Theory and Applications of Inelasticity

Wei Cai
Stanford University
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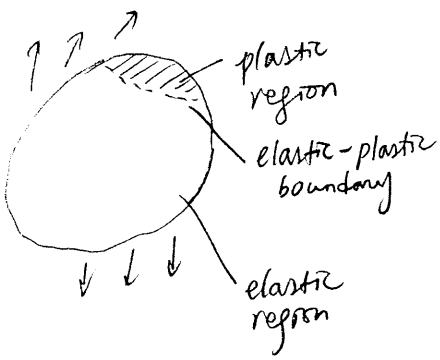
In this course, we discuss plasticity theory, its applications, and fracture.

Study of (ductile) fracture in engineering materials requires an understanding of plasticity in front of the crack tip.

This course is the sequel of ME340 Elasticity, in which we limit the load such that the stress is below yield stress everywhere.

In this course, we allow the load to reach (or exceed) yield stress and allow the structure to develop (permanent) plastic strain.



The goal is (still) to find the stress, strain, and displacement fields.

Usually, only a part of the structure will undergo plastic deformation, while the rest remains in the elastic regime.

So as an intermediate step, we also need to determine the elastic-plastic boundary.

The other complication is that, in the plastic region, the total strain is the sum of elastic and plastic strain.

We shall assume that the elastic strain is proportional to stress. But the plastic strain and total strain are in general not proportional to stress. The plastic strain and total strain depend on the history of loading.

Goal of this course:

1. The theory of plasticity

   i.e. the physics behind the phenomena and how to describe it mathematically (PDE)

2. Method of solution (not including FEM)

   Analytical method

   Semi-analytical method involving Matlab

3. Solution of classical plasticity problems

A major goal of this course is to develop an 'intuitive feel' of plasticity problems, so that

- when you encounter a simple plasticity problem in your research, you can use an analytical solution, or write a simple numerical code to solve it.
- when you encounter an finite element method (FEM) solution, you can quickly tell whether the solution 'makes sense'.

Relationship with other courses

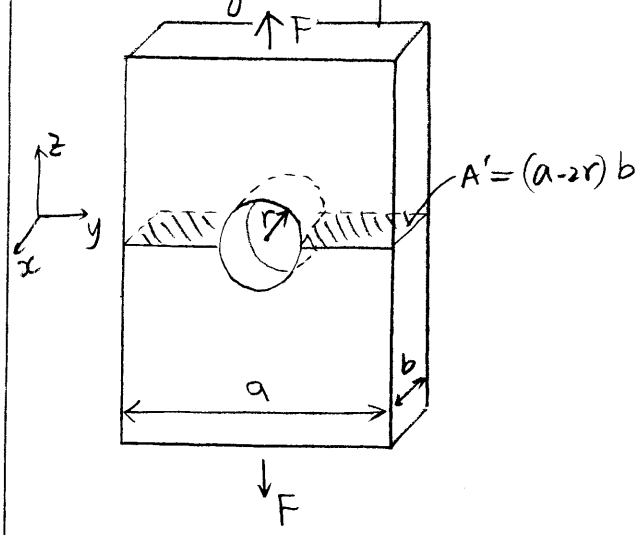
- ME340 "Elasticity" (or equivalent as prerequisite)
- CEE292 "Computational Micromechanics" (Spring 2013)

  deals with the same physical phenomenon — plasticity

  focuses on numerical algorithm

This course focuses on the physics of the solution, analytical method.

Motivating Examples



From Elasticity theory, we know

$$(\sigma_{zz})_{max} = K \cdot \frac{F}{A'}$$

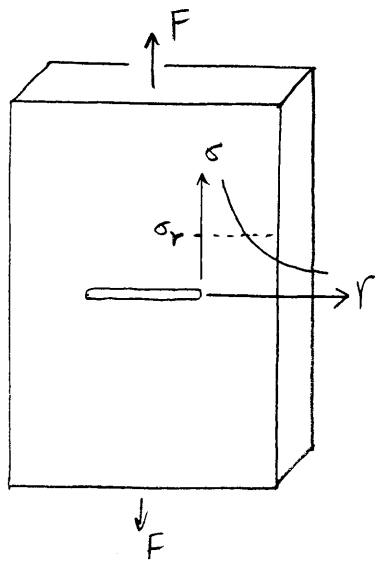
Stress concentration factor

when stress is below σ_Y (yield stress) every where.

$K = 3$ when $a \gg r$.

Q: what is the stress distribution when $(\sigma_{zz})_{max} \geq \sigma_Y$?

Q: what is the size of the plastic zone?



Slit like crack

Elasticity theory predicts

$$\sigma \sim \frac{1}{\sqrt{r}}$$

which surely will exceed σ_Y for arbitrarily small F.

Q: How is stress field modified when σ_Y is accounted for ?

Q: what is the size (shape) of the plastic zone ?

Q: What is the effect of σ_Y on fracture toughness ?

The purpose of this lecture is to gather all equations dealing with a <u>perfectly plastic, isotropic</u> medium, with brief derivations and discussions.

The types of relations include:

- equilibrium condition for stress (§4)
- compatibility condition for total strain (§5)
- Stress-strain relation in Elastic Regime (§7)
- Yield condition (§8)
- stress-strain relation (flow rule) in Plastic Regime (§9)

A summary of key formulas is given in §10.

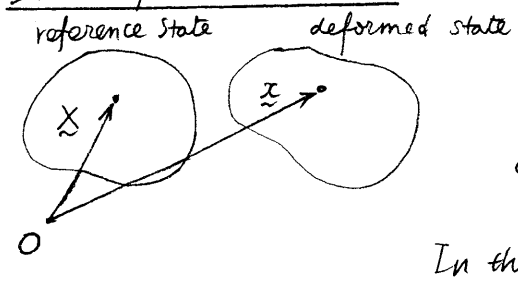
The goal is to provide a global view of a (simplest) plasticity theory.

More discussions on yield condition and flow rule will be given, with examples, in subsequent lectures.

A perfectly plastic medium (i.e. no hardening) is assumed here to keep the discussion simple. Various hardening rules will be discussed later.

Similar to the theory of elasticity, in the theory of plasticity we will also be dealing with stress, strain and displacement field. The main difference is that the stress-strain relationship is non-linear (and history-dependent) in the theory of plasticity.

§1. Displacement Field



Displacement $\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X}$

displacement field $\underset{\sim}{u}(\underset{\sim}{x})$

In this class, we will assume $|\underset{\sim}{u}| \ll 1$ and ignore the difference between $\underset{\sim}{u}(\underset{\sim}{x})$ and $\underset{\sim}{u}(\underset{\sim}{X})$.

displacement field in component form:

$$u_x(x, y, z)$$
$$u_y(x, y, z)$$
$$u_z(x, y, z)$$

§2. Strain Field

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

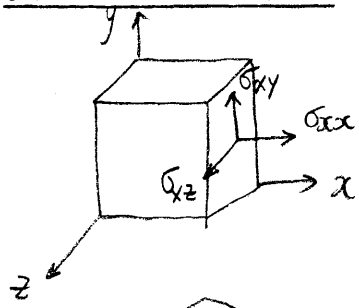
$i, j = 1, 2, 3$
$x_1 = x,$
$x_2 = y,$
$x_3 = z$

e.g. $\varepsilon_{xx} = u_{x,x}$

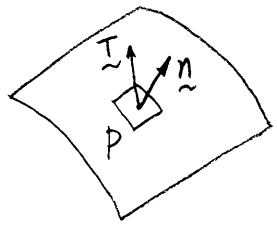
$\varepsilon_{xy} = \frac{1}{2}(u_{x,y} + u_{y,x})$

The engineering strain, e.g. $\gamma_{xy} = u_{x,y} + u_{y,x}$, is not a component of the strain tensor.

§3. Stress Field



σ_{ij} : force per unit area on i-th face in j-th direction



Given the stress at point P, the traction force $\underset{\sim}{T}$ per unit area on any surface element with normal vector $\underset{\sim}{n}$ is

$$T_j = \sigma_{ij} n_i$$

u_i is a vector

ε_{ij}, σ_{ij} are (symmetric) second order tensors.

The components change due to transformation of coordinate system as

$$u_i' = Q_{ip} u_p \qquad \text{where} \quad Q_{ip} \equiv (\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_p)$$

$$\varepsilon_{ij}' = Q_{ip} Q_{jq} \varepsilon_{pq}$$

$$\sigma_{ij}' = Q_{ip} Q_{jq} \sigma_{pq}$$

unit vector of new coordinate system

unit vector of original coordinate system

(For more details, see ME340 Lecture Notes Winter 2013.)

§4. Equilibrium Condition for Stress

$$\sigma_{ij,i} + F_j = 0$$

F_j: body force per unit volume.

in tensor notation $\nabla \cdot \underset{\approx}{\sigma} + \underset{\sim}{F} = 0$

in component form:

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx} + F_x = 0$$

$$\frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + \frac{\partial}{\partial z}\sigma_{zy} + F_y = 0$$

$$\frac{\partial}{\partial x}\sigma_{xz} + \frac{\partial}{\partial y}\sigma_{yz} + \frac{\partial}{\partial z}\sigma_{zz} + F_z = 0$$

Note: The equilibrium condition is satisfied regardless of whether the material is elastic or plastic.

§5. Compatibility condition for total strain.

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the total strain

$$\varepsilon_{ij} = \underset{\substack{\uparrow \\ \text{elastic} \\ \text{strain}}}{\varepsilon_{ij}^{el}} + \underset{\substack{\uparrow \\ \text{plastic} \\ \text{strain}}}{\varepsilon_{ij}^{pl}}$$

ε_{ij} has to satisfy the compatibility condition if the material is not ruptured.

compatibility condition: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$

The compatibility condition gaurentees that ε_{ij} (6 DOF at every point) can be written in terms of spatial derivatives of u_i (3 DOF).

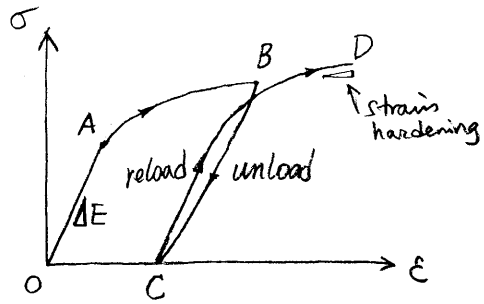
The compatibility condition is automatically satisfied if we start from u_i and obtain ε_{ij} as $\frac{1}{2}(u_{i,j} + u_{j,i})$.

(For more details, see ME 340 Lecture Notes Winter 2013).

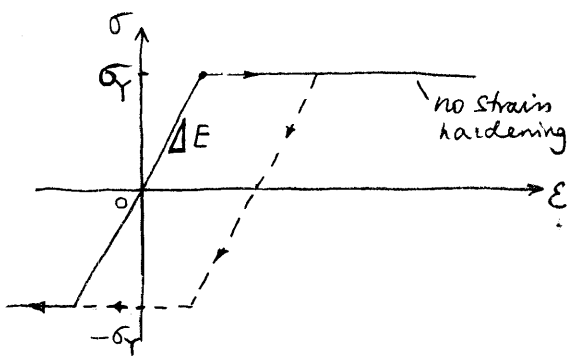
Note: The elastic strain field ε_{ij}^{el} does not have to satisfy compatibility condition if $\varepsilon_{ij}^{pl} \neq 0$.

* up to this point, every equation is the same as in elasticity theory (provided $\varepsilon_{ij}^{pl} = 0$).

§6. Tensile Stress-Strain Curve



Stress-strain curve of a ductile material in a tensile test



Stress-strain curve of a perfectly plastic material (idealization)

We will discuss more complex (realistic) models of plastic material later.

σ_Y: yield stress

Note: After the yield point is reached, the stress-strain curve becomes history dependent.

§7 Constitutive (Stress-Strain) Relation in the Elastic Regime

If the stress never reaches the yield condition (i.e. $\sigma < \sigma_y$ in tensile test) then the stress-strain relation is linear and history-independent.

Generalized Hooke's Law: $\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}^{el}$

$\uparrow$ elastic stiffness tensor

$$\varepsilon_{ij} = \varepsilon_{ij}^{el}, \qquad \varepsilon_{ij}^{pl} = 0$$

Isotropic elasticity: $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \lambda = \frac{2\mu\nu}{1-2\nu}$$

μ: shear modulus

ν: Poisson's ratio

In component form

$$\sigma_{xx} = (\lambda+2\mu)\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \qquad \sigma_{xy} = 2\mu\,\varepsilon_{xy}$$

$$\sigma_{yy} = \lambda\,\varepsilon_{xx} + (\lambda+2\mu)\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \qquad \sigma_{yz} = 2\mu\,\varepsilon_{yz}$$

$$\sigma_{zz} = \lambda\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + (\lambda+2\mu)\,\varepsilon_{zz} \qquad \sigma_{zx} = 2\mu\,\varepsilon_{zx}$$

notice factor of 2

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \qquad \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \qquad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} \qquad \varepsilon_{zx} = \frac{1}{2\mu}\sigma_{zx}$$

$E = 2\mu(1+\nu)$ — <u>Young's modulus</u>

Define hydrostatic stress (mean normal stress), hydrostatic strain (mean normal strain)

$$\bar{\sigma} \equiv \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) \qquad \bar{\varepsilon} \equiv \frac{1}{3}\varepsilon_{ii} = \frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})$$

$$\sigma_{xx}+\sigma_{yy}+\sigma_{zz} = (3\lambda+2\mu)(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})$$

$$K = \frac{1}{3}(3\lambda+2\mu) = \frac{2\mu(1+\nu)}{3(1-2\nu)} = \frac{E}{3(1-2\nu)}$$ — <u>Bulk modulus</u>

$$\boxed{\bar{\sigma} = 3K\,\bar{\varepsilon}}$$

Define deviatoric stress tensor, deviatoric strain tensor

$$s_{ij} \equiv \sigma_{ij} - \bar{\sigma}\,\delta_{ij} \qquad e_{ij} \equiv \varepsilon_{ij} - \bar{\varepsilon}\,\delta_{ij}$$

$$s_{xx} = \sigma_{xx}-\bar{\sigma},\quad s_{yy} = \sigma_{yy}-\bar{\sigma},\quad s_{zz} = \sigma_{zz}-\bar{\sigma} \qquad e_{xx} = \varepsilon_{xx}-\bar{\varepsilon},\quad e_{yy} = \varepsilon_{yy}-\bar{\varepsilon},\quad e_{zz} = \varepsilon_{zz}-\bar{\varepsilon}$$

$$s_{xy} = \sigma_{xy},\quad s_{yz} = \sigma_{yz},\quad s_{zx} = \sigma_{zx} \qquad e_{xy} = \varepsilon_{xy},\quad e_{yz} = \varepsilon_{yz},\quad e_{zx} = \varepsilon_{zx}$$

The relation between deviatoric stress and deviatoric strain is very simple

$$\boxed{s_{ij} = 2\mu\,e_{ij}} \qquad \text{for all } i, j = 1, 2, 3$$

* obviously, the equations here are also the same as in elasticity theory. An emphasis on deviatoric stress-strain is given here to better compare with plastic strain (which is all deviatoric).

§8 Yield Condition

We postulate that the yield condition (i.e. onset of plastic deformation) can be described by a function of stress tensor

$$f(\{\sigma_{ij}\}) = 0 \quad , \quad \text{i.e.} \quad f(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}) = 0$$

At this point, f is a general function.
Note that time and rates ($\dot{\sigma}_{ij} \equiv \frac{d}{dt}\sigma_{ij}$, $\dot{\varepsilon}_{ij} \equiv \frac{d}{dt}\varepsilon_{ij}$) do not enter f.
So this is still an idealization, which is well justified by experimental observations of ductile metals.

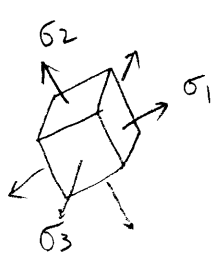
We shall assume that the material is isotropic, then the value of f should not change by a coordinate transformation, i.e.

$$f(\{\sigma_{ij}\}) = f(\{\sigma_{ij}'\}), \quad \text{where} \quad \sigma_{ij}' = Q_{ip} Q_{jq} \sigma_{pq}$$

This means that we must be able to write f in terms stress invariants — functions of σ_{ij} that does not change by coordinate transformation.

For example, the mean normal stress $\bar{\sigma} \equiv \frac{1}{3}\sigma_{ii}$ is a stress invariant,
i.e. $\bar{\sigma} = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\sigma_{ii}'$

Other stress invariants can be defined in terms of the principal stress values: σ_1, σ_2, σ_3



They are the normal stress values in a special coordinately system such that all shear stresses vanish.

Mathematically, $\sigma_1, \sigma_2, \sigma_3$ are the eigenvalues of the stress matrix

$$\{\sigma_{ij}\} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

i.e. $\sigma_1, \sigma_2, \sigma_3$ are the three solutions of the eigen-equation

$$\det\left[\underset{\approx}{\sigma} - \hat{\lambda}\underset{\approx}{I}\right] = \begin{vmatrix} \sigma_{xx}-\hat{\lambda} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-\hat{\lambda} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\hat{\lambda} \end{vmatrix} = 0 \qquad \text{------ (1)}$$

The eigen-equation is a polynomial equation of $\hat{\lambda}$:

$$\hat{\lambda}^3 - I_1\hat{\lambda}^2 - I_2\hat{\lambda} - I_3 = 0 \qquad \text{------- (2)}$$

where I_1, I_2, I_3 are stress invariants.

Because $\sigma_1, \sigma_2, \sigma_3$ are solutions of the eigen-equation, we must have

$$(\hat{\lambda}-\sigma_1)(\hat{\lambda}-\sigma_2)(\hat{\lambda}-\sigma_3) = 0$$

$$\hat{\lambda}^3 - (\sigma_1+\sigma_2+\sigma_3)\hat{\lambda} - (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\hat{\lambda} - \sigma_1\sigma_2\sigma_3 = 0 \quad \text{---- (3)}$$

Comparing Eq (2) and Eq (3), we get

$$\begin{cases} I_1 = \sigma_1+\sigma_2+\sigma_3 = 3\bar{\sigma} \\ I_2 = -(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) \\ I_3 = \sigma_1\sigma_2\sigma_3 \end{cases}$$

Three stress invariants in terms of the principal stress values.

(Note: some books choose opposite sign in the definition of I_2)

The stress invariants can also be computed from $\{\sigma_{ij}\}$ of an arbitrary coordinate system (for which shear stresses do not vanish).

Comparing Eq (1) and Eq (2), we get

$$\begin{cases} I_1 = \sigma_{xx}+\sigma_{yy}+\sigma_{zz} = 3\bar{\sigma} \\ -I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} = (\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}) - (\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2) \\ \qquad\qquad = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \\ I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} \end{cases}$$

Because the yield condition is experimentally found to be independent of pressure (i.e. hydrostatic stress $\bar{\sigma}$), it is more convenient to express function f in terms of the invariants of deviatoric stress s_{ij}.

