C_{IK})\mathcal{S}^E_{KM} + C_{IM}. \tag{4}$$

We can also write this equation in matrix form

$$\mathbf{A} = \mathbf{C}'\mathcal{S}^E + \mathbf{C}(\mathbf{I} - \mathcal{S}^E), \tag{5}$$

where $\mathbf{I}$ is the symmetric identity matrix and $\mathcal{S}^E$ is not symmetric. Note that all matrices here are real valued.

A real symmetric matrix $\mathbf{B}$ is positive definite (PD) if [6]

$$\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x} \geq 0 \qquad \forall \mathbf{x} \quad \text{real} \tag{6}$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x} = 0 \qquad \text{iff} \quad \mathbf{x} = \mathbf{0}. \tag{7}$$

A *non-symmetric* matrix $\mathbf{A}$ is also PD if its symmetric portion ($\mathbf{B} = \frac{\mathbf{A}+\mathbf{A}^{\mathrm{T}}}{2}$) is PD [7] (see Appendix B). A PD matrix has all positive eigenvalues (i.e. no zero or negative eigenvalues) [8]. Because the matrix does not have a zero eigenvalue, it is invertible and its determinant is non-zero. We will show that A is always invertible by utilizing the properties of PD matrices and the fact that the energy of an infinite elastic medium containing a finite-sized inclusion is inherently non-negative and bounded.

The paper is organized as follows. First, we briefly review Eshelby's method in Section 2. Section 3 presents our proof that $\mathbf{A}$ is always invertible. This is accomplished by first proving that both $\mathbf{C}(\mathbf{I} - \mathcal{S}^E)$ and $\mathbf{C}\mathcal{S}^E$ are PD and then using energy arguments to show that $\mathbf{A}$ is non-singular.

2. Eshelby's method

2.1. Eshelby's inclusion problem

Eshelby [9] showed that the strain energy associated with a transformed ellipsoidal inclusion with a uniform eigenstrain e^*_I in an infinite matrix is

$$E = -\frac{1}{2}\sigma_I e^*_I V_0, \tag{8}$$

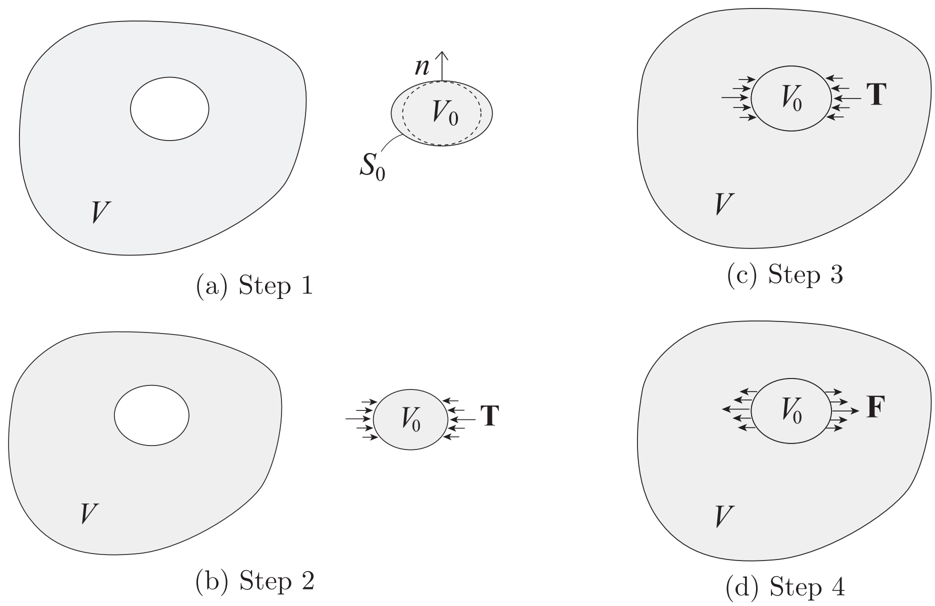

Figure 2. Steps to solve Eshelby's inclusion problem.

where σ_I is the stress in the transformed inclusion and V_0 is the volume of the inclusion. In the following, we give a brief review of Eshelby's method leading to Equation (8), because the derivation will be useful for our proof.