Let s_1, s_2, s_3 be the principal values of $\{s_{ij}\}$.

i.e. they are the solution of the eigen-equation

$$\det(\underset{\approx}{s} - \hat{\lambda}\underset{\approx}{I}) = \begin{vmatrix} s_{xx}-\hat{\lambda} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy}-\hat{\lambda} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz}-\hat{\lambda} \end{vmatrix} = 0 \quad \text{-----} (4)$$

$$\hat{\lambda}^3 - J_1\hat{\lambda}^2 + J_2\hat{\lambda} + J_3 = 0 \quad \text{-----} (5)$$

where J_1, J_2, J_3 are invariants of deviatoric stress $\{s_{ij}\}$.

$$(\hat{\lambda}-s_1)(\hat{\lambda}-s_2)(\hat{\lambda}-s_3) = 0$$

$$\hat{\lambda}^3 - (s_1+s_2+s_3)\hat{\lambda}^2 + (s_1s_2+s_2s_3+s_3s_1)\hat{\lambda} - s_1s_2s_3 = 0 \quad \text{-----} (6)$$

Comparing Eq (4) and Eq (5), we get

(because $s_1+s_2+s_3=0$
$0=(s_1+s_2+s_3)^2 = s_1^2+s_2^2+s_3^2+2s_1s_2+2s_2s_3+$

$$\begin{cases} J_1 = s_1+s_2+s_3 = 0 \\ J_2 = -(s_1s_2+s_2s_3+s_3s_1) = \frac{1}{2}(s_1^2+s_2^2+s_3^2) \\ J_3 = s_1s_2s_3 = \frac{1}{3}(s_1^3+s_2^3+s_3^3) \end{cases}$$

(* sign convention for J_2 consistent with Prager-Hodge

$s_1^3+s_2^3+s_3^3 = s_1^3+s_2^3-(s_1+s_2)^3$
$= -3s_1s_2^2 - 3s_1^2s_2 = -3s_1s_2(s_1+s_2)$
$= 3s_1s_2s_3$)

comparing Eq (4) and Eq (6), we get

(because $s_{xx}+s_{yy}+s_{zz}=0$
$0=(s_{xx}+s_{yy}+s_{zz})^2 = s_{xx}^2+s_{yy}^2+s_{zz}^2$
$+2s_{xx}s_{yy}+2s_{yy}s_{zz}+2s_{zz}$

$$\begin{cases} J_1 = s_{xx}+s_{yy}+s_{zz} = 0 \\ J_2 = -(s_{xx}s_{yy}+s_{yy}s_{zz}+s_{zz}s_{xx}) + (s_{yz}^2+s_{zx}^2+s_{xy}^2) \\ \quad = \frac{1}{2}(s_{xx}^2+s_{yy}^2+s_{zz}^2) + (s_{yz}^2+s_{zx}^2+s_{xy}^2) \\ \quad = \frac{1}{2}s_{ij}s_{ij} \\ J_3 = \begin{vmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{vmatrix} \end{cases}$$

($J_2 = \frac{1}{2}s_{ij}s_{ij}$ is a measure of the magnitude of the deviatoric stress, similar to the norm of a vector squared.)

Here are the stress invariants we have obtained

$$I_1 = 3\bar{\sigma}, \quad I_2 = -\frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}), \quad I_3 = \det(\sigma_{ij})$$

$$\bar{\sigma}, \quad J_1 = 0, \quad J_2 = \frac{1}{2} S_{ij} S_{ij}, \quad J_3 = \det(S_{ij})$$

The yield condition can be written as either

$$f(I_1, I_2, I_3) = 0$$

or

$$f(\bar{\sigma}, J_2, J_3) = 0$$

Experimentally, it was found that $\bar{\sigma}$ has no significant effect on the yield condition (and plastic deformation in general, which conserves volume).

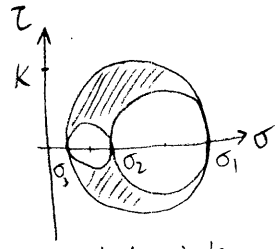
So the yield condition of isotropic medium can be written as

$$f(J_2, J_3) = 0$$

In the most widely used yield condition (J_2-plasticity) the dependence on J_3 is also dropped.

$$f(J_2) = J_2 - k^2 = 0 \qquad \text{— } \underline{\text{Von Mises criterion}}$$

In another widely used yield condition, k_T is the greatest shear stress.



3D Mohr's circle

If $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then $\sigma_1 - \sigma_3 = 2k_T$ i.e. $S_1 - S_2 = 2k_T$ — <u>Tresca criterion</u>

In general, $\left[(S_1-S_2)^2 - 4k_T^2\right]\left[(S_3-S_1)^2 - 4k_T^2\right]\left[(S_1-S_2)^2 - 4k_T^2\right] = 0$

$$\rightarrow f(J_2, J_3) = 4J_2^3 - 27J_3^2 - 36k_T^2 J_2^2 + 96k_T^4 J_2 - 64k_T^6 = 0$$

Mathematically much more complex.
More difficult to use if principal stress not already known.

In practice, the difference between Von Mises and Tresca criteria is less than 15%, within typical experimental error.

The simple form of von Mises criterion makes it most popular.
(For experimental verification of Von Mises criterion, see <u>Lecture Note 2</u>)

We now relate the parameter k to the yield stress σ_Y in tensile test.

In tensile test, $\sigma_{xx} > 0$, $\sigma_{yy} = \sigma_{zz} = 0$

$$\bar{\sigma} = \frac{1}{3}\sigma_{xx}$$

$$S_{xx} = \frac{2}{3}\sigma_{xx}, \quad S_{yy} = -\frac{1}{3}\sigma_{xx}, \quad S_{zz} = -\frac{1}{3}\sigma_{xx}$$

$$J_2 = \frac{1}{2}(S_{xx}^2 + S_{yy}^2 + S_{zz}^2) = \frac{1}{2}\cdot\frac{6}{9}\sigma_{xx}^2 = \frac{1}{3}\sigma_{xx}^2$$

At yield point $J_2 = \frac{1}{3}\sigma_Y^2 = k^2$

$$\therefore \sigma_Y = \sqrt{3}\,k, \qquad k = \frac{\sigma_Y}{\sqrt{3}} \qquad \text{(Von Mises)}$$

(For Tresca yield condition, $k_T = \frac{\sigma_Y}{2}$)

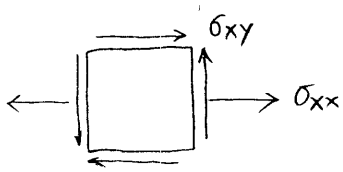
If the sample is subjected to pure shear stress σ_{xy} (all other stress component=0

then $J_2 = \sigma_{xy}$

At onset of yield, $\sigma_{xy} = \tau_c$, $J_2 = \tau_c^2 = k^2$

$\therefore \tau_c = k$, i.e. k is the critical shear stress when a pure shear stress is applied. (Von Mises)

(For Tresca yield condition, $k_T = \tau_c$ also)

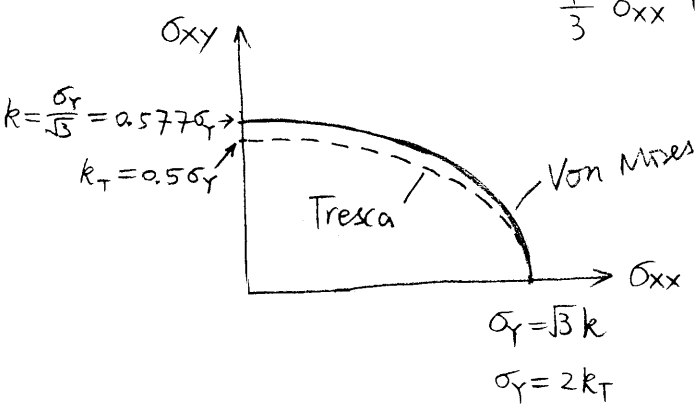


when σ_{xx} and σ_{xy} are the two non-zero stress components,

$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2$$

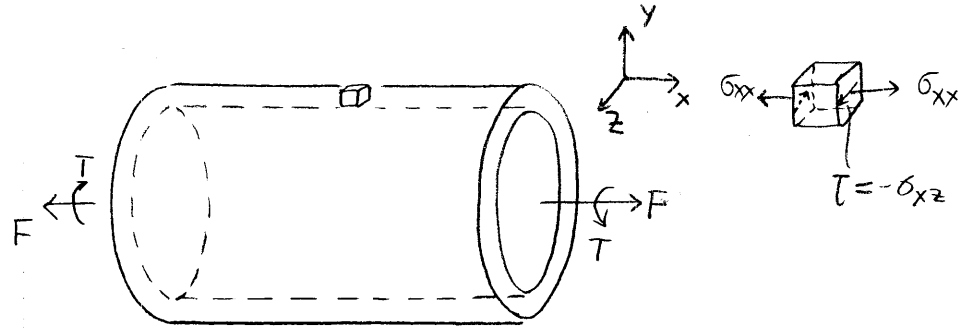
Von Mises yield condition

$$\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$$



(exercise: draw the curve corresponds to the Tresca yield condition on the $\sigma_{xx} - \sigma_{xy}$ plane using Matlab.

Testing Yield Criterion by Experiments



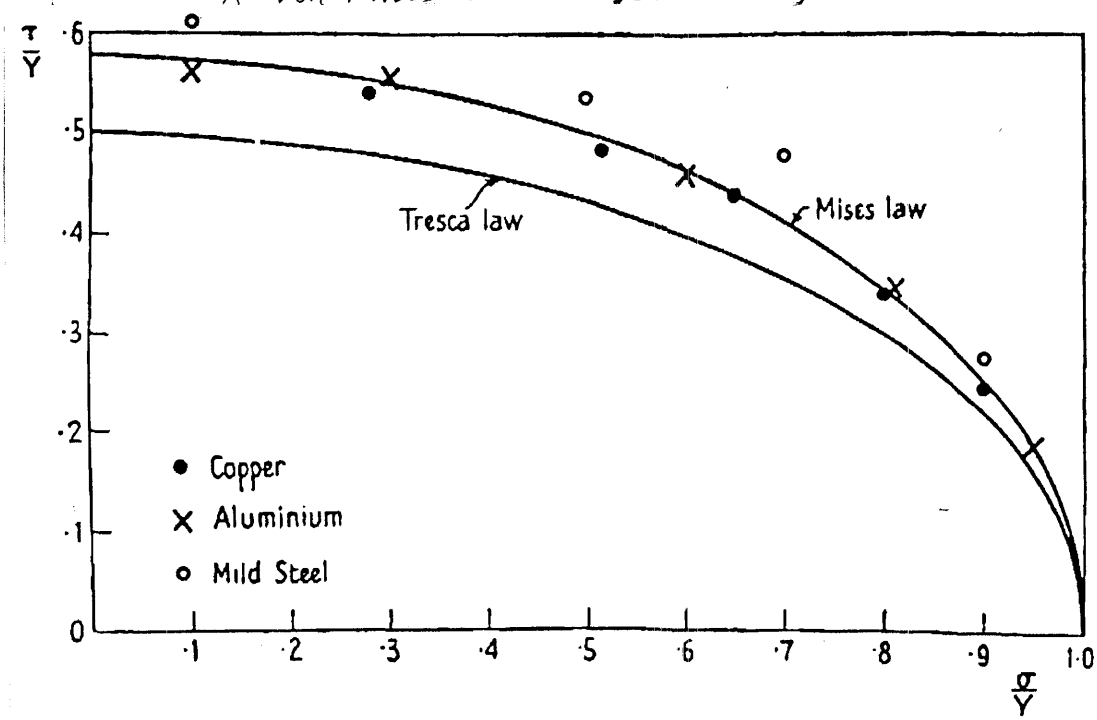
consider a tube subjected to both tension force F and torque T.

The element shown above will be subjected to both normal stress σ_{xx} and shear stress σ_{xz}.

Increasing F and T together while keeping $\frac{T}{F}$ at constant allows us to increase σ_{xx} and σ_{xz} together with $\frac{\sigma_{xz}}{\sigma_{xx}} = \text{const}$.

Measuring the critical values of σ_{xx}, σ_{xz} for various $\frac{\sigma_{xz}}{\sigma_{xx}}$ allows us to map the yield surface and compare with theoretical models.

* Von Mises criterion seem to agree better with experiments than Tresca.



From: R. Hill, The Mathematical Theory of Plasticity, Oxford Univ. Press 1950 P.22

Taylor and Quinney Phil. Trans. Roy. Soc. A 230, 323 (1931)

FIG. 4. Experimental results of Taylor and Quinney from combined torsion and tension tests, each metal being work-hardened to the same state for all tests. The Mises law is $\sigma^2 + 3\tau^2 = Y^2$, while the Tresca law is $\sigma^2 + 4\tau^2 = Y^2$, where σ = tensile stress, τ = shear stress, Y = tensile yield stress.

§9 Constitutive (stress-strain) Relation in the Plastic Regime

For simplicity, here we consider a perfectly plastic material (no strain hardening.

Then in the plastic regime $f(\{\sigma_{ij}\}) = J_2 - k^2 = 0$, k remains constant

i.e. $J_2 = k^2$, $\dot{J}_2 = 0$

However, this does not mean that all stress components stays constant.

$J_2 = k^2$ defines a 'yield surface' in the stress space.

The stress components can vary within the yield surface

Recall $J_2 = \frac{1}{2}(s_{xx}^2 + s_{yy}^2 + s_{zz}^2) + (s_{yz}^2 + s_{zx}^2 + s_{xy}^2)$

$$\dot{J}_2 = s_{xx}\dot{s}_{xx} + s_{yy}\dot{s}_{yy} + s_{zz}\dot{s}_{zz} + 2 s_{yz}\dot{s}_{yz} + 2 s_{zx}\dot{s}_{zx} + 2 s_{xy}\dot{s}_{xy} = s_{ij}\dot{s}_{ij} = 0$$

This is a constraint on the stress rate $\dot{s}_{ij}$.

While the stress stays on the yield surface, the material will undergo plastic strain ε_{ij}^{pl}, so that $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$

ε_{ij}^{el} remain proportional to stress σ_{ij} : $\sigma_{ij} = \lambda \varepsilon_{kk}^{el}\delta_{ij} + 2\mu\varepsilon_{ij}^{el}$

$$\boxed{s_{ij} = 2\mu e_{ij}^{el}}$$

Q: How to determine ε_{ij}^{pl}?

It took a long time for scientists to find out the correct way to compute ε_{ij}^{pl} (after many wrong starts).

It became know that ε_{ij}^{pl} is history dependent, i.e. it cannot be determined by the current stress state (σ_{ij}).

Instead, ε_{ij}^{pl} must be determined by accumulating small increments.

$$\varepsilon_{ij}^{pl} = \int_0^t \dot{\varepsilon}_{ij}^{pl}(t)\,dt$$

In addition, it has been found that plastic deformation produces negligible volume change: $\varepsilon_{ii}^{pl} = 0$, $\dot{\varepsilon}_{ii}^{pl} = 0$

$\therefore$ deviatoric plastic strain is the same as plastic strain

Assumption of "associative flow":

when yield condition is reached, $\dot{\varepsilon}_{ij}^{pl}$ follow the direction of s_{ij}

$$\boxed{2\mu\,\dot{\varepsilon}_{ij}^{pl} = \tilde{\lambda}\, s_{ij}}$$

where λ is a positive scalar factor to be determined.

compare this with $2\mu\, e_{ij}^{el} = s_{ij}$, $2\mu\,\dot{e}_{ij}^{el} = \dot{s}_{ij}$

Note both the deviatoric elastic strain (e_{ij}^{el})
and the plastic strain rate $(\dot{\varepsilon}_{ij}^{pl})$
follow the deviatoric stress (s_{ij}).

Total deviatoric strain rate

$$2\mu\,\dot{e}_{ij} = 2\mu\left(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}\right) = \dot{s}_{ij} + \lambda s_{ij}$$

To determine the factor $\tilde{\lambda}$, we need to introduce

$$\dot{W} = s_{ij}\,\dot{e}_{ij}$$

← rate of work done associated with shape change
part of which goes to stored elastic energy
part of which dissipated as heat

$$= s_{ij}\left(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}\right)$$

(work done associated with volume change is entirely stored as elastic energy)

$$\dot{W} = s_{xx}\dot{e}_{xx} + s_{yy}\dot{e}_{yy} + s_{zz}\dot{e}_{zz} + 2\,s_{yz}\dot{e}_{yz} + 2\,s_{zx}\dot{e}_{zx} + 2\,s_{xy}\dot{e}_{xy}$$

$$2\mu\,\dot{W} = s_{ij}\left(\dot{s}_{ij} + \tilde{\lambda} s_{ij}\right)$$

Recall $\dot{J}_2 = s_{ij}\dot{s}_{ij} = 0$ for perfectly plastic material

$$2\mu\,\dot{W} = \tilde{\lambda} s_{ij} s_{ij} = 2\tilde{\lambda} J_2 = 2\tilde{\lambda} k^2$$

$$\therefore \boxed{\tilde{\lambda} = \frac{2\mu}{2k^2}\dot{W}}$$

Q: How to determine $\dot{\sigma}_{ij}$?

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2}\cdot s_{ij} \quad , \quad \dot{s}_{ij} = 2\mu\,\dot{e}_{ij}^{el} = 2\mu\left(\dot{e}_{ij} - \dot{\varepsilon}_{ij}^{pl}\right)$$

$$\dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \frac{\dot{W}}{2k^2}s_{ij}\right)$$

describes how stress responds to imposed strain rate $\dot{\varepsilon}_{ij}$

The hydrostatic stress-strain response is purely elastic

$$\bar{\sigma} = 3K\,\bar{\varepsilon} \quad , \quad \dot{\bar{\sigma}} = 3K\,\dot{\bar{\varepsilon}}$$

Summary: How stress changes under a specified (total) strain rate $\dot{\varepsilon}_{ij}$ in the plastic regime?

Given $\dot{\varepsilon}_{ij}$ and current stress σ_{ij}, to find $\dot{\sigma}_{ij}$ (imagine a metal forming process)

$$\begin{tikzcd}
 & \dot{\bar{\varepsilon}} = \frac{1}{3}\dot{\varepsilon}_{ii} \arrow[r] & \dot{\bar{\sigma}} = 3K\dot{\bar{\varepsilon}} \arrow[rr] & & \dot{\sigma}_{ij} = \dot{s}_{ij} + \dot{\bar{\sigma}}\delta_{ij} \\
\dot{\varepsilon}_{ij} \arrow[ur] \arrow[dr] & & & & \\
 & \dot{e}_{ij} = \dot{\varepsilon}_{ij} - \dot{\bar{\varepsilon}}\delta_{ij} \arrow[dr] \arrow[rr, dashed] & & \dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \frac{\dot{W}}{2k^2}s_{ij}\right) \arrow[ur] & \\
 & & \dot{W} = s_{ij}\dot{e}_{ij} \arrow[ur] & & \\
\sigma_{ij} \arrow[ur] \arrow[dr] & \bar{\sigma} = \frac{1}{3}\sigma_{ii} & & & \\
 & s_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij} \arrow[uur] \arrow[rr, dashed] & & &
\end{tikzcd}$$

§10 Summary of All Formulas

Elastic Regime: $\bar{\sigma} = 3K\bar{\varepsilon}$ (hydrostatic part)

$s_{ij} = 2\mu e_{ij}$ (deviatoric part)

Yield condition: $J_2 = \frac{1}{2}s_{ij}s_{ij} = k^2$ (Von Mises)

$\sigma_Y = \sqrt{3}\,k$ (yield stress in tension)

Plastic Regime: $\dot{e}_{ij}^{el} = \frac{1}{2\mu}\dot{s}_{ij}$ (deviatoric elastic strain rate)

$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2}s_{ij}$ (plastic strain rate)

$\dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \frac{\dot{W}}{2k^2}s_{ij}\right)$ (deviatoric stress rate)

$\dot{\bar{\sigma}} = 3K\dot{\bar{\varepsilon}}$ (hydrostatic stress rate)

In this lecture, we give more discussions on the yield conditions, (Von Mises and Tresca), including their graphical representation and experimental verification.

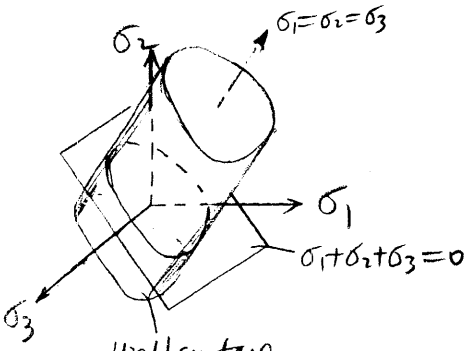
§1 Yield Surface in the space of Principal Stresses

If the material is isotropic, the three pricipal stress values contains all the information about whether the material has reached the yield condition or not.

In other words, the yield condition is reached on a 2D surface, i.e. the yield surface, in the 3D space spanned by σ_1, σ_2, σ_3

$$f(\sigma_1, \sigma_2, \sigma_3) = 0$$

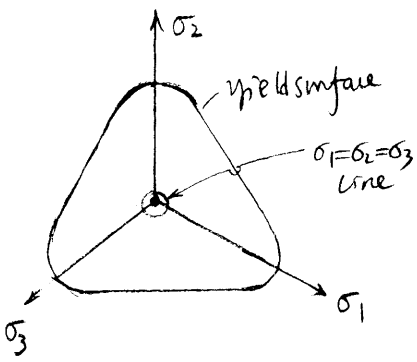
Further more, experiments have shown that hydrostatic stress $\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ plays no role in yield.



yield surface $f(\sigma_1, \sigma_2, \sigma_3) = 0$

This means that the yield surface must have a prismatic shape, with the axes along the line defined by $\sigma_1 = \sigma_2 = \sigma_3$ (i.e. [111] direction)

In other words, the intersection between the yield surface $f(\sigma_1, \sigma_2, \sigma_3) = 0$ and any plane defined by $\sigma_1 + \sigma_2 + \sigma_3 = c$ gives a curve of the same shape



$\sigma_1+\sigma_2+\sigma_3=0$ plane

Imagine looking down on the $\sigma_1+\sigma_2+\sigma_3=0$ then the entire yield surface looks like a curve.

In other words, all we need to do is to specify the shape of this curve. The yield surface can be constructed by translating this curve along [111].

Material isotropy require the yield surface be symmetric with respect to 120° rotation around the $\sigma_1=\sigma_2=\sigma_3$ line, i.e. $\sigma_1 \to \sigma_2$, $\sigma_2 \to \sigma_3$, $\sigma_3 \to \sigma_1$,

as well as mirror reflection, e.g. $\sigma_1 \to \sigma_3$, $\sigma_3 \to \sigma_1$

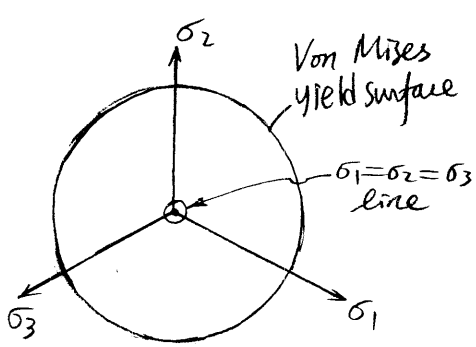
Other restrictions can be imposed on the yield surface. For example, the yield surface is usually assumed to be convex, which we will discuss later.

§2 Von Mises Yield Condition

$$J_2 \equiv \frac{1}{2}\left(S_1^2+S_2^2+S_3^2\right) = k^2$$

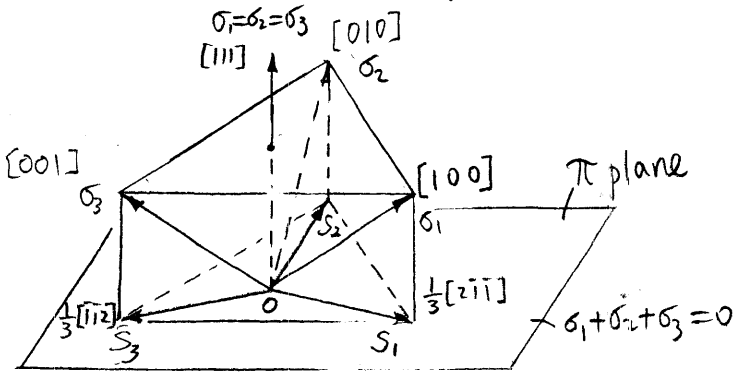
Note $\bar{\sigma}=\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$, $S_1=\sigma_1-\bar{\sigma}$, $S_2=\sigma_2-\bar{\sigma}$, $S_3=\sigma_3-\bar{\sigma}$

$$\begin{aligned} J_2 &= \frac{1}{2}\left[(\sigma_1-\bar{\sigma})^2+(\sigma_2-\bar{\sigma})^2+(\sigma_3-\bar{\sigma})^2\right] \\ &= \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-3\bar{\sigma}^2\right) \\ &= \frac{1}{3}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] \\ &= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \end{aligned}$$



The Von Mises yield surface is a cylinder with a circular cross section.

The proof is given in the next page.



To be more precise, we shall express J in terms of vectors on the π-plane, i.e.

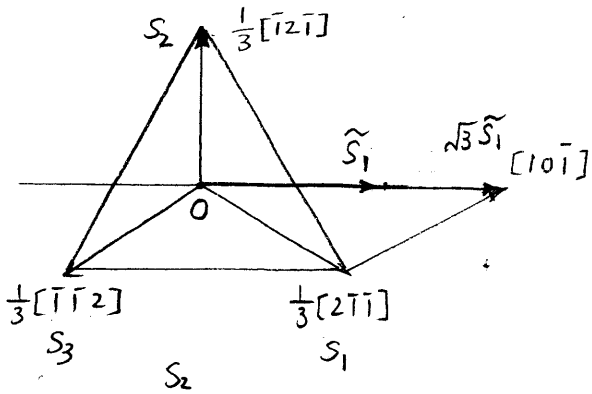
$S_1,\ S_2,\ S_3,$

Note: $S_1+S_2+S_3=0$

$$S_1=\frac{1}{3}(2\ \ -1\ \ -1)\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}=\frac{2\sigma_1-\sigma_2-\sigma_3}{3}$$

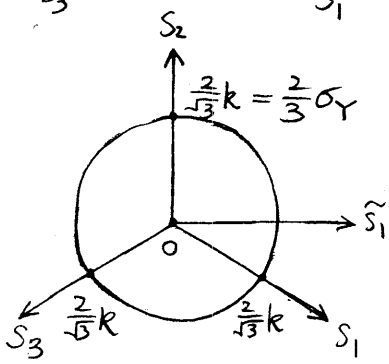
$$S_2=\frac{1}{3}(-1\ \ 2\ \ -1)\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}=\frac{-\sigma_1+2\sigma_2-\sigma_3}{3}$$

$$S_3=\frac{1}{3}(-1\ \ -1\ \ 2)\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}=\frac{-\sigma_1-\sigma_2+2\sigma_3}{3}$$



Note that S_1, S_2, S_3 are not independent.

The axis of S_1, S_2, S_3 are not perpendicular to each other.



Introduce $\tilde{S}_1$ whose axis is perpendicular to the axis of S_2.

$$\tilde{S}_1=\frac{1}{\sqrt{3}}(1\ \ 0\ \ -1)\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}=\frac{\sigma_1-\sigma_3}{\sqrt{3}}$$

Because

$$\frac{1}{2}\left\{(1\ \ 0\ \ -1)+\frac{1}{3}(1\ \ -2\ \ 1)\right\}=\frac{1}{3}(2\ \ -1\ \ -1)$$

$$\frac{1}{2}\left\{(-1\ \ 0\ \ 1)+\frac{1}{3}(1\ \ -2\ \ 1)\right\}=\frac{1}{3}(-1\ \ -1\ \ 2)$$

we have

$$S_1=\frac{\sqrt{3}}{2}\tilde{S}_1-\frac{1}{2}S_2$$

$$S_3=-\frac{\sqrt{3}}{2}\tilde{S}_1-\frac{1}{2}S_2$$

$$J_2=\frac{1}{2}(S_1^2+S_2^2+S_3^2)=\frac{3}{4}(\tilde{S}_1^2+S_2^2)=k^2$$

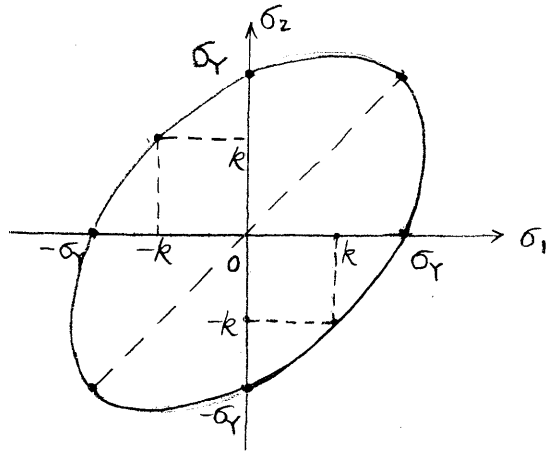
$$\tilde{S}_1^2+S_2^2=\left(\frac{2}{\sqrt{3}}k\right)^2=\left(\frac{2}{3}\sigma_Y\right)^2$$

Von Mises yield condition is a circle of radius

$$\frac{2}{\sqrt{3}}k=\frac{2}{3}\sigma_Y$$

in the plane of S_1, S_2, S_3.

Von Mises Criterion in plane stress



$\sigma_3 = 0$

$$J_2 = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

$$= \frac{1}{6}\left(\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_2^2 + \sigma_1^2\right)$$

$$= \frac{1}{3}\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right) = k^2$$

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = (\sqrt{3}\,k)^2 = \sigma_Y^2$$

The intersection of yield surface cylinder with $\sigma_3 = 0$ plane is an ellipse

several special points on yield surface

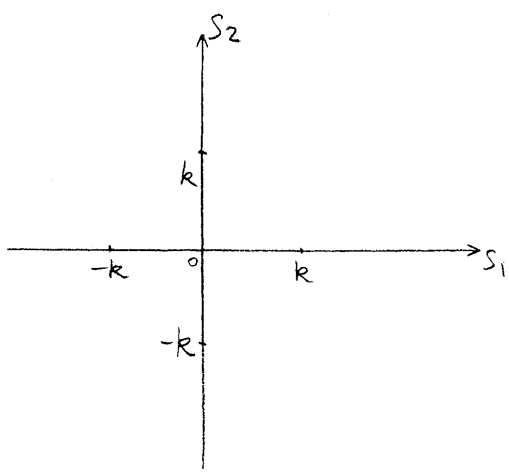
$\sigma_1 = \sigma_Y \quad \sigma_2 = 0$

$\sigma_1 = 0 \quad \sigma_2 = \sigma_Y$

$\sigma_1 = \sigma_Y \quad \sigma_2 = \sigma_Y$

$\sigma_1 = k \quad \sigma_2 = -k$

Von Mises Criterion in the plane of S_1-S_2



by definition, $S_1 + S_2 + S_3 = 0$

$$S_3 = -(S_1 + S_2)$$

$$J_2 = \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right)$$

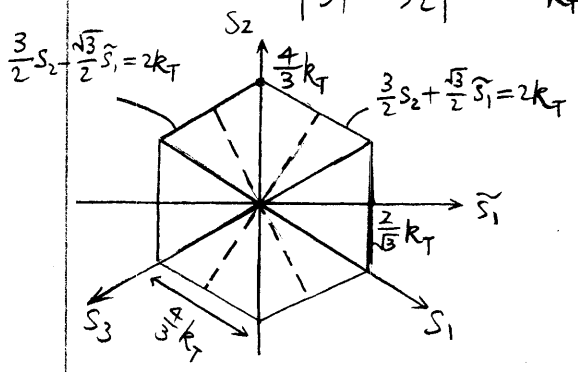
$$= \frac{1}{2}\left(S_1^2 + S_2^2 + (S_1+S_2)^2\right)$$

$$= S_1^2 + S_2^2 + S_1 S_2$$

Sketch the yield surface on S_1-S_2 plane

§3. Tresca Yield Condition

$$|s_1 - s_2| = 2k_T, \quad \text{or} \quad |s_2 - s_3| = 2k_T, \quad \text{or} \quad |s_3 - s_1| = 2k_T$$



In terms of $\tilde{s}_1$ and s_2,

$$\left|\frac{\sqrt{3}}{2}\tilde{s}_1 - \frac{3}{2}s_2\right| = 2k_T$$

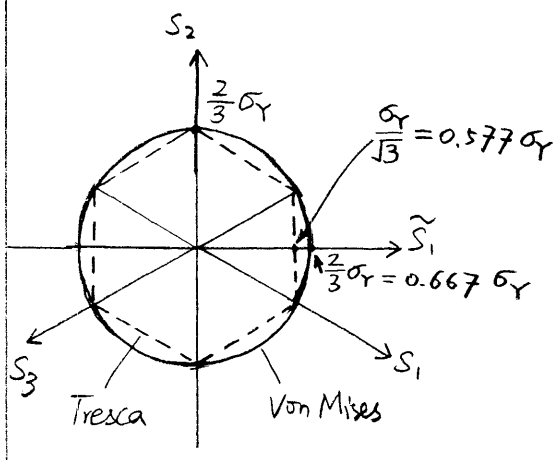
$$\left|\frac{\sqrt{3}}{2}\tilde{s}_1 + \frac{3}{2}s_2\right| = 2k_T$$

$$\left|\sqrt{3}\,\tilde{s}_1\right| = 2k_T$$

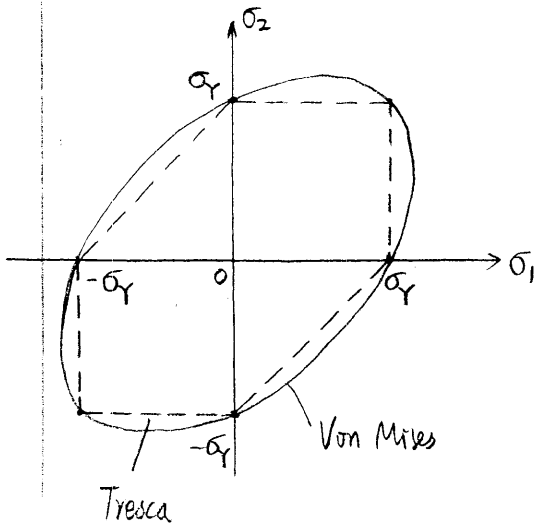
Tresca yield condition is a hexagon of side length

$$\frac{4}{3}k_T = \frac{2}{3}\sigma_Y$$

in the plane of s_1, s_2, s_3



Tresca yield condition is a hexagon inscribed in the circle corresponding to the Von Mises condition.



Tresca yield condition in plane stress ($\sigma_3 = 0$): $|\sigma_1 - \sigma_2| = \sigma_Y$, $|\sigma_1| = \sigma_Y$, $|\sigma_2| = \sigma_Y$ is a polygon inscribed in the ellipse corresponding to the Von Mises condition.

§4. Flow Rule for Perfectly Plastic Material (no hardening)

elastic response: $e_{ij}^{el} = \frac{1}{2\mu} S_{ij}$

deviatoric elastic strain — deviatoric stress

$$de_{ij}^{el} = \frac{1}{2\mu} dS_{ij}$$

incremental deviatoric elastic strain — incremental deviatoric stress

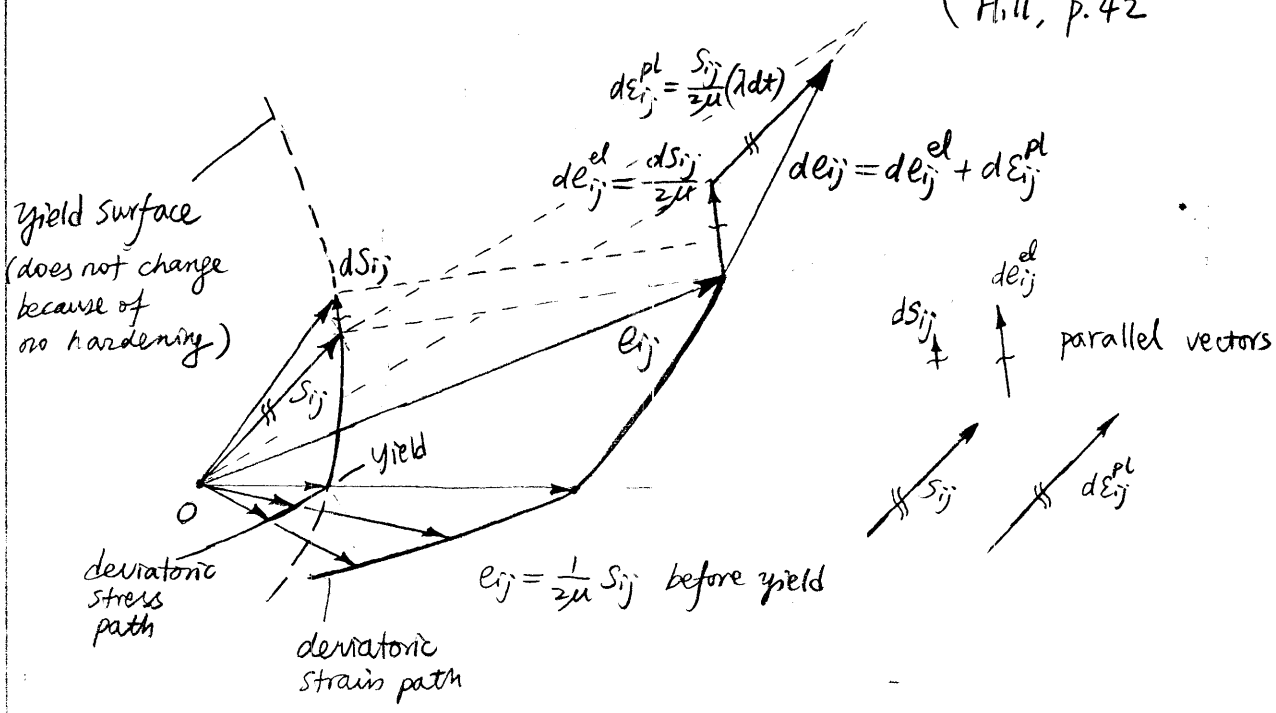
plastic response: $\dot{\varepsilon}_{ij}^{pl} = \frac{1}{2\mu} \lambda S_{ij}$

plastic strain rate (automatically deviatoric)

$$d\varepsilon_{ij}^{pl} = \frac{1}{2\mu} S_{ij} \cdot (\lambda dt)$$

incremental plastic strain

(analogus to Fig 7 of Hill, p. 42)



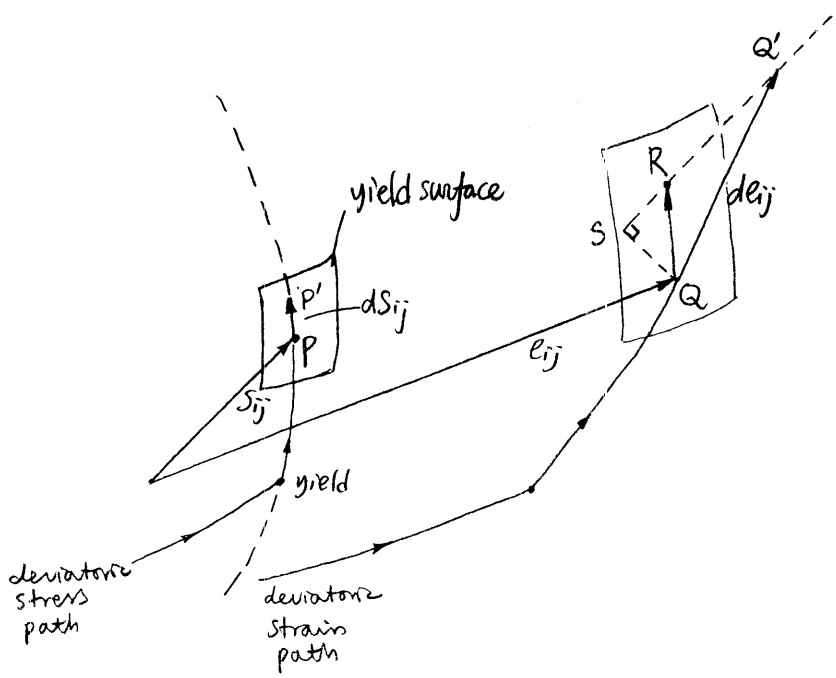
- before S_{ij} reaches yield surface, e_{ij} is parallel to S_{ij}, $de_{ij} \parallel dS_{ij}$
- after yield condition is reached, e_{ij} is no longer parallel to S_{ij}.
  de_{ij} is no longer parallel to dS_{ij}
- $de_{ij} = de_{ij}^{el} + d\varepsilon_{ij}^{pl}$, $de_{ij}^{el} \parallel dS_{ij}$, $d\varepsilon_{ij}^{pl} \parallel S_{ij}$

In the plastic regime,

given s_{ij}, e_{ij}, de_{ij} (change of total deviatoric strain) elastic+plastic

find ds_{ij} (change of deviatoric stress)

- draw line SQ' parallel to s_{ij}
- Construct the patch of yield surface at point P
- Draw a parallel surface patch at point Q
- Find the intersection point R between line SQ' and surface patch
- $RQ' = d\varepsilon_{ij}^{pl}$, $QR = de_{ij}^{el}$, $\underline{PP' = ds_{ij} = 2\mu \cdot QR}$

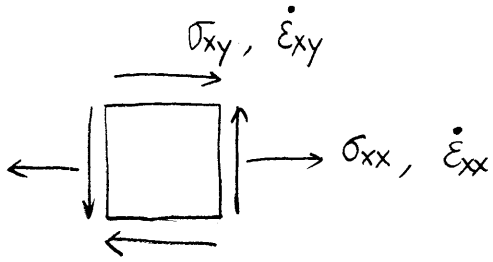


We now work out a simple example problem in which the material is subjected to both tension and shear (which can be achieved in a tube under tension and torsion).