Initially both matrix and the ellipsoidal inclusion are undeformed. The inclusion and matrix are separated (Figure 2(a)) and the inclusion alone is given the uniform stress-free transformation strain e^*_I. As e^*_I is inelastic strain and stress-free, there is no strain energy change in either the inclusion or the matrix. In Step 2, tractions $-\sigma^*_{ij}n_i$ (where $\sigma^*_{ij} = C_{ijkl}e^*_{kl}$ and n_i is the outward facing unit normal of the inclusion) are applied to the inclusion boundary to 'elastically' back-strain it to its original shape and size (Figure 2(b)), while the matrix remains undeformed. The strain energy change at the end of Step 2 is

$$\Delta W_1 = \frac{1}{2}\int_{V_0} \sigma^*_I e^*_I \, \mathrm{d}V = \frac{1}{2}\sigma^*_I e^*_I V_0, \tag{9}$$

where V_0 is the volume of the inhomogeneity.

In Step 3, the inclusion is inserted back into the ellipsoidal 'hole' in the matrix (Figure 2(c)), and the discontinuity in traction across the matrix–inhomogeneity interface is equivalent to a 'body force'

$$f_j = \sigma^*_{ij} n_i \, \mathrm{d}S \tag{10}$$

distributed over S_0, the interface, where dS is a differential surface element of the interface. In Step 4, relaxing this body force to zero (or, equivalently, adding on a solution to remove this body force layer everywhere on S_0 (Figure 2(d)), produces an additional total energy change (strain energy change plus body force work)

$$\Delta W_2 = -\frac{1}{2}\int_{S_0} \sigma^*_{ij} n_i u^c_j \, \mathrm{d}S = -\frac{1}{2}\int_{V_0} \sigma^*_{ij} e^c_{ij} \mathrm{d}V_0 = -\frac{1}{2}\sigma^*_I e^c_I V_0 = -\frac{1}{2}\sigma^c_I e^*_I V_0 \tag{11}$$

which is simply the average work done on S_0 in relaxing the layer of body force to zero. Here $e^c_I = \mathcal{S}^E_{IK} e^*_K$ is the 'constrained' elastic strain (in Eshelby's terminology) associated with e^*, and is thus uniform in V_0, and $\mathcal{S}^E$ is the Eshelby tensor for the inclusion.

In its final state the transformed inclusion has undergone an elastic strain given by $e^c_I - e^*_I$ and its stress state is given by $\sigma^I_K = C_{KJ}(e^c_J - e^*_J)$, where the superscript 'I' stands for inclusion. The total strain energy change in the system of inclusion and surrounding matrix is the sum of Equation (9) and Equation (11), namely

$$\Delta W = -\frac{1}{2}\sigma_K^I e_K^* V_0. \tag{12}$$

2.2. Eshelby's transformed inhomogeneity problem

Eshelby's most clever method of solving the 'transformed' inhomogeneity problem was to note that an ellipsoidal inhomogeneity (elastic stiffnesses $\mathbf{C}'$ and eigenstrain $e^{*\prime}$) can be replaced by an 'equivalent' inclusion with elastic stiffness $\mathbf{C}$ and eigenstrain e^*. We can apply Eshelby's procedure in §2.1 to find the total strain and elastic strain inside the inhomogeneity and the equivalent inclusion:

$$e_I^{\text{total}} = e_I^c \tag{13}$$

$$e_I^{\text{elastic}} = e_I^c - e_I^* \quad \text{(for the inclusion)} \tag{14}$$

$$e_I^{\text{elastic}} = e_I^c - e_I^{*\prime} \quad \text{(for the inhomogeneity).} \tag{15}$$

As the total strains are the same, the inclusion may be replaced by the inhomogeneity with displacement continuity maintained at the inhomogeneity/matrix interface. Traction continuity is maintained with this switch if the stress in both inclusion and inhomogeneity are equal, i.e. if

$$C'_{IK}(e_K^c - e_K^{*\prime}) = C_{IK}(e_K^c - e_K^*). \tag{16}$$

This is the equation from which Eshelby's equivalent inclusion method stems and is equivalent to Equation (1) and Equation (3). Eshelby assumed that an equivalent inclusion (e^*) can always be found for each transformed inhomogeneity (e_I^*, C'_{IK}). This requires $\mathbf{A} = \mathbf{C}'\mathcal{S}^E + \mathbf{C}(\mathbf{I} - \mathcal{S}^E)$ to be invertible, which is the property we aim to prove in this paper.