This example illustrates how to solve a plasticity problem, as well as the <u>history-dependent</u> nature of plasticity.

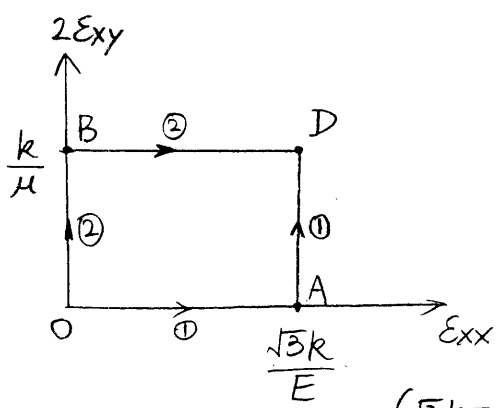
This example is extracted from Dr. Paul Paslay's lecture video "Introduction to Theoretical and Applied Plasticity"
For copies of this DVD, send email to <u>info@blade-energy.com</u>.

§1. Problem Statement



Assume no work hardening, von Mises yield criterion, incompressible material ($\nu = 0.5$)

Incompressibility means even the elastic strain ε_{ij}^{el} conserves volume, so that $\varepsilon_{ij}^{el} = e_{ij}^{el}$ (deviatoric)
This simplifies discussions.
Given $\varepsilon_{ij}^{pl} = e_{ij}^{pl}$ (deviatoric)
we have $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl} = e_{ij} = e_{ij}^{el} + e_{ij}^{pl}$



($\sqrt{3}k = \sigma_Y$)

We shall consider <u>two different strain paths</u>:

① OA–AD First load in tension to exactly the yield point then load in shear

② OB–BD First load in shear to exactly the yield point then load in tension

Note: we have assumed incompressibility ($\nu = 0.5$)

So $E = 2\mu(1+\nu) = 3\mu$

The goal is to find the stress (as a function of strain) along these two strain paths.

§2. Strain Path ①: Tension – Shear

Obviously, along path OA, all strains are elastic.

$$\sigma_{xx} = E \cdot \varepsilon_{xx} \quad (\text{goes from } 0 \text{ to } \sqrt{3}k = \sigma_Y)$$

$$\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$$

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\,\varepsilon_{xx} = -\frac{1}{2}\varepsilon_{xx}$$

Along path AD, the stress need to stay on yield surface

i.e. $J_2 = k^2$, $\dot{J}_2 = 0$

$$\dot{\varepsilon}^{pl}_{ij} = \frac{\dot{W}}{2k^2} s_{ij}$$

$\dot{W} = s_{ij}\dot{e}_{ij}$: rate of work done associated with shape change

$$\dot{W}_{tot} \equiv \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_{xy} \cdot 2\dot{\varepsilon}_{xy}$$

(In this example, $\dot{W}$ is also to rate of work done because volume stays constant)

simplification due to incompressibility $\longrightarrow$ $\therefore \underline{\dot{W} = \sigma_{xy} \cdot 2\dot{\varepsilon}_{xy}}$

Since $\sigma_{yy} = \sigma_{zz} = 0$, $\bar{\sigma} = \frac{\sigma_{xx}}{3}$, $s_{xx} = \frac{2}{3}\sigma_{xx}$, $s_{yy} = -\frac{1}{3}\sigma_{xx}$, $s_{zz} = -\frac{1}{3}\sigma_{xx}$

$$J_2 = \frac{1}{2}(s_{xx}^2 + s_{yy}^2 + s_{zz}^2) + (s_{yz}^2 + s_{zx}^2 + s_{xy}^2)$$

$$= \underline{\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2}$$

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}^{el}_{xy} + \dot{\varepsilon}^{pl}_{xy} \underset{\text{(incompressibility)}}{=} \dot{e}^{el}_{xy} + \dot{\varepsilon}^{pl}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{W}}{2k^2}\sigma_{xy}$$

$$\dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{\varepsilon}_{xy}}{k^2}\sigma_{xy}^2$$

$$\left(1-\frac{\sigma_{xy}^2}{k^2}\right)\dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu}$$

$$2\mu\,\frac{\dot{\varepsilon}_{xy}}{k} = \frac{\dfrac{\dot{\sigma}_{xy}}{k}}{1-\left(\dfrac{\sigma_{xy}}{k}\right)^2}$$

The solution to this equation is

$$2\mu\,\frac{\varepsilon_{xy}(t)}{k} = \operatorname{arctanh}\left(\frac{\sigma_{xy}(t)}{k}\right),$$

$$\left(\frac{d}{dx}\operatorname{arctanh}(x) = \frac{1}{1-x^2}\right)$$

i.e.

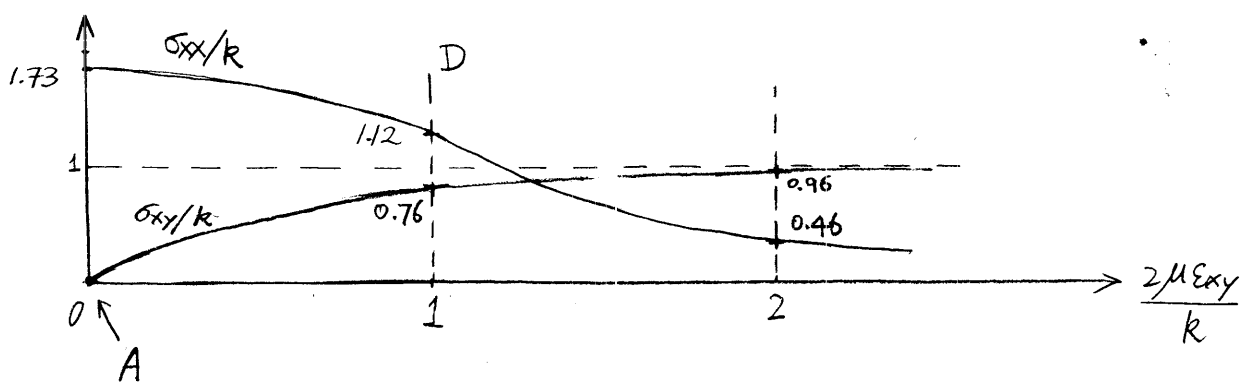
$$\frac{\sigma_{xy}(t)}{k} = \tanh\left(2\mu\,\frac{\varepsilon_{xy}(t)}{k}\right)$$

Given that $\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$

$$\left(1-[\tanh(x)]^2 = \frac{(\cosh x)^2-(\sinh x)^2}{(\cosh x)^2} = \frac{1}{(\cosh x)^2}\right)$$

$$\frac{\sigma_{xx}}{k} = \sqrt{3}\cdot\sqrt{1-\left(\frac{\sigma_{xy}}{k}\right)^2}$$

$$\frac{\sigma_{xx}(t)}{k} = \frac{\sqrt{3}}{\cosh\left(\dfrac{2\mu\,\varepsilon_{xy}(t)}{k}\right)}$$



At point A, $\sigma_{xx} = \sqrt{3}\,k$ and $\sigma_{xy} = 0$

With increasing shear strain, the normal stress σ_{xx} decreases while the shear stress σ_{xy} increases.

At point D, $\boxed{\sigma_{xx} = 1.12\,k \text{ and } \sigma_{xy} = 0.76\,k}$

If shear strain keeps on increasing beyond D, eventually $\sigma_{xx} \to 0$, $\sigma_{xy} \to k$ (yield condition in pure shear)

§3. Strain Path ②: Shear - Tension

Obviously, along path OB, all strains are elastic

$$\sigma_{xy} = 2\mu \cdot \varepsilon_{xy} \quad (\text{goes from } o \text{ to } k)$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

Along path BD, the stress need to stay on yield surface

$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$$

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2} S_{ij}, \quad \dot{\varepsilon}_{xx}^{pl} = \frac{\dot{W}}{2k^2} S_{xx} = \frac{\dot{W}}{2k^2}\cdot\frac{2}{3}\sigma_{xx} = \frac{\dot{W}}{3k^2}\sigma_{xx}$$

simplification due to incompressibility $\longrightarrow$ $\dot{W} = \dot{W}_{tot} = \sigma_{xx}\dot{\varepsilon}_{xx}$

$$\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{xx}^{el} + \dot{\varepsilon}_{xx}^{pl} = \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{W}}{3k^2}\sigma_{xx}$$

$$\dot{\varepsilon}_{xx} = \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{\varepsilon}_{xx}}{3k^2}\sigma_{xx}^2$$

$$\left(1 - \frac{\sigma_{xx}^2}{3k^2}\right)\dot{\varepsilon}_{xx} = \frac{\dot{\sigma}_{xx}}{E}, \quad E\frac{\dot{\varepsilon}_{xx}}{\sqrt{3}k} = \frac{\frac{\dot{\sigma}_{xx}}{\sqrt{3}k}}{1 - \left(\frac{\sigma_{xx}}{\sqrt{3}k}\right)^2}$$

The solution to this equation is

$$E\frac{\varepsilon_{xx}(t)}{\sqrt{3}k} = \operatorname{arctanh}\left(\frac{\sigma_{xx}(t)}{\sqrt{3}k}\right), \text{ i.e. } \frac{\sigma_{xx}(t)}{\sqrt{3}k} = \tanh\left(E\frac{\varepsilon_{xx}(t)}{\sqrt{3}k}\right) = \tanh\left(\sqrt{3}\mu\frac{\varepsilon_{xx}(t)}{k}\right)$$

Note: $\sqrt{3}k = \sigma_Y$, $E = 3\mu$ (incompressibility)

Given that $\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$

$$\frac{\sigma_{xy}}{k} = \sqrt{1 - \left(\frac{\sigma_{xx}}{\sqrt{3}k}\right)^2}$$

$$\frac{\sigma_{xy}(t)}{k} = \frac{1}{\cosh\left(E\frac{\varepsilon_{xx}(t)}{\sqrt{3}k}\right)} = \frac{1}{\cosh\left(\sqrt{3}\mu\frac{\varepsilon_{xx}(t)}{k}\right)}$$

So along path BD,

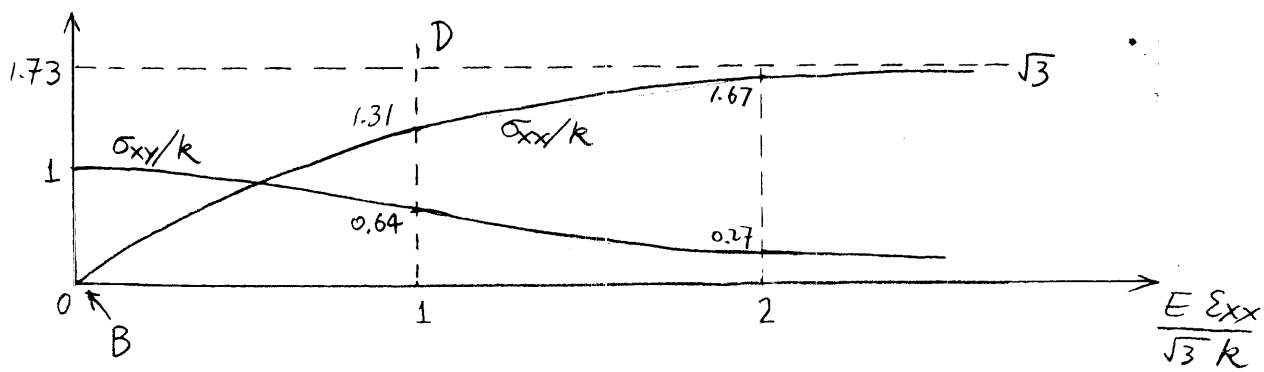
$$\frac{\sigma_{xx}}{k} = \sqrt{3}\tanh\left(\sqrt{3}\mu\frac{\varepsilon_{xx}}{k}\right)$$

$$\frac{\sigma_{xy}}{k} = \frac{1}{\cosh\left(\sqrt{3}\mu\frac{\varepsilon_{xx}}{k}\right)}$$

compared with path AD, the roles of σ_{xx} and σ_{xy} are reversed.

(* exercise: plot plastic strain ε_{xx}^{pl}, ε_{xy}^{pl} as functions of ε_{xx}.)

(* exercise: consider a strain path in which ε_{xx} goes from 0 to $\frac{\sigma_Y}{E}$ and then ε_{yy} goes from 0 to $\frac{\sigma_Y}{E}$, find σ_{xx} and σ_{yy})



At point B, $\sigma_{xx} = 0$, and $\sigma_{xy} = k$

With increasing normal strain, the shear stress σ_{xy} decreases while the normal stress increases.

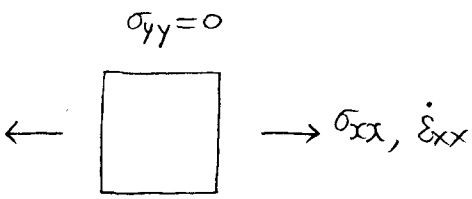
At point D, $\boxed{\sigma_{xx} = 1.31\,k \text{ and } \sigma_{xy} = 0.64\,k}$

* different from the values on p.3 — history dependence.

If normal strain keeps on increasing beyond D, eventually $\sigma_{xx} \to \sqrt{3}\,k$ (yield stress in tension), $\sigma_{xy} \to 0$.

We now work through an example problem that requires a numerical method, to illustrate how numerical methods can be applied to plasticity problems.

§1. Problem Statement



Plane strain condition: $\varepsilon_{zz}=0$

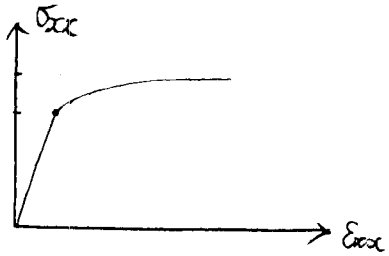
The material is loaded along x

y-surface is traction free: $\sigma_{yy}=0$

Assume von Mises yield criterion, no hardening

$\nu < 0.5$

(If $\nu=0.5$, i.e. incompressible, then the problem can be solved analytically. The condition $\nu<0.5$ is what makes numerical method necessary)



Find the stress-strain relation

$\sigma_{xx}(\varepsilon_{xx})$

§2. Elastic Regime

Before yield occurs, the stress-strain relation is linear

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \cancel{\frac{\nu}{E}\sigma_{yy}} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \cancel{\frac{1}{E}\sigma_{yy}} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \cancel{\frac{\nu}{E}\sigma_{yy}} - \frac{\nu}{E}\sigma_{zz} = 0$$

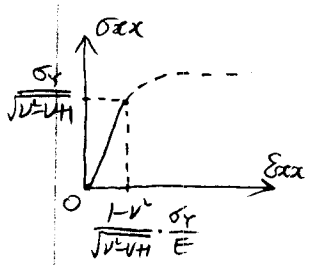
$$\therefore \quad \sigma_{zz} = \nu\,\sigma_{xx}, \qquad \bar{\sigma} = \frac{1+\nu}{3}\sigma_{xx}$$

$$S_{xx} = \sigma_{xx} - \bar{\sigma} = \frac{2-\nu}{3}\sigma_{xx}, \quad S_{yy} = \sigma_{yy} - \bar{\sigma} = -\frac{1+\nu}{3}\sigma_{xx},$$

$$S_{zz} = \sigma_{zz} - \bar{\sigma} = \frac{2\nu-1}{3}\sigma_{xx}$$

$$J_2 = \frac{1}{2}\left(S_{xx}^2 + S_{yy}^2 + S_{zz}^2\right) = \frac{\sigma_{xx}^2}{3}\left(\nu^2 - \nu + 1\right)$$

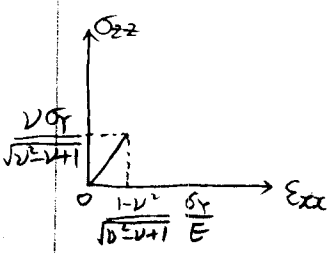
At onset of yield, $J_2 = k^2 = \frac{\sigma_Y^2}{3}$

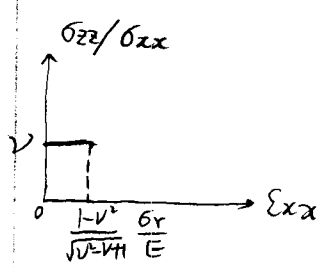


$$\sigma_{xx} = \frac{\sigma_Y}{\sqrt{\nu^2-\nu+1}}$$

(This is higher than σ_Y due to the constraint $\varepsilon_{zz} = 0$.)

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{zz}\right) = \frac{1-\nu^2}{E}\sigma_{xx} = \frac{1-\nu^2}{\sqrt{\nu^2-\nu+1}}\cdot\frac{\sigma_Y}{E}$$





The ratio $\frac{\sigma_{zz}}{\sigma_{xx}}$ remains constant ν within the elastic regime.