3. Proof

3.1. Proving $\mathbf{C}(\mathbf{I} - \mathcal{S}^E)$ is PD

Energy arguments will be used to investigate the possible PD nature of $\mathbf{C}(\mathbf{I} - \mathcal{S}^E)$. Using Equation (12) the inclusion energy can be written as

$$\begin{aligned} E &= \frac{1}{2}\sigma_I^*(e_I^* - \mathcal{S}_{IK}^E e_K^*)V_0 \\ &= \frac{1}{2}\sigma_I^*(I_{IK} - \mathcal{S}_{IK}^E)e_K^* V_0 \\ &= \frac{1}{2}C_{IM}e_M^*(I_{IK} - \mathcal{S}_{IK}^E)e_K^* V_0 \\ &= \left[\frac{1}{2}e_M^* e_K^* V_0\right][C_{IM}(I_{IK} - \mathcal{S}_{IK}^E)]. \end{aligned} \tag{17}$$

Because the left-hand side of Equation (17) is a strain energy (an inherently non-negative quantity for stable media), the right-hand side must also be positive for any real eigenstrain e_I^*. Therefore, the matrix $\mathbf{C}(\mathbf{I} - \mathcal{S}^E)$ is PD.

3.2. Proving $\mathbf{C}\mathcal{S}^E$ is PD

It is possible to show that $\mathbf{C}\mathcal{S}^E$ is also PD. The energy change ΔW_2 in Eshelby's last step (see Figure 2(d)), must be negative, because we remove a constraint on the system (the layer of distributed body force) which then relaxes spontaneously to its final state:

$$\Delta W_2 = -\frac{1}{2}\sigma_I^* e_I^c V_0 = -\frac{1}{2}C_{IK}e_K^*\mathcal{S}_{IM}^E e_M^* V_0 < 0. \tag{18}$$

This leads to

$$e_K^* C_{KI} \mathcal{S}_{IM}^E e_M^* > 0, \tag{19}$$

meaning that $\mathbf{C}\mathcal{S}^E$ is PD. According to Appendix B, $\det[\mathbf{C}\mathcal{S}^E] > 0$, even though $\mathbf{C}\mathcal{S}^E$ is not a symmetric matrix [8].

In the special case that $\mathbf{C}' = \lambda\mathbf{C}$, with $\lambda > 0$, we have

$$\mathbf{A} = \lambda\mathbf{C}\mathcal{S}^E + \mathbf{C}(\mathbf{I} - \mathcal{S}^E), \tag{20}$$

so that $\mathbf{A}$ is the sum of two PD matrices and must be PD. This is a sufficient condition for $\mathbf{A}$ being non-singular. In general. $\mathbf{C}'\mathcal{S}^E$ is not PD and $\mathbf{A}$ is not PD. However, we can still show that $\mathbf{A}$ is always non-singular (i.e. invertible).

3.3. Proving $\mathbf{A}$ *is invertible*

The method of proof by contradiction will be used to show that $\mathbf{A}$ is invertible. Suppose (for some $\mathbf{C}'$) $\mathbf{A}$ is singular (i.e. its determinant = 0), and $\mathbf{A}^{-1}$ does not exist. Without the inverse, the equivalent inclusion strain, e_M^* cannot be found for a given C'_{IK}. Now consider C'_{IK} altered through a perturbation, i.e.

$$C'_{IK} \longrightarrow C'_{IK} + \epsilon C_{IK}, \text{ where } \epsilon \geq 0. \tag{21}$$

Substituting this into Equation (4) leads to a perturbed $\mathbf{A}$ matrix as a function of ϵ:

$$\mathbf{A}(\epsilon) \equiv \mathbf{C}'\mathcal{S}^E + \epsilon\mathbf{C}\mathcal{S}^E + \mathbf{C}(\mathbf{I} - \mathcal{S}^E). \tag{22}$$

Note by assumption

$$\det[\mathbf{A}(\epsilon)] = 0 \quad \text{when } \epsilon = 0. \tag{23}$$

For sufficiently large values of ϵ, $\mathbf{A}$ is dominated by the $\epsilon\mathbf{C}\mathcal{S}^E$ term, and

$$\det[\mathbf{A}(\epsilon)] \longrightarrow \epsilon^6 \det[\mathbf{C}\mathcal{S}^E] > 0. \tag{24}$$