§3. Plastic Regime

Our strategy is to choose the 'principal' unknowns of the problem, express every other unknowns in terms of them, and find sufficient number of equations to solve for the 'principal' unknowns.

We shall choose three 'principal' unknowns: σ_{xx}, σ_{zz}, $\frac{\tilde{\lambda}}{2\mu}$

Every other quantity can be expressed in terms of these three variables and other known quantities.

We then need to identify three equations to solve the problem.

Recall $\sigma_{yy} = 0$

$$\bar{\sigma} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3}(\sigma_{xx} + \sigma_{zz})$$

$$S_{xx} = \sigma_{xx} - \bar{\sigma}, \qquad S_{yy} = -\bar{\sigma}, \qquad S_{zz} = \sigma_{zz} - \bar{\sigma}$$

$$\bar{\varepsilon}^{el} = \frac{\bar{\sigma}}{3K}, \quad \text{where } K = \frac{E}{3(1-2\nu)} \text{ is bulk modulus}$$

$$\varepsilon_{xx}^{el} = \bar{\varepsilon}^{el} + e_{xx}^{el} = \frac{\bar{\sigma}}{3K} + \frac{S_{xx}}{2\mu}, \quad \varepsilon_{yy}^{el} = \frac{\bar{\sigma}}{3K} + \frac{S_{yy}}{2\mu}, \quad \varepsilon_{zz}^{el} = \frac{\bar{\sigma}}{3K} + \frac{S_{zz}}{2\mu}$$

$$\dot{\varepsilon}_{xx}^{pl} = \frac{\tilde{\lambda}}{2\mu} S_{xx}, \qquad \dot{\varepsilon}_{yy}^{pl} = \frac{\tilde{\lambda}}{2\mu} S_{yy}, \qquad \dot{\varepsilon}_{zz}^{pl} = \frac{\tilde{\lambda}}{2\mu} S_{zz}$$

Given that plasticity is path dependent, we need to solve the problem in small time increments

Assume we know everything at time t:

$$\sigma_{xx}(t),\ \sigma_{zz}(t),\ \text{etc}$$

and we need to find the solution at $t+\Delta t$:

$$\sigma_{xx}(t+\Delta t),\ \sigma_{zz}(t+\Delta t),\ \text{etc.}$$

We shall also treat $\varepsilon_{xx}(t+\Delta t)$, $\varepsilon_{zz}(t+\Delta t)$ as known quantities because we wish to express everything in terms of ε_{xx}, and the plane strain condition requires $\varepsilon_{zz}=0$.

Known: $\sigma_{xx}(t)$, $\sigma_{zz}(t)$, $\varepsilon_{xx}(t)$, $\varepsilon_{zz}(t)$, $\varepsilon_{xx}(t+\Delta t)$, $\varepsilon_{zz}(t+\Delta t)$,

unknown: $\sigma_{xx}(t+\Delta t)$, $\sigma_{zz}(t+\Delta t)$, $\frac{\tilde{\lambda}}{2\mu}$ (in the period $[t, t+\Delta t]$)

The change of stress from t to $t+\Delta t$ is

$$\Delta\bar{\sigma} = \bar{\sigma}(t+\Delta t) - \bar{\sigma}(t), \quad \text{where } \bar{\sigma}(t+\Delta t) = \frac{1}{3}\left[\sigma_{xx}(t+\Delta t) + \sigma_{zz}(t+\Delta t)\right]$$

$$\bar{\sigma}(t) = \frac{1}{3}\left[\sigma_{xx}(t) + \sigma_{zz}(t)\right]$$

$$\Delta S_{xx} = S_{xx}(t+\Delta t) - S_{xx}(t)$$

$$\Delta S_{zz} = S_{zz}(t+\Delta t) - S_{zz}(t)$$

This leads to a change of elastic strain

$$\Delta\varepsilon_{xx}^{el} = \frac{\Delta\bar{\sigma}}{3K} + \frac{\Delta S_{xx}}{2\mu}$$

$$\Delta\varepsilon_{zz}^{el} = \frac{\Delta\bar{\sigma}}{3K} + \frac{\Delta S_{zz}}{2\mu}$$

The plastic strain increment from t to $t+\Delta t$ can be approximated by

$$\Delta \varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} \cdot \frac{s_{xx}(t) + s_{xx}(t+\Delta t)}{2}$$

$$\Delta \varepsilon_{zz}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} \cdot \frac{s_{zz}(t) + s_{zz}(t+\Delta t)}{2}$$

Notice that the average deviatoric stress during period $[t, t+\Delta t]$ is approximated by the average value at t and $t+\Delta t$.

This corresponds to the Trapezoid rule of numerical quadrature.

Compared with an alternative approximation based on forward Euler

$$\Delta \varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} s_{xx}(t), \quad \Delta \varepsilon_{zz}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} s_{zz}(t)$$

the Trapezoid (or mid-point) rule is not only more accurate, but also (more importantly) more stable (hence tolerating larger Δt).

Now that everything is expressed in terms of $\sigma_{xx}(t+\Delta t)$, $\sigma_{zz}(t+\Delta t)$, $\frac{\tilde{\lambda}}{2\mu}$ we are ready to write down the three equations:

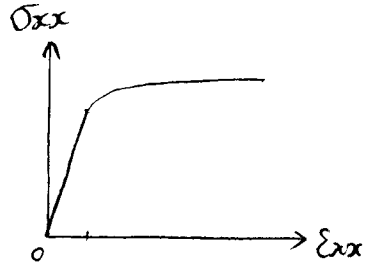
$$\begin{cases} \varepsilon_{xx}(t) + \Delta \varepsilon_{xx}^{el} + \Delta \varepsilon_{xx}^{pl} - \varepsilon_{xx}(t+\Delta t) = 0 & \text{(imposed strain)} \\ \varepsilon_{zz}(t) + \Delta \varepsilon_{zz}^{el} + \Delta \varepsilon_{zz}^{pl} - \varepsilon_{zz}(t+\Delta t) = 0 & \text{(imposed strain)} \\ \frac{1}{2}\left[s_{xx}^2(t+\Delta t) + s_{yy}^2(t+\Delta t) + s_{zz}^2(t+\Delta t) \right] - k^2 = 0 & \text{(yield condition)} \end{cases}$$

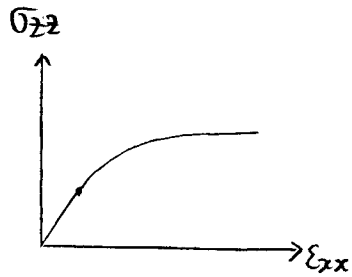
This is a set of 3 non-linear equations for 3 unknowns.

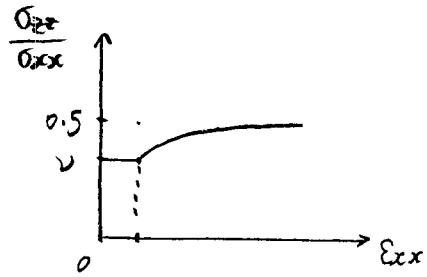
It can be conveniently solved by the 'fsolve' command in Matlab.

The Matlab files are attached below (also on Courseworks).

eqns_plane_strain_uniaxial_plast.m implements the 3 equations which are solved by calling 'fsolve' in plane-strain_uniaxial.m







From the numerical solution, we can see that $\frac{\sigma_{zz}}{\sigma_{xx}}$ transitions from ν to 0.5 during the plastic regime.

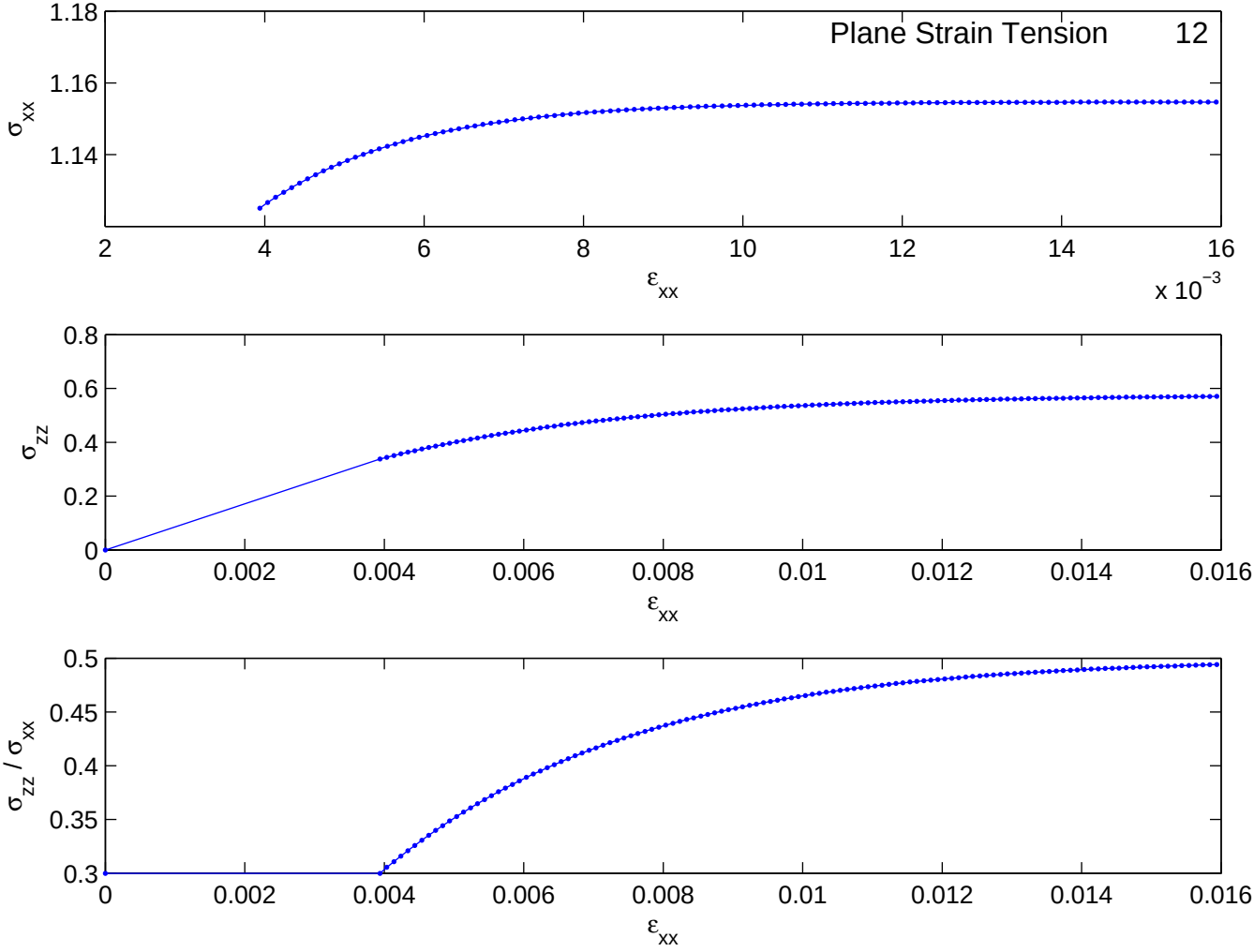
```
% ME342 Theory and Applications of Inelasticity
% Wei Cai, caiwei@stanford.edu
%
% Example: plane strain uniaxial loading

% Material parameters
mu = 100;                   % shear modulus
nu = 0.3;                   % Poisson's ratio
sig_Y = 1;                  % yield stress

K = 2*mu*(1+nu)/3/(1-2*nu); % bulk modulus
E = 2*mu*(1+nu);            % Young's modulus
k = sig_Y/sqrt(3);

% condition at yield
sig_xx = sig_Y/sqrt(nu^2-nu+1); % see HW 1.1(a)
sig_yy = 0;
sig_zz = nu*sig_xx;

% hydrostatic stress
sig_bar   = (sig_xx + sig_yy + sig_zz)/3;

% deviatoric stress
s_xx = sig_xx - sig_bar;
s_yy = sig_yy - sig_bar;
s_zz = sig_zz - sig_bar;
J2 = (s_xx^2 + s_yy^2 + s_zz^2)/2;  % J2 should equal k^2 at yield

% strain (elastic at onset of yield)
eps_xx = sig_bar/(3*K) + s_xx/(2*mu);
eps_yy = sig_bar/(3*K) + s_yy/(2*mu);
eps_zz = sig_bar/(3*K) + s_zz/(2*mu);

% strain array (going beyond yield point)
eps_xx_data = eps_xx + [0:120]*1e-4;

% initialize arrays to store results
sig_xx_data = zeros(size(eps_xx_data));
sig_zz_data = zeros(size(eps_xx_data));
lambda_dt_over_2mu_data = zeros(size(eps_xx_data));
s_xx_data = zeros(size(eps_xx_data));
s_yy_data = zeros(size(eps_xx_data));
s_zz_data = zeros(size(eps_xx_data));
J2_data = zeros(size(eps_xx_data));

% the first data point is the yield point considered above
sig_xx_data(1) = sig_xx;
sig_zz_data(1) = sig_zz;
s_xx_data(1)   = s_xx;
s_yy_data(1)   = s_yy;
s_zz_data(1)   = s_zz;
```

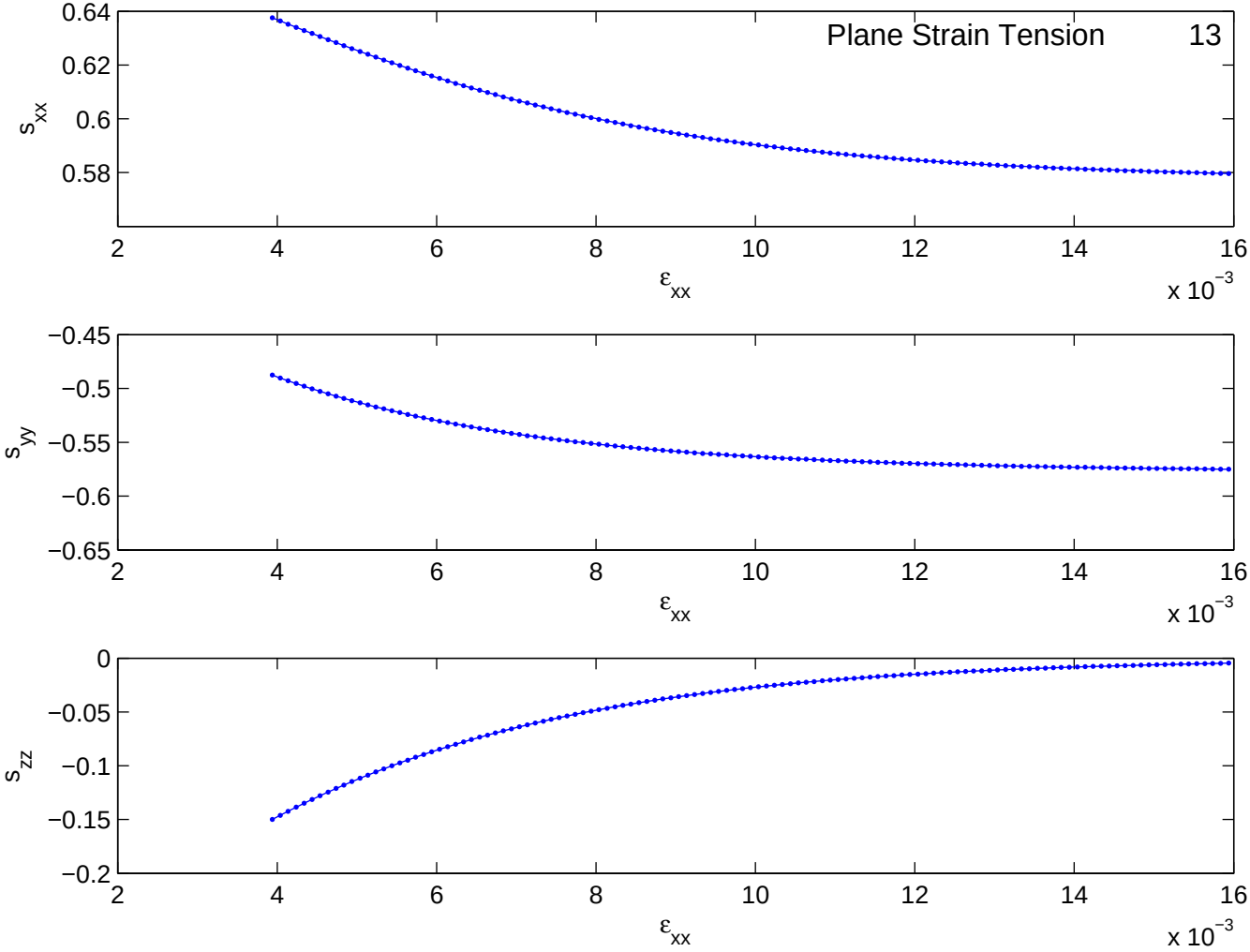
```
J2_data(1)     = J2;

for i = 2:length(eps_xx_data),
    param = [mu, nu, k, sig_xx_data(i-1), sig_zz_data(i-1), ...
             eps_xx_data(i-1), eps_xx_data(i)];
    %options = optimset('Display','iter','TolFun',1e-10);
    options = optimset('Display','off','TolFun',1e-10);
    trial = [sig_xx_data(i-1), sig_zz_data(i-1), 0];
    sol = fsolve('eqns_plane_strain_uniaxial_plast', trial, options, param);
    sig_xx = sol(1); sig_zz = sol(2); lambda_dt_over_2mu = sol(3);

    % compute hydrostatic stress and deviatoric stress
    sig_bar = (sig_xx + sig_zz)/3;
    s_xx = sig_xx - sig_bar;
    s_yy = sig_yy - sig_bar;
    s_zz = sig_zz - sig_bar;
    J2 = (s_xx^2 + s_yy^2 + s_zz^2)/2;

    % save results to array
    sig_xx_data(i) = sig_xx; sig_zz_data(i) = sig_zz;
    lambda_dt_over_2mu_data(i) = lambda_dt_over_2mu;
    s_xx_data(i) = s_xx; s_yy_data(i) = s_yy; s_zz_data(i) = s_zz;
    J2_data(i) = J2;
end

% plot solution
% plot stress
figure(1);
subplot(3,1,1);
plot(eps_xx_data, sig_xx_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('\sigma_{xx}');
subplot(3,1,2);
plot([0 eps_xx_data], [0 sig_zz_data], '.-');
xlabel('\epsilon_{xx}');
ylabel('\sigma_{zz}');
subplot(3,1,3);
plot([0 eps_xx_data], [nu sig_zz_data./sig_xx_data], '.-');
xlabel('\epsilon_{xx}');
ylabel('\sigma_{zz} / \sigma_{xx}');

% plot deviatoric stress
figure(2);
subplot(3,1,1);
plot(eps_xx_data, s_xx_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('s_{xx}');
subplot(3,1,2);
plot(eps_xx_data, s_yy_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('s_{yy}');
```

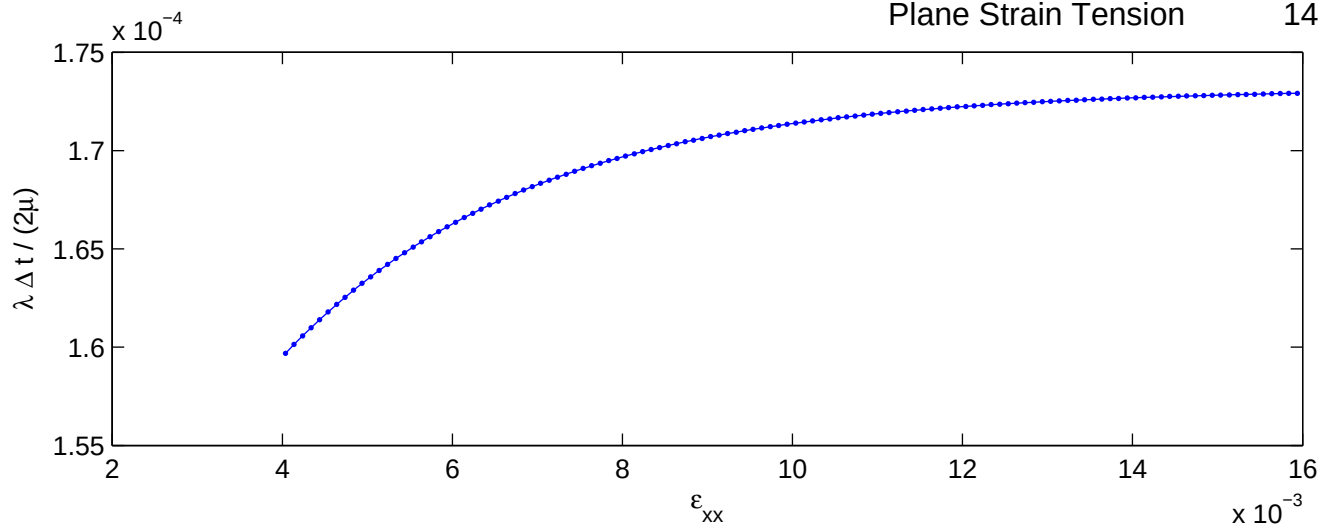
```
subplot(3,1,3);
plot(eps_xx_data, s_zz_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('s_{zz}');

% plot lambda_dt_over_2mu and J2 (should equal k^2)
figure(3);
subplot(2,1,1);
plot(eps_xx_data, [NaN lambda_dt_over_2mu_data(2:end)], '.-');
xlabel('\epsilon_{xx}');
ylabel('\lambda \Delta t / (2\mu)');
subplot(2,1,2);
plot(eps_xx_data, J2_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('J_2');
ylim([0.9 1.1]*k^2);
```

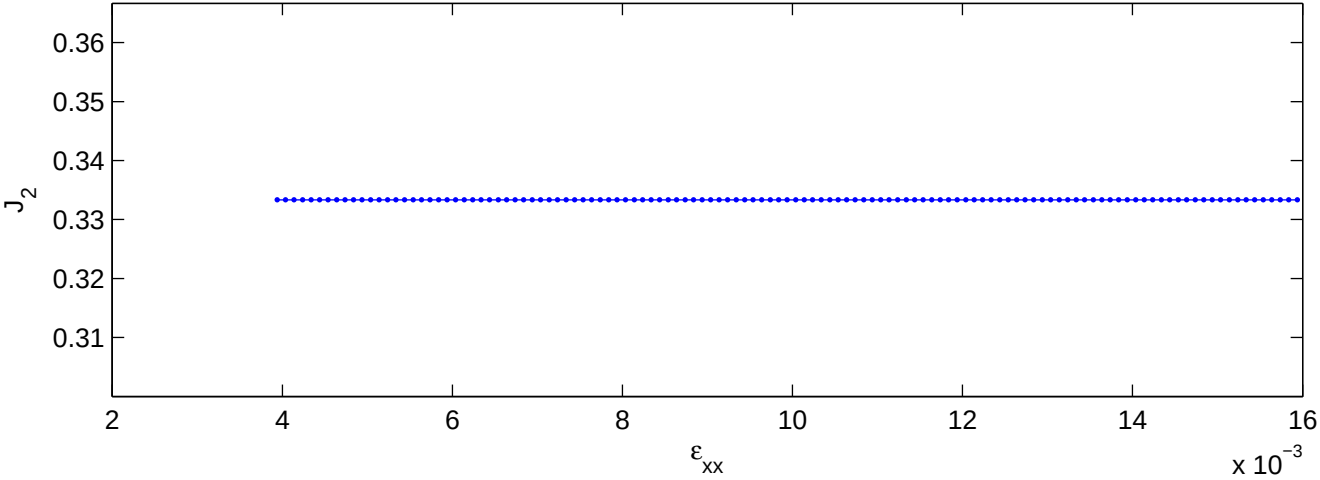
```
% ME342 Theory and Applications of Inelasticity
% Wei Cai, caiwei@stanford.edu
%
% equations to be solved by plane_strain_uniaxial.m

function F = eqns_plane_strain_uniaxial_plast(vars, param)
% F(1): eps_xx_tot (specified)
% F(2): eps_zz_tot = 0
% F(3): yield condition

sig_xx = vars(1);
sig_zz = vars(2);
lambda_dt_over_2mu = vars(3);
sig_yy = 0;

mu = param(1);
nu = param(2);
k  = param(3);
sig_xx_0 = param(4);
sig_zz_0 = param(5);
sig_yy_0 = 0;
eps_xx_0 = param(6); % total strain at previous time step
eps_xx   = param(7); % total strain at current  time step
eps_zz_0 = 0;
eps_zz   = 0;

K = 2*mu*(1+nu)/3/(1-2*nu); % bulk modulus

% change of stress
dsig_xx = sig_xx - sig_xx_0;
dsig_yy = sig_yy - sig_yy_0;
dsig_zz = sig_zz - sig_zz_0;

dsig_bar = (dsig_xx + dsig_yy + dsig_zz)/3;
ds_xx = dsig_xx - dsig_bar;
ds_yy = dsig_yy - dsig_bar;
ds_zz = dsig_zz - dsig_bar;

% increment of elastic strain
deps_elast_xx = dsig_bar/(3*K) + ds_xx/(2*mu);
deps_elast_yy = dsig_bar/(3*K) + ds_yy/(2*mu);
deps_elast_zz = dsig_bar/(3*K) + ds_zz/(2*mu);

% hydrostaic stress
sig_bar   = (sig_xx + sig_yy + sig_zz)/3;
sig_bar_0 = (sig_xx_0 + sig_yy_0 + sig_zz_0)/3;

% deviatoric stress
s_xx = sig_xx - sig_bar;
s_yy = sig_yy - sig_bar;
s_zz = sig_zz - sig_bar;
```

```
s_xx_0 = sig_xx_0 - sig_bar_0;
s_yy_0 = sig_yy_0 - sig_bar_0;
s_zz_0 = sig_zz_0 - sig_bar_0;

% average deviatoric stress
ave_s_xx = (s_xx + s_xx_0)/2;
ave_s_yy = (s_yy + s_yy_0)/2;
ave_s_zz = (s_zz + s_zz_0)/2;

% increment of plastic strain
deps_plast_xx = lambda_dt_over_2mu * ave_s_xx;
deps_plast_yy = lambda_dt_over_2mu * ave_s_yy;
deps_plast_zz = lambda_dt_over_2mu * ave_s_zz;

% construct equation
F = [0 0 0]';

% total strain in xx
F(1) = (eps_xx_0 + deps_elast_xx + deps_plast_xx) - eps_xx;

% total strain in zz
F(2) = (eps_zz_0 + deps_elast_zz + deps_plast_zz) - eps_zz;

% yield condition
J2 = (s_xx^2 + s_yy^2 + s_zz^2)/2;
F(3) = J2 - k^2;
```


Plane Strain Tension
12
1.18
1.16
1.14
σxx
2
4
6
8
10
12
14
16
εxx
x 10−3
0.8
0.6
0.4
0.2
0
σzz
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
εxx
0.5
0.45
0.4
0.35
0.3
σzz / σxx
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
εxx


Plane Strain Tension
13
0.64
0.62
0.6
0.58
s_{xx}
2
4
6
8
10
12
14
16
ε_{xx}
x 10^{-3}
−0.45
−0.5
−0.55
−0.6
−0.65
s_{yy}
2
4
6
8
10
12
14
16
ε_{xx}
x 10^{-3}
0
−0.05
−0.1
−0.15
−0.2
s_{zz}
2
4
6
8
10
12
14
16
ε_{xx}
x 10^{-3}

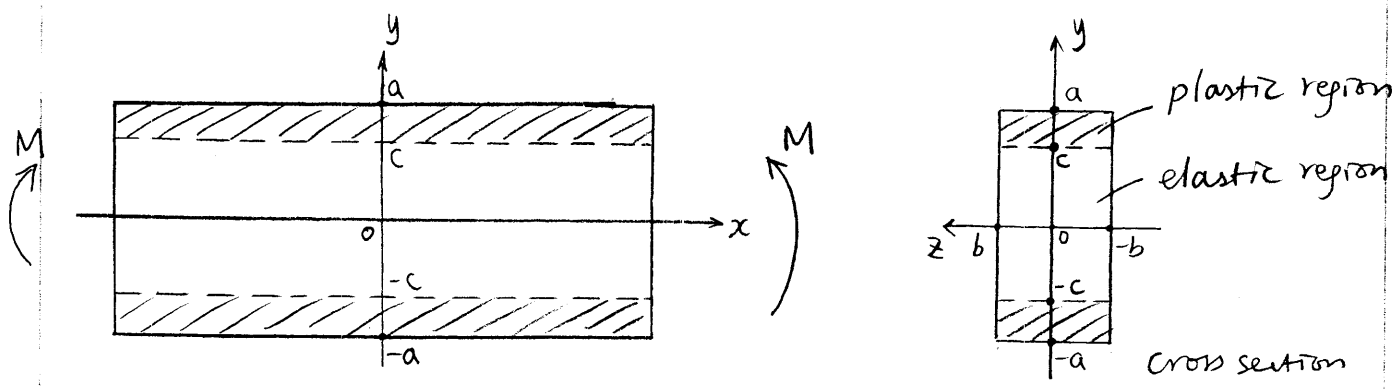

x 10^-4
1.75
1.7
1.65
1.6
1.55
λ Δ t / (2μ)
2
4
6
8
10
12
14
16
ε_xx
x 10^-3


0.36
0.35
0.34
0.33
0.32
0.31
J_2
2
4
6
8
10
12
14
16
ε_xx
x 10^-3

This is our first example in which plastic and elastic regions coexist in the specimen.

This example is taken from Hill's book "The Mathematical Theory of Plasticity", IV.7 p.81

§1. Problem Statement



For a rectangular beam subjected to pure bending moment M, find the thickness of plastic region (a-c) and bending curvature κ as functions of M.

We expect the plastic region to appear if $M > M_Y$, where M_Y is a threshold value. (onset of yield)

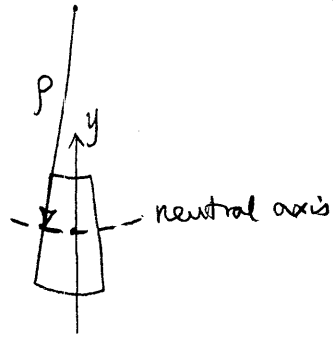
We also expect a maximum value M_{max}, at which the plastic region extends to cover the entire cross section.

The beam collapses at $M = M_{max}$ and cannot support a greater bending moment.

Find M_{max}/M_Y.

For simplicity, we shall again assume that the material is incompressible ($\nu = 0.5$).

§2. Elastic Bending ($M < M_Y$)



From strength of materials analysis

$$\varepsilon_{xx} = -\frac{y}{\rho} \qquad \left(\rho = \frac{1}{K} : \text{curvature radius of neutral axis}\right)$$

$$\sigma_{xx} = -\frac{Ey}{\rho}, \text{ all other stresses zero}$$

These expressions follow from the assumption that planar cross sections remain planar.

For $\nu = 0.5$, the above expression are exact (see Hill, p. 82)

$$M = \int_{-a}^{a} -\sigma_{xx}\, y\, dy \cdot (2b) = E \cdot \frac{2b}{\rho} \int_{-a}^{a} y^2 dy$$

define $I_z = (2b) \cdot \int_{-a}^{a} y^2 dy = (2b)\frac{(2a)^3}{12} = \frac{4a^3 b}{3}$

$$M = E \cdot \frac{1}{\rho} \cdot I_z$$

$$K = \frac{1}{\rho} = \frac{M}{EI_z}, \qquad \sigma_{xx} = -\frac{My}{I_z}$$

Maximum stress magnitude occurs at $y = \pm a$

take $y = a$, $\sigma_{xx} = -\frac{Ma}{I_z}$, $\sigma_{yy} = \sigma_{zz} = 0$

$$J_2 = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] = \frac{\sigma_{xx}^2}{3} = \frac{1}{3}\left(\frac{Ma}{I_z}\right)^2$$

At onset of yield, $M = M_Y$ $J_2 = k^2$

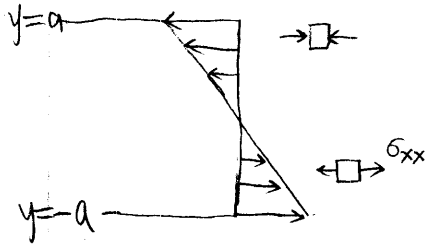
$$\frac{1}{3}\left(\frac{Ma}{I_z}\right)^2 = k^2 \qquad |\sigma_{xx}| = \frac{M_c a}{I_z} = \sqrt{3}\,k = \sigma_Y$$

$$M_Y = \frac{I_z \sigma_Y}{a}$$

The critical curvature at onset of yielding is

$$\kappa_Y = \frac{1}{\rho_Y} = \frac{M_Y}{EI_z} = \frac{\sigma_Y}{Ea}$$

$$\rho_Y = \frac{Ea}{\sigma_Y}.$$



At $M = M_Y$, the plastic region is still infinitesimal, i.e. $a - c = 0$.

Stress field is still linear with y.

§3. Plastic Bending $(M > M_Y)$

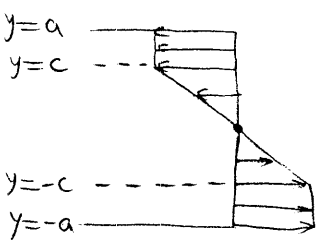
Hill showed that if $\nu = 0.5$, then

$$\varepsilon_{xx} = -\frac{y}{\rho}, \quad \sigma_{xx} \text{ is the only non-zero stress}$$

are still true even in the plastic regime $(M > M_c)$. (Hill, p.83)

Since σ_{xx} is the only non-zero stress component, every point in the beam is in the state of simple tension (or compression).

Hence
$$\begin{cases} |\sigma_{xx}| = \sigma_Y & \text{in the plastic region: } [-a, -c] \text{ and } [c, a] \\ \sigma_{xx} = -\dfrac{Ey}{\rho} & \text{in the elastic region: } [-c, c] \end{cases}$$



For continuity at $y = c$, $\dfrac{Ec}{\rho} = \sigma_Y$

$$M = \int_{-a}^{a} -\sigma_{xx} \cdot y \, dy \cdot (2b) = \left(\int_{-c}^{c} \frac{Ey^2}{\rho} dy + 2\int_{c}^{a} \sigma_Y \cdot y \, dy \right) \cdot (2b)$$

$$= \left[\frac{E}{\rho} \frac{(2c)^3}{12} + \frac{Ec}{\rho}(a^2 - c^2) \right] \cdot (2b)$$

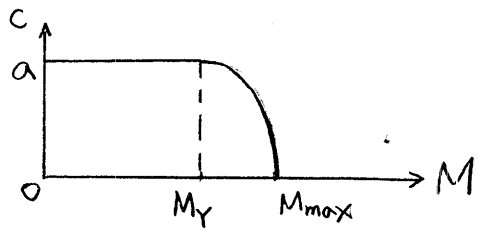
$$= \frac{Ec}{\rho}\left(a^2 - \frac{c^2}{3}\right) \cdot 2b = \sigma_Y \left(a^2 - \frac{c^2}{3}\right) \cdot 2b$$

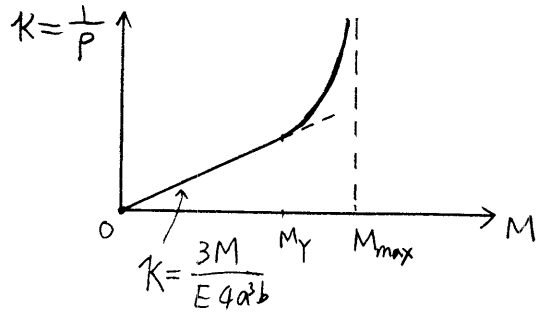
$$C = \sqrt{3}\cdot\sqrt{a^2 - \frac{M}{\sigma_Y\cdot 2b}} \qquad \text{for } M > M_c = \frac{I_z\sigma_Y}{a}$$

$$K = \frac{1}{\rho} = \frac{\sigma_Y}{EC} = \frac{\sigma_Y}{E\cdot\sqrt{3}\cdot\sqrt{a^2 - M/(\sigma_Y\cdot 2b)}}$$

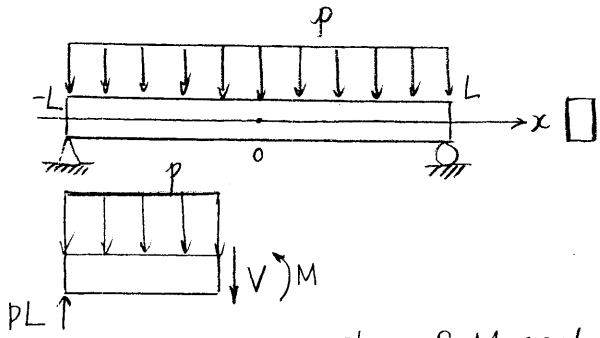
$M = M_{max}$ when $C = 0$, $a^2 = \frac{M_{max}}{\sigma_Y\cdot 2b}$, $M_{max} = \sigma_Y\cdot 2a^2b$

$$\frac{M_{max}}{M_c} = \frac{\sigma_Y\cdot 2a^2b}{\sigma_Y\cdot\frac{I_z}{a}} = \frac{2a^2b}{\frac{4}{3}a^2b} = \frac{3}{2}$$





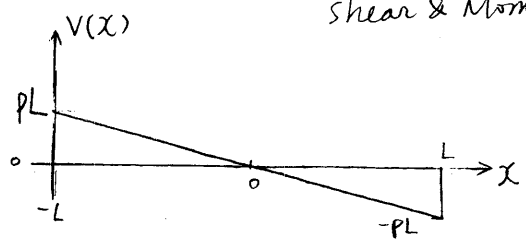
§4. Simply supported beam under uniform load



The result obtained above can be used to study a beam in which the internal moment is not uniform.