This means that $\det[\mathbf{A}(\epsilon)]$ must be positive for sufficiently large ϵ. In addition, note that $\det[\mathbf{A}(\epsilon)]$ is a smooth (polynomial) function of ϵ. Hence, $\det[\mathbf{A}(\epsilon)]$ must also deviate from zero for arbitrarily small $\epsilon > 0$. For any infinitesimal ϵ, there exists a unique equivalent inclusion eigenstrain by solving Equation (22):

$$\begin{aligned} A_{IK}(\epsilon) e_K^*(\epsilon) &= C'_{IK} e_K^{*\prime} \\ e^*(\epsilon) &= \mathbf{A}^{-1}(\epsilon)\mathbf{C}' e^{*\prime} \qquad (\epsilon \to 0^+). \end{aligned} \tag{25}$$

Note that $\mathbf{C}'$ is always invertible, meaning that $\mathbf{C}'e^{*\prime}$ spans the full six-dimensional space. Because $\det[\mathbf{A}(\epsilon = 0)] = 0$, then $\det[\mathbf{A}^{-1}(\epsilon = 0)] \longrightarrow \infty$. Applying this to the right-hand side of Equation (25) as $\epsilon \to 0^+$ yields

$$\lim_{\epsilon \to 0^+} \mathbf{A}^{-1}(\epsilon)\mathbf{C}' e^{*\prime} \longrightarrow \infty. \tag{26}$$

This would mean that

$$\lim_{\epsilon \to 0^+} e^*(\epsilon) \longrightarrow \infty. \tag{27}$$

This implies that the equivalent inclusion eigenstrain is unbounded for $\mathbf{A}$ close to singular. We assume the problem at hand is not the trivial case of a singularity caused by $a \to 0$, $b \to 0$, or $c \to 0$ (a, b, c are the major/minor axes of the ellipsoid) and that $\mathbf{C}$ and $\mathbf{C}'$ are always invertible. Then $\lim_{\epsilon \to 0^+} e^*(\epsilon) \longrightarrow \infty$ would imply that the total strain of the equivalent inclusion e^I, which is the same as the total strain of the inhomogeneity $e^{I\prime}$, will go to infinity as $\epsilon \to 0$. This would mean the elastic energy stored in the matrix will become unbounded as $\epsilon \to 0$, which is physically impossible because the energy in the matrix must be smaller than the energy stored in the inhomogeneity when it is subjected to an elastic strain that exactly cancels the transformed strain (i.e. when $e^{I\prime,\text{total}} = e^{I\prime,\text{elastic}} + e^{*\prime} = 0$). This means $E < \frac{1}{2}\sigma_I^{*\prime} e_I^{*\prime} V_0$.

Because $e^*(\epsilon)$ cannot get arbitrarily large from infinitesimal changes of $\mathbf{C}'$, the possibility that $\lim_{\epsilon \to 0} \det[\mathbf{A}(\epsilon)] = 0$ must be discarded, a contradiction to the initial assumption. Thus, $\mathbf{A}$ must always be invertible.

4. Conclusions

We used the PD nature of two tensors combined with energy arguments used by Eshelby in his original approach to show that the matrix **A** defined in Equation (5) must be invertible. Since **A** has this property (for non-trivial **C**, i.e. the matrix is not rigid or infinitely pliant), Equation (1) can always be solved for e^*. The method outlined above only applies for the one scenario corresponding to Equation (1) (initially strained inhomogeneity with different elastic constants than the surrounding infinite matrix, with no applied external fields), however a similar approach can be used to explore the solvability of other situations.

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Conflict of Interest

None declared.

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Appendices

A. Voigt notation

Our proof used arguments based on properties (such as PD) and operations (such as determinants) traditionally associated with matrices. A symmetric fourth-rank tensor may be reduced to a symmetric matrix using the Voigt form. However, this reduction in order requires the use of weights to recover losses associated with the symmetries of the tensor. There are different approaches to the weights, but in this paper the Nye notation will be used [10]. In the fourth-rank tensors presented below the 81 components reduce to 36 distinct components represented in a 6×6 matrix due to minor symmetries (i.e. $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$):