(Prager & Hodge, §7, P.44)

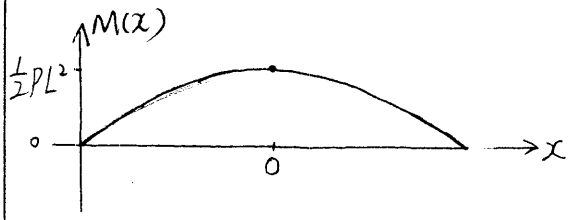
Shear & Moment diagram



$V(-L) = PL, \quad V(L) = -PL$

$V'(x) = -p$

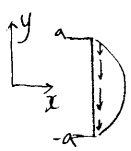
$V(x) = -px$



$M(-L) = 0, \quad M(L) = 0$

$M'(x) = V(x)$

$M(x) = \frac{1}{2}p(L^2 - x^2)$



shear force gives rise to σ_{xy}

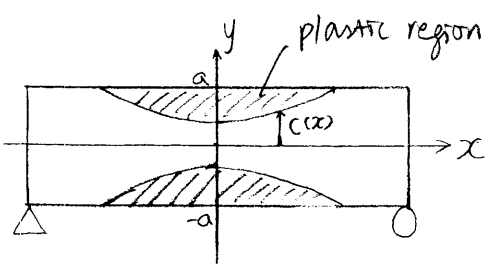
if entire beam is in elastic regime,

$$\sigma_{xy} = -V(x) \cdot \frac{3}{2}(a^2 - y^2)$$

bending moment gives rise to σ_{xx}.

if entire beam is in elastic regime,

$$\sigma_{xx} = -\frac{M(x)\,y}{I_z}$$



yield condition: $J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$

in practice: $|\sigma_{xy}| \ll |\sigma_{xx}|$

$\therefore J_2 \approx \frac{1}{3}\sigma_{xx}^2$

Hence yield condition is the same as in §2.

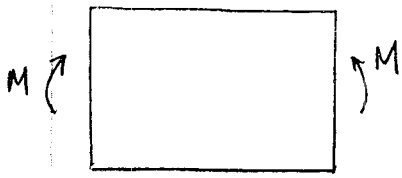
Assuming the maximum bending moment ($\frac{1}{2}PL^2$ at $x=0$) already exceeds M_Y, then in the plastic section (see §3, p.3)

$$M(x) = \sigma_Y\left(a^2 - \frac{c^2}{3}\right)2b = \frac{1}{2}p(L^2 - x^2)$$

$$c(x) = \sqrt{3}\cdot\sqrt{a^2 - \frac{P}{4b\sigma_Y}(L^2 - x^2)}$$

(* exercise: find deflection of the beam's neutral axis)

§5. Unloading



Consider again a beam in pure bending

Suppose M has increased from 0 to $M_1 > M_Y$

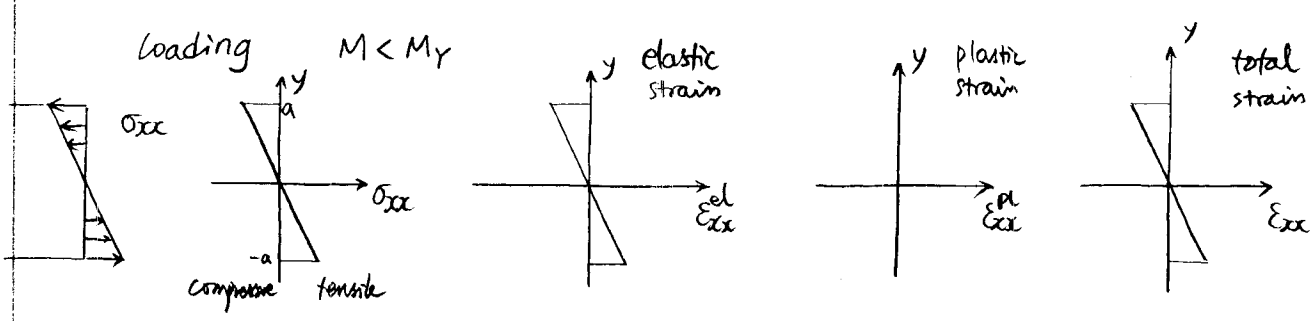
We now let M decrease from M_1 back to 0.

Q: What is the residual stress in the beam?

What is the curvature of the beam when it is fully unloaded ($M=0$)?

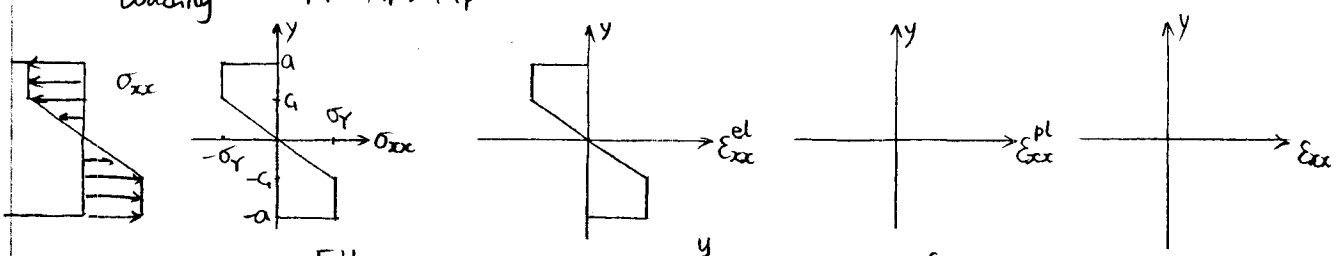
Is the residual stress strong enough to cause plastic flow (in the reverse direction)?

Loading $M < M_Y$



$$\sigma_{xx} = -\frac{My}{I_z} \qquad \varepsilon_{xx}^{el} = -\frac{My}{EI_z} \qquad \varepsilon_{xx}^{pl} = \qquad \varepsilon_{xx} =$$

Loading $M = M_1 > M_Y$

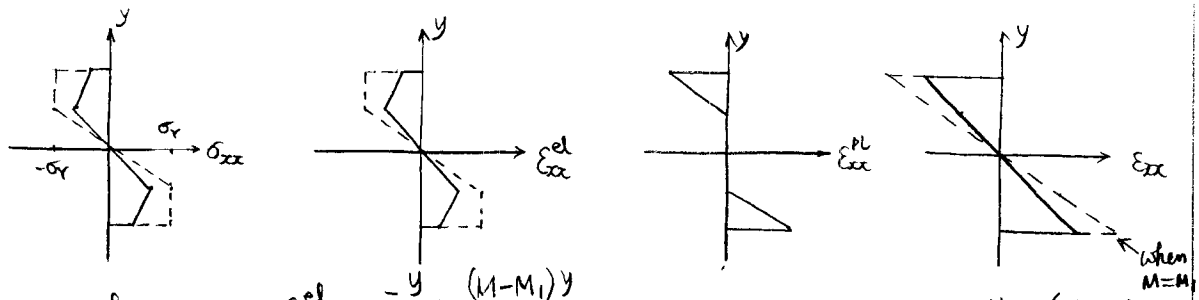


$$\sigma_{xx} = \begin{cases} -\dfrac{Ey}{\rho_1} & -c_1 \le y \le c_1 \\ \pm\sigma_Y & |y| > c_1 \end{cases} \qquad \varepsilon_{xx}^{el} = \begin{cases} -\dfrac{y}{\rho_1} \\ \pm\dfrac{\sigma_Y}{E} \end{cases} \qquad \varepsilon_{xx}^{pl} = \Bigg\{ \qquad \varepsilon_{xx} = -\frac{y}{\rho_1}$$

$$c_1 = \sqrt{3}\cdot\sqrt{a^2 - \frac{M_1}{\sigma_Y \cdot 2b}}$$

$$\frac{1}{\rho_1} = \frac{\sigma_Y}{Ec_1} = \frac{\sigma_Y}{E\sqrt{3}\cdot\sqrt{a^2 - M_1/(\sigma_Y\cdot 2b)}}$$

Unloading $0 < M < M_1$



$$\sigma_{xx} = E \cdot \varepsilon_{xx}^{el}$$

$$\varepsilon_{xx}^{el} = \begin{cases} -\dfrac{y}{\rho_1} - \dfrac{(M-M_1)y}{EI_z} \\ \pm\dfrac{\sigma_Y}{E} - \dfrac{(M-M_1)y}{EI_z} \end{cases}$$

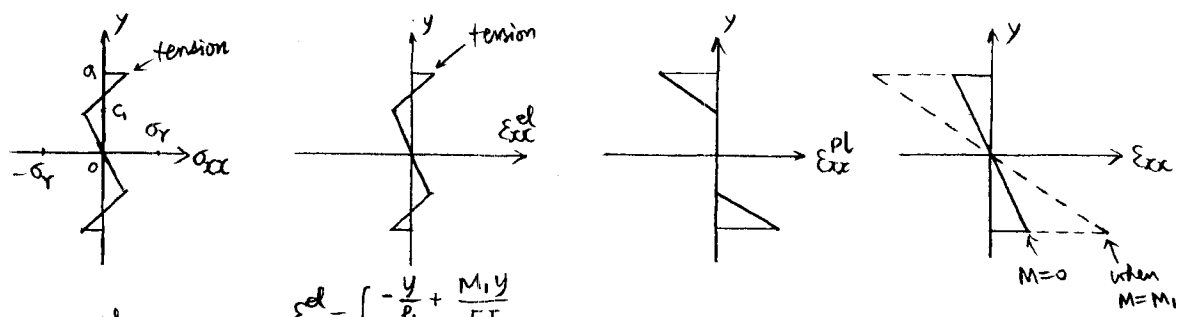
$$\varepsilon_{xx} = -\frac{y}{\rho_1} - \frac{(M-M_1)y}{EI_z}$$

During unloading, as long as stress stay below σ_Y,

ε_{xx}^{pl} remains unchanged.

all strain changes are accommodated by ε_{xx}^{el}

Unloading $M = 0$



$$\sigma_{xx} = E \cdot \varepsilon_{xx}^{el}$$

$$\varepsilon_{xx}^{el} = \begin{cases} -\dfrac{y}{\rho_1} + \dfrac{M_1 y}{EI_z} \\ \pm\dfrac{\sigma_Y}{E} + \dfrac{M_1 y}{EI_z} \end{cases}$$

$$\varepsilon_{xx} = -\frac{y}{\rho_1} + \frac{M_1 y}{EI_z} = -\frac{y}{\rho_0}$$

residual curvature $\dfrac{1}{\rho_0} = \dfrac{1}{\rho_1} - \dfrac{M_1}{EI_z} = \dfrac{\sigma_Y}{E \cdot \sqrt{3} \cdot \sqrt{a^2 - M_1/(\sigma_Y \cdot 2b)}} - \dfrac{M_1}{E \cdot I_z}$

residual stress $\sigma_{xx}\Big|_{y=c_1} = -\sigma_Y + \dfrac{M_1 c_1}{I_z}$

$$\sigma_{xx}\Big|_{y=a} = -\sigma_Y + \frac{M_1 a}{I_z}$$

Recall $M_Y \le M_1 \le M_{max}$ $\left(M_{max} = \frac{3}{2} M_Y\right)$ $\left(M_Y = \frac{I_z \sigma_Y}{a}\right)$

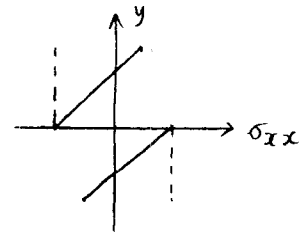
When $M_1 = M_Y$ (lower limit) $c_1 = a$.

$$\sigma_{xx}\big|_{y=c_1} = \sigma_{xx}\big|_{y=a} = -\sigma_Y + \frac{M_Y a}{I_z} = 0 \quad \rightarrow \text{no residual stress}$$

When $M_1 = M_{max}$ (upper limit) $c_1 = 0$.

$$\sigma_{xx}\big|_{y=0} = -\sigma_Y$$

$$\sigma_{xx}\big|_{y=a} = -\sigma_Y + \frac{\frac{3}{2} I_z \sigma_Y}{a} \cdot \frac{a}{I_z} = \frac{1}{2}\sigma_Y$$



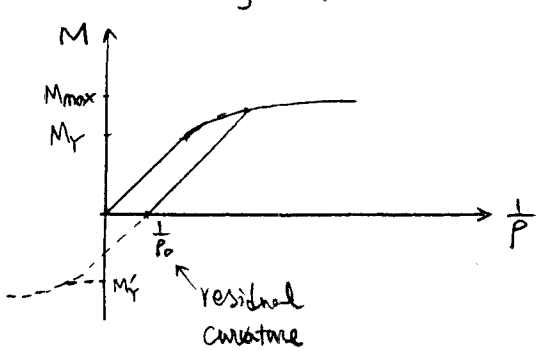
In general, for $M_Y < M < M_{max}$

$$-\sigma_Y < \sigma_{xx}\big|_{y=c_1} < 0$$

$$0 < \sigma_{xx}\big|_{y=a} < \frac{\sigma_Y}{2}$$

Hence plastic flow $\left\{ \begin{array}{l} \text{should} \\ \text{should not} \end{array} \right\}$ (choose one) occur during unloading

Q: After unloading, suppose we apply a bending moment in the reverse direction, at which bending moment M_Y' will plastic flow occur again? Where will yield occur first?



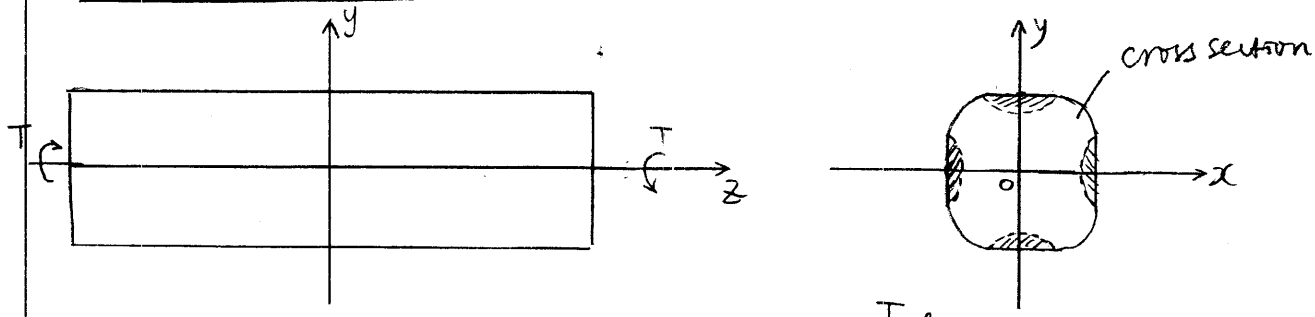
(Bauschinger effect if $|M_Y'| < M_Y$)

We will study the torsion of prismatic bar beyond yield condition.

Similar to the bending example, we expect the plastic region to instate at the periphery and grow inward on the cross section.

We shall start with a quick review of the elastic solution using Prandtl's stress function and then extend it to plastic regime.

§1. Problem Statement



A prismatic bar is subjected to torque T along its (z) axis.

Find the plastic region (shaded) and twist per unit length as functions of T.

We expect the plastic region to appear if $T > T_Y$, where T_Y is a threshold value (onset of yield)

We also expect a maximum value T_{max}, at which the plastic region extends to cover the entire cross section.

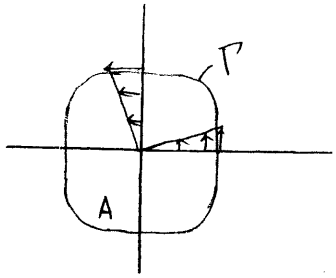
The bar collapses at $T = T_{max}$ and cannot support a greater torque.

Find $\dfrac{T_{max}}{T_Y}$.

§2. Elastic Torsion ($T < T_Y$)

(For more details, see ME340 Winter 2013 Lecture Notes "Torsion".)

Trial solution: $\theta(z) = \beta \cdot z$ β: twist per unit length



displacement field

$$\begin{cases} u_x = -\theta y = -\beta z y \\ u_y = \theta x = \beta z x \\ u_z = \beta \cdot f(x,y) \end{cases}$$

strain field

$$\begin{cases} \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0 \\ \varepsilon_{xy} = 0 \\ \varepsilon_{xz} = \frac{1}{2}(-\beta y + \beta f_{,x}) \\ \varepsilon_{yz} = \frac{1}{2}(\beta x + \beta f_{,y}) \end{cases}$$

stress field

$$\begin{cases} \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \\ \sigma_{xy} = 0 \\ \sigma_{xz} = \mu\beta(-y + f_{,x}) \\ \sigma_{yz} = \mu\beta(x + f_{,y}) \end{cases}$$

Introduce Prandtl's stress function: $\phi(x,y)$ such that

$$\sigma_{zx} = \phi_{,y} \qquad \sigma_{zy} = -\phi_{,x}$$

so that the equilibrium condition $\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0$ is automatically satisfied.

$\phi(x,y)$ satisfies PDE:

$$\boxed{\nabla^2\phi \equiv \phi_{,xx} + \phi_{,yy} = -2\mu\beta} \quad \text{(Poisson equation)}$$

traction free boundary condition on outer surface of bar

$\rightarrow \phi = \text{const}$ on Γ

without loss of generality, we choose $\boxed{\phi = 0 \text{ on } \Gamma.}$

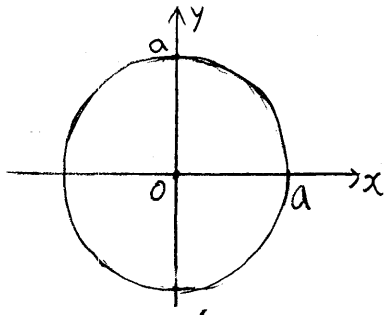
Torque T is related to ϕ through:

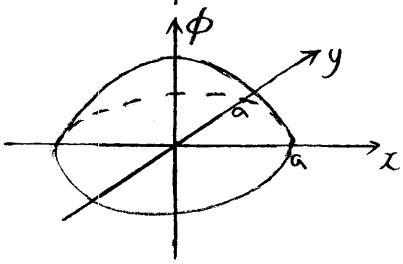
$$T = \iint_A (x\sigma_{zy} - y\sigma_{zx})\,dx\,dy = \iint_A (-x\phi_{,x} - y\phi_{,y})\,dx\,dy$$

$$\boxed{T = 2\iint_A \phi\,dx\,dy}$$

$$\tau = \sqrt{(\phi_{,x})^2 + (\phi_{,y})^2} \qquad S_1 = \tau,\ S_2 = -\tau,\ S_3 = 0,\ J_2 = \tau^2$$

§2.1 Circular cross section





$$\phi(x,y) = -\frac{\mu\beta}{2}(x^2+y^2-a^2) \qquad r \equiv \sqrt{x^2+y^2}$$

$$\phi(r) = -\frac{\mu\beta}{2}(r^2-a^2)$$

$$T = \mu\beta K_t \qquad K_t : \text{torsional rigidity}$$

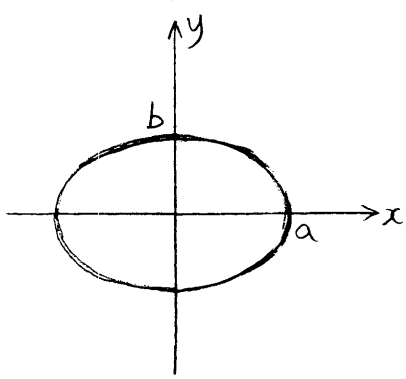
$$K_t = \frac{\pi}{2}a^4 = J \quad \text{for circular cross section}$$

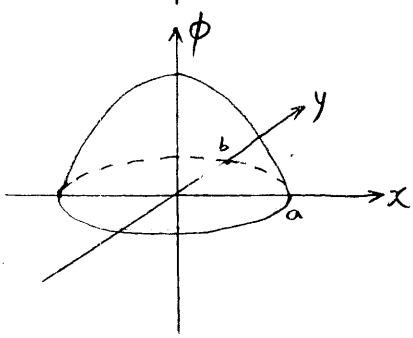
$$\tau \equiv \sqrt{\sigma_{xz}^2+\sigma_{yz}^2} = \left|\frac{\partial\phi}{\partial r}\right| = \mu\beta r$$

$$\tau_{max} = \mu\beta a = \frac{T}{K_t}a = \frac{2T}{\pi a^3} \quad \text{at } r=a$$

when $T = T_Y$, $\tau_{max} = k = \frac{\sigma_Y}{\sqrt{3}}$, $\therefore \underline{T_Y = \frac{\pi}{2}ka^3}$