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} =$$

$$\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{S}^E_{11} & \mathcal{S}^E_{12} & \mathcal{S}^E_{13} & \mathcal{S}^E_{14} & \mathcal{S}^E_{15} & \mathcal{S}^E_{16} \\ \mathcal{S}^E_{21} & \mathcal{S}^E_{22} & \mathcal{S}^E_{23} & \mathcal{S}^E_{24} & \mathcal{S}^E_{25} & \mathcal{S}^E_{26} \\ \mathcal{S}^E_{31} & \mathcal{S}^E_{32} & \mathcal{S}^E_{33} & \mathcal{S}^E_{34} & \mathcal{S}^E_{35} & \mathcal{S}^E_{36} \\ \mathcal{S}^E_{41} & \mathcal{S}^E_{42} & \mathcal{S}^E_{43} & \mathcal{S}^E_{44} & \mathcal{S}^E_{45} & \mathcal{S}^E_{46} \\ \mathcal{S}^E_{51} & \mathcal{S}^E_{52} & \mathcal{S}^E_{53} & \mathcal{S}^E_{54} & \mathcal{S}^E_{55} & \mathcal{S}^E_{56} \\ \mathcal{S}^E_{61} & \mathcal{S}^E_{62} & \mathcal{S}^E_{63} & \mathcal{S}^E_{64} & \mathcal{S}^E_{65} & \mathcal{S}^E_{66} \end{pmatrix} =$$

$$\begin{pmatrix} \mathcal{S}^E_{1111} & \mathcal{S}^E_{1122} & \mathcal{S}^E_{1133} & \mathcal{S}^E_{1123} & \mathcal{S}^E_{1113} & \mathcal{S}^E_{1112} \\ \mathcal{S}^E_{2211} & \mathcal{S}^E_{2222} & \mathcal{S}^E_{2233} & \mathcal{S}^E_{2223} & \mathcal{S}^E_{2213} & \mathcal{S}^E_{2212} \\ \mathcal{S}^E_{3311} & \mathcal{S}^E_{3322} & \mathcal{S}^E_{3333} & \mathcal{S}^E_{3323} & \mathcal{S}^E_{3313} & \mathcal{S}^E_{3312} \\ 2\mathcal{S}^E_{2311} & 2\mathcal{S}^E_{2322} & 2\mathcal{S}^E_{2333} & 2\mathcal{S}^E_{2323} & 2\mathcal{S}^E_{2313} & 2\mathcal{S}^E_{2312} \\ 2\mathcal{S}^E_{1311} & 2\mathcal{S}^E_{1322} & 2\mathcal{S}^E_{1333} & 2\mathcal{S}^E_{1323} & 2\mathcal{S}^E_{1313} & 2\mathcal{S}^E_{1312} \\ 2\mathcal{S}^E_{1211} & 2\mathcal{S}^E_{1222} & 2\mathcal{S}^E_{1233} & 2\mathcal{S}^E_{1223} & 2\mathcal{S}^E_{1213} & 2\mathcal{S}^E_{1212} \end{pmatrix}.$$

Note that in the Nye notation, the matrix notation of Eshelby's tensor is different from that of the elastic stiffness tensor **C**. This is because **C** describes the relation between stress and strain, while $\mathcal{S}^E$ describes the relation between two strain tensors.

The second rank tensors can also be reduced to a vector formulation.

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{13} \\ 2e_{12} \end{pmatrix}, \quad \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}.$$

B. Positive definite non-symmetric matrices

B.1. Definition

The positive definiteness of non-symmetric matrices depends on the positive definiteness of the symmetric part of the non-symmetric matrix [7]. As mentioned earlier, a matrix (**A**) can be separated into a symmetric (**B**) and an antisymmetric (**D**) part as

$$\mathbf{A} = \mathbf{B} + \mathbf{D}$$
$$\mathbf{B} = \frac{\mathbf{A} + \mathbf{A}^{\mathrm{T}}}{2}$$
$$\mathbf{D} = \frac{\mathbf{A} - \mathbf{A}^{\mathrm{T}}}{2}$$

where **B** and **D** can be substituted into the definition of PD, yielding

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}(\mathbf{B} + \mathbf{D})\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{D}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}$$

because $\mathbf{x}^{\mathrm{T}}\mathbf{D}\mathbf{x} = 0$. Thus, if the symmetric term is PD, the entire non-symmetric matrix is PD.

The fact that all eigenvalues are positive comes immediately from this result. If $\mathbf{x}$ is a right eigenvector of non-symmetric PD matrix $\mathbf{A}$ with eigenvalue λ, then

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} &= \mathbf{x}^{\mathrm{T}}\lambda\mathbf{x} \\ \lambda &= \frac{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \end{aligned}$$

where the numerator is positive due to $\mathbf{A}$ being PD and the denominator is simply the norm of $\mathbf{x}$, a non-negative quantity. Thus, λ must be positive.

B.2. Determinant

In this appendix, we show that the determinant of a non-symmetric PD matrix is positive. We give the proof here for completeness, because most of the discussion on positive definiteness in the literature focuses on symmetric matrices, with the exception of [8]. Suppose that $\mathbf{H}$ is an $N \times N$, real, symmetric, PD matrix (i.e. $\mathbf{H} = \mathbf{H}^{\mathrm{T}}$, $\mathbf{x}^{+}\mathbf{H}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq 0$, and $\det(\mathbf{H}) > 0$, where $\mathbf{x}^{+}$ indicates the complex transpose, or adjoint.) and suppose that $\mathbf{D}$ is a real, antisymmetric matrix (i.e. $\mathbf{D}^{\mathrm{T}} = -\mathbf{D}$). To show that the determinant of a PD, real, non-symmetric matrix has to be positive, it is equivalent to showing that $\det(\mathbf{H} + \mathbf{D}) > 0$.

Since $\mathbf{H}$ is real, symmetric, and PD, $\mathbf{H}^{-1}$ is also real, symmetric, and PD. This means that $-\mathbf{D}\mathbf{H}^{-1}\mathbf{D}$ is a real, symmetric, semi-definite matrix because

$$-\mathbf{x}^{+}\mathbf{D}\mathbf{H}^{-1}\mathbf{D}\mathbf{x} = \begin{cases} (\mathbf{D}\mathbf{x})^{+}\mathbf{H}^{-1}(\mathbf{D}\mathbf{x}) > 0 & \text{iff } \mathbf{D}\mathbf{x} \neq 0 \\ 0 & \text{iff } \mathbf{D}\mathbf{x} = 0. \end{cases}$$

We can also state that $\mathbf{x}^{\mathrm{T}}\mathbf{D}\mathbf{x}$ is purely imaginary or zero because

$$(\mathbf{x}^{+}\mathbf{D}\mathbf{x})^{*} = x_i D_{ij}^{*} x_j^{*} = x_i D_{ij} x_j^{*} = x_j^{*} D_{ij} x_i = x_j^{*}(-D_{ji}) x_i = -(\mathbf{x}^{+}\mathbf{D}\mathbf{x}).$$

This is an expansion of what was found in the first part of this appendix.

We can also state that $\mathbf{H}^{-1}\mathbf{D}$ has no real eigenvalues λ, with the possible exception of $\lambda = 0$. This can be shown by supposing $\mathbf{H}^{-1}\mathbf{D}\mathbf{x} = \lambda\mathbf{x}$, with $\lambda \neq 0$, which also indicates that $\mathbf{D}\mathbf{x} \neq 0$. Then

$$0 < -\mathbf{x}^{+}\mathbf{D}\mathbf{H}^{-1}\mathbf{D}\mathbf{x} = -\mathbf{x}^{+}\mathbf{D}(\lambda\mathbf{x}) = -\lambda\mathbf{x}^{+}\mathbf{D}\mathbf{x}$$

because $\mathbf{x}^{+}\mathbf{D}\mathbf{x}$ is purely imaginary (it is not zero in this case). Thus, λ must be purely imaginary to make $-\lambda\mathbf{x}^{+}\mathbf{D}\mathbf{x} > 0$.

Hence, $\det(\lambda\mathbf{I} - \mathbf{H}^{-1}\mathbf{D}) = 0$ has no real roots, except possibly $\lambda = 0$. Equivalently, $\det(\lambda\mathbf{I} + \mathbf{H}^{-1}\mathbf{D}) = 0$ has no real roots, except possibly $\lambda = 0$. Excluding the possibility that $\lambda = 0$ yields

$$0 \neq \det(\lambda\mathbf{I} + \mathbf{H}^{-1}\mathbf{D}) \equiv f(\lambda).$$

In fact, we also know that

$$f(\lambda) > 0 \qquad \lambda \to \infty.$$

Since there are no real roots and $f(\lambda) > 0$ at some point, we must have $f(\lambda) > 0$ for $\forall\lambda > 0$. From here, we can let $\lambda = 1$ to get $f(1) = \det(\mathbf{I} + \mathbf{H}^{-1}\mathbf{D}) > 0$. Thus,

$$\det(\mathbf{H} + \mathbf{D}) = \det[\mathbf{H}(\mathbf{I} + \mathbf{H}^{-1}\mathbf{D})] = \det(\mathbf{H}) \cdot \det(\mathbf{I} + \mathbf{H}^{-1}\mathbf{D}) > 0,$$

since $\det(\mathbf{H}) > 0$ and $\det(\mathbf{I} + \mathbf{H}^{-1}\mathbf{D}) > 0$.