§2.2 Elliptical cross section





$$\phi(x,y) = -\frac{\mu\beta a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

$$T = \mu\beta K_t$$

(equality reached for $a=b$)

$$K_t = \frac{\pi a^3b^3}{a^2+b^2} \leqslant J = \frac{\pi ab}{4}(a^2+b^2)$$

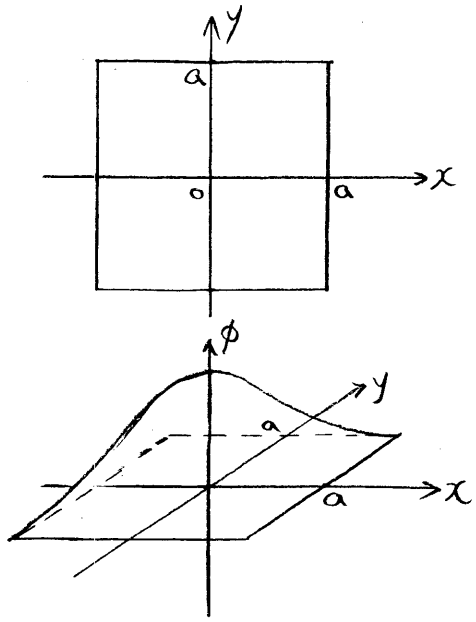
$J \equiv \int r^2 dA$ polar moment of inertia

$$\tau_{max} = |\sigma_{xz}|\Big|_{\substack{x=0\\y=b}} = \left|\frac{\partial\phi}{\partial y}\right|\Big|_{\substack{x=0\\y=b}} = \frac{\mu\beta a^2b^2}{a^2+b^2}\cdot\frac{2b}{b^2}$$

$$\tau_{max} = \frac{2\mu\beta a^2b}{a^2+b^2} = \frac{T}{K_t}\frac{2a^2b}{a^2+b^2} = \frac{2T}{\pi ab^2}$$

when $T = T_Y$, $\tau_{max} = k = \frac{\sigma_Y}{\sqrt{3}}$, $\therefore \underline{T_Y = \frac{\pi}{2}kab^2}$

§2.3 Square cross section



$\phi(x,y)$ must be solved numerically.

$$T = \mu\beta K_t$$

$$K_t \approx 2.24\,a^4 \leq J = \frac{8}{3}a^4$$

$$\frac{K_t}{J} \approx 0.84$$

$$\tau_{max} = |\sigma_{yz}|\Big|_{\substack{x=\pm a\\ y=0}} = |\sigma_{xz}|\Big|_{\substack{x=0\\ y=\pm a}}$$

$$\approx 0.60\frac{T}{a^3}$$

when $T = T_Y$, $\tau_{max} = k = \frac{\sigma_Y}{\sqrt{3}}$ $\therefore$ $T_Y \approx 1.7\,k a^3$

* exercise: plot contour of $\phi(x,y)$ for circular, elliptic and square cross sections.

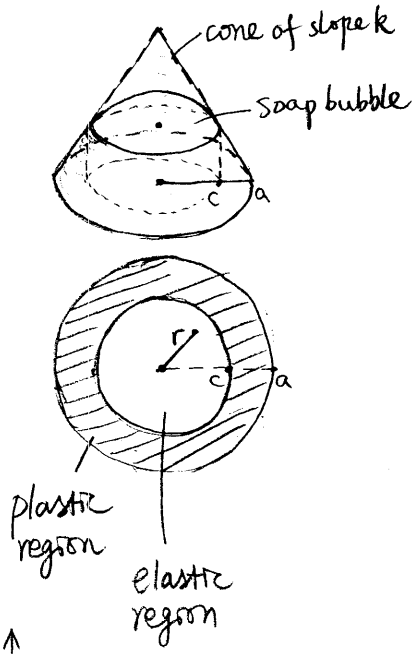
§3. Plastic Torsion ($T > T_Y$)

For perfectly plastic material (no hardening), the shear stress τ inside the torsion bar cannot exceed $k = \frac{\sigma_Y}{\sqrt{3}}$ (Von Mises).

In the plastic region $\tau = k$, i.e. $|\nabla\phi(x,y)| = k$.

In other words, the slope of function $\phi(x,y)$ cannot exceed k.

§3.1 Circular cross section



$$r = \sqrt{x^2+y^2}$$

$$\phi(x,y) = k(a-r) \qquad c \le r \le a$$

$$\phi(x,y) = -\frac{\mu\beta}{2}(r^2-c^2) + k(a-r) \qquad 0 \le r \le c$$

continuity of $d\phi/dr$ at $r=c$

$$k = \mu\beta c \qquad \beta = \frac{k}{\mu c}$$

known unknown

$$T = \int_0^a \left|\frac{\partial\phi}{\partial r}\right| \cdot r \cdot 2\pi r\, dr$$

$$= \int_0^c \mu\beta r^3 \cdot 2\pi\, dr + \int_c^a k \cdot r^2 \cdot 2\pi\, dr$$

$$= \frac{\pi}{2}\mu\beta c^4 + \frac{2\pi}{3}k(a^3-c^3)$$

$$= \frac{\pi}{2}k c^3 + \frac{2\pi}{3}k(a^3-c^3)$$

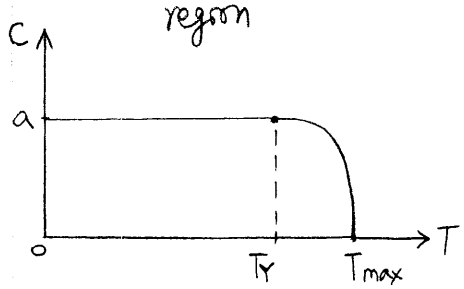
$$= \pi k\left(\frac{2}{3}a^3 - \frac{1}{6}c^3\right) = \frac{\pi k}{6}(4a^3-c^3)$$

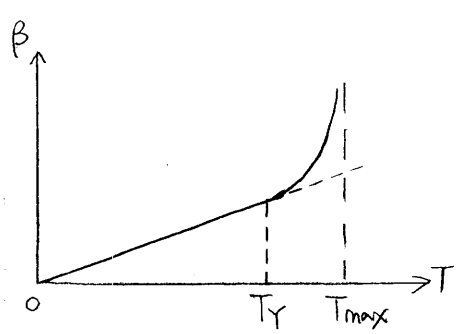
$$c = \left(4a^3 - \frac{6T}{\pi k}\right)^{1/3}, \qquad c=0 \text{ at } T = T_{max} = \frac{2\pi}{3}ka^3$$

$$\beta = \frac{k}{\mu}\left(4a^3 - \frac{6T}{\pi k}\right)^{-1/3}$$

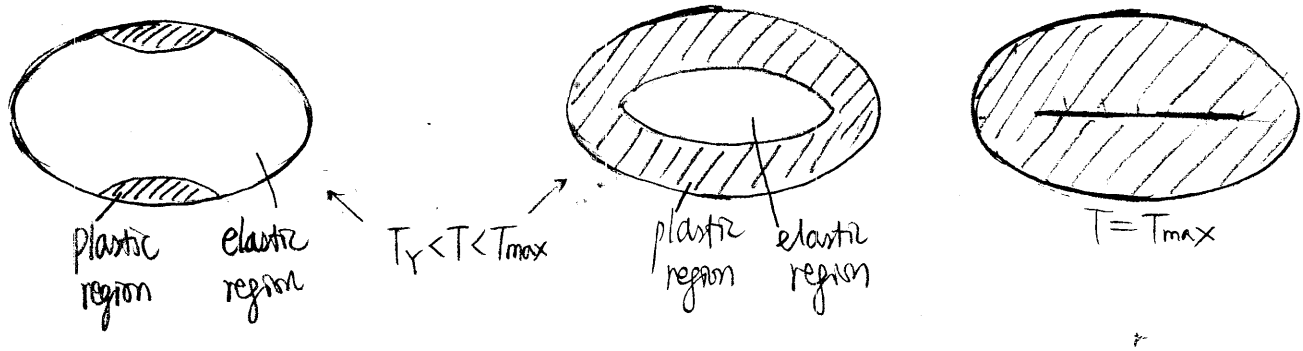
$$\boxed{\frac{T_{max}}{T_Y} = \frac{4}{3}}$$

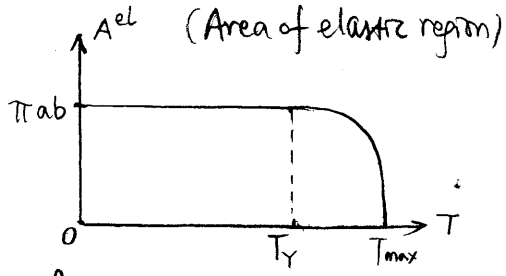
*compare with bending



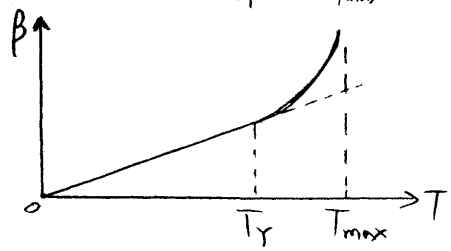


§3.2 Elliptic cross section



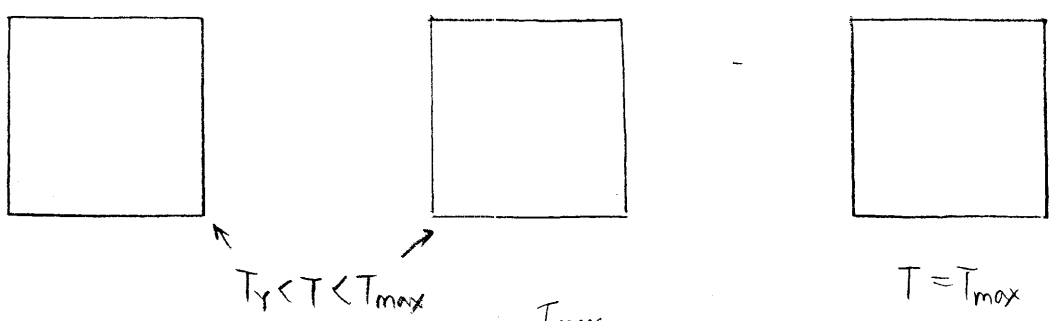


(Numerical solution required)



§3.3 Square cross section

(Left as exercise problem.) sketch elastic & plastic regions



(Numerical solution required)

$\frac{T_{max}}{T_Y} =$ _____

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2D Cylinder Example: Torsion with plasticity using triangular elements
%
% Wei Cai caiwei@stanford.edu
% William Kuykendall wpkuyken@stanford.edu
%
% adapted from FeCalc.m written by Peter Pinsky pinsky@stanford.edu
% universal meshing from Michael Hunsweck
%
% First Adapted 01/03/2013
%
% Last Modified 04/22/2013
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Part 1: Set simulation parameters
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%clear;
global nNodes nElements Coord u IEN K F Nxe Nye EBC ID g penalty_factor
nDoF = 1;         % number of degrees of freedom
nDim = 2;         % number of dimensions
% L: [l w pl pw]
%    l, w:   domain length and width [m]
%    pl, pw: density of triangles along length and width

% Circular cylinder [x0 y0 r]--[center coordinates, radius] [m m m]
%shape = 'circle'; L = [0.8 0.8 10 10]; c = [0 0 0.2];
% Elliptical cylinder [x0 y0 a b]--[(x-x0)/a]^2+[(y-y0)/b]^2=1
%shape = 'ellipse'; L = [1.3 1.3  10 10]; c = [0 0 0.6 0.2];
% Rectangular rod [x0 y0 a b]-- [x_center, y_center, length, width]
%shape = 'rectangle'; L = [0.4 0.4 5 5];  c = [0 0 0.4 0.4];
shape = 'rectangle'; L = [0.6 0.4 5 5];  c = [0 0 0.6 0.4];
E = 200e9;
nu = 0.32;
mu = E/(2*(1+nu));% shear modulus [ N/m^2 ]

tau_max = 4.0e6;  % maximum shear stress
%beta   = 3.5e-4;   % twist per unit length [1/m]
beta   = 4e-4;   % twist per unit length [1/m]

% options for solving plasticity problem by iteration
Niter = 2000; dt = 5e6; plotfreq = 100;  penalty_factor = 2e-14;

% set axis limits for plotting
if strcmp(shape,'circle')
    clim = c(3);
elseif strcmp(shape,'ellipse')
    clim = max(c(3:4));
elseif strcmp(shape,'rectangle')
    clim = max(c(3:4))/2;
else
    fprintf('Unrecognized shape for plotting.\n');
    clim = 1;
```

```
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Part 2: Generate Mesh, Initialize Arrays, Set Up Boundary Conditions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Step 1: Generate solution mesh
[Coord,IEN,c,shape]   = CreateMesh(L,c,shape);  % Generates the mesh
IEN = IEN';
nNodes        = size(Coord,1);            % Number of nodes
nElements     = size(IEN,2);              % Number of elements
nNodesElement = size(IEN,1);              % Number of nodes per element
nEdgesElement = 3;                        % Number of edges per element

% Step 2: Set Tolerance for detecting boundary nodes
boundary_tol = 1e-8;
onSurface = logical(zeros(1,nNodes));

% Step 3: Allocate Arrays
C   = spalloc(nElements,nDoF,nElements);
f   = spalloc(nElements,nDoF,nElements);
g   = spalloc(nNodes,nDoF,round(nNodes/4));
EBC = spalloc(nNodes,nDoF,round(nNodes/4));
% Step 4: Set Essential Boundary Condition
    % Step 4.1: Get coordinates of all nodes
    X = Coord(:,1);                         % x-position of all nodes
    Y = Coord(:,2);                         % y-position of all nodes
    % Step 4.2: Define essential boundary condition value(s)
    T  = [ 0 ]; % Traction
    % Step 4.3: Set g, EBC for all nodes on the surface
    % Step 4.3a: Find all nodes on external surface
    if strcmp(shape,'circle')
        % distance squared of node from circle center
        D2 = (X-c(1)).^2 + (Y-c(2)).^2;
        onSurface = (D2 >=  c(3)^2-boundary_tol);
    elseif strcmp(shape,'ellipse')
        % distance squared of node from ellipse center
        D2 = ((X-c(1))/c(3)).^2 + ((Y-c(2))/c(4)).^2;
        onSurface = (D2 >= 1-boundary_tol);
    elseif strcmp(shape,'rectangle')
        for ii = 1:nNodes
            onSurface(ii) = abs(X(ii)-c(1))>(c(3)/2-boundary_tol) ...
                         || abs(Y(ii)-c(2))>(c(4)/2-boundary_tol) ;
        end
    else
        error('Code should not get here.  Shape should have been set.\n');
    end

% Adjust the interior mesh coords
[newCoord] = AdjustMesh(Coord, IEN, onSurface, 200, 0e-1, 200);
Coord = newCoord;
```

```
    % Step 4.3b: Set g, EBC for node on surface
    EBC(onSurface) = 1;
    g(onSurface)   = T;

% Step 5: Define the Natural Boundary Condition
    % Step 5.1: Allocate h array
    h = spalloc(nEdgesElement,nElements,round(nElements/4));
    % In this problem there is not any natural boundary condition applied
% Step 6: Define the Load Value (often RHS of equation you are solving)
    % Step 1: Set f for all elements
    f(:) = 0.1;
    f(:) = 10;
    f(:) = 1;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Part 3: Create Destination Array (ID) and Location Matrix (LM)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Step 1: Create Data Processing Arrays
    % Step 1.1: Allocate arrays
    ID = zeros(nNodes,nDoF)';
        % Destination Array: ID(A) = P if A is not on essential boundary
        %                             0 if A is on the essential boundary
        %                    index is global node number (A)
        %                    result is global equation number (P)
    LM = zeros(nNodesElement*nDoF,nElements);
        % Location Matrix: P = LM(a,e)
        %                  row is degree of freedom (a)
        %                  column is element number (e)
        %                  result is global equation number (P)
    % Step 1.2: Populate ID Array
    I = full(EBC' == 0);          % Creates logical indexing array
    nEquations = sum(sum(I));     % Count number of DoF's
    ID(I) = 1:nEquations;         % Assign Equation numbers
    ID    = ID';
    % Step 1.3: Populate LM Array
    P     = ID(IEN,:)';
    LM(:) = P(:);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Part 4: Assembly of K and F Matrices
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Step 1: Allocate K and F
K = spalloc(nEquations,nEquations,20*nDoF*nEquations);
F = spalloc(nEquations,1,nEquations);
% Step 2: Assemble K and F
    % Step 2.1: Compute element contributiontions
    for e = 1:nElements
        % Step 2.2: Get coordinates of element nodes
        x = Coord(IEN(:,e),1);
        y = Coord(IEN(:,e),2);
        % Step 2.3: Define Shorthand
```

```
        x12 = x(1) - x(2); x23 = x(2) - x(3); x31 = x(3) - x(1);
        y12 = y(1) - y(2); y23 = y(2) - y(3); y31 = y(3) - y(1);
        % Step 2.4: Compute element Jacobian
        J_e = [ -x31, x23; -y31 y23 ];
        % Step 2.5: Compute element size
        elementSize = 0.5*det(J_e);
        Ae = elementSize;
        %=====================================================================
        % Subsection 1: Computation of k_e
        %=====================================================================
        % For isotropic case.
        k_e = 1/(4*2*mu*beta*Ae)* ...
                [y23^2+x23^2,        y23*y31+x23*x31,    y23*y12+x23*x12;
                 y23*y31+x23*x31,    y31^2+x31^2,        y31*y12+x31*x12;
                 y23*y12+x23*x12,    y31*y12+x31*x12,    y12^2+x12^2];
        %=====================================================================
        % Subsection 2: Computation of f_e
        %=====================================================================
        ff = f(IEN(1,e));
        f_e = ff*Ae/3*[1;1;1];
        %=====================================================================
        % Subsection 3: Computation of f_g
        %=====================================================================
        g_e = g(IEN(:,e));
        f_g = -k_e*g_e;
        %=====================================================================
        % Subsection 4: Computation of f_h
        %=====================================================================
        % Step 4.1a: Edge load intensity (homogeneous boundary)
        f_h = h(:,e);
        % Step 4.1b: Edge load intensity (non-homogeneous boundary)
        l12 = sqrt(x12^2 + y12^2);
        l23 = sqrt(x23^2 + y23^2);
        l31 = sqrt(x31^2 + y31^2);
        h_e = h(:,e);
        f_h = h_e(1)*l12/2*[1;1;0] + h_e(2)*l23/2*[0;1;1] + h_e(3)*l31/2*[1;0;1];
        %=====================================================================
        % End of Subsections
        %=====================================================================
        % Step 2.6: Get Global equation numbers
        P = LM(:,e);
        % Step 2.7: Eliminate Essential DOFs
        I = (P ~= 0);
        P = P(I);
        % Step 2.8: Insert k_e, f_e, f_g, f_h
        K(P,P) =  K(P,P) + k_e(I,I);
        F(P)   =  F(P) + f_e(I) + f_g(I) + f_h(I);
    end

% Global matrices for computing spatial gradients and area of elements
Nxe = zeros(nElements,nNodes); Nye = zeros(nElements,nNodes); Ae = zeros(nElements,1);
```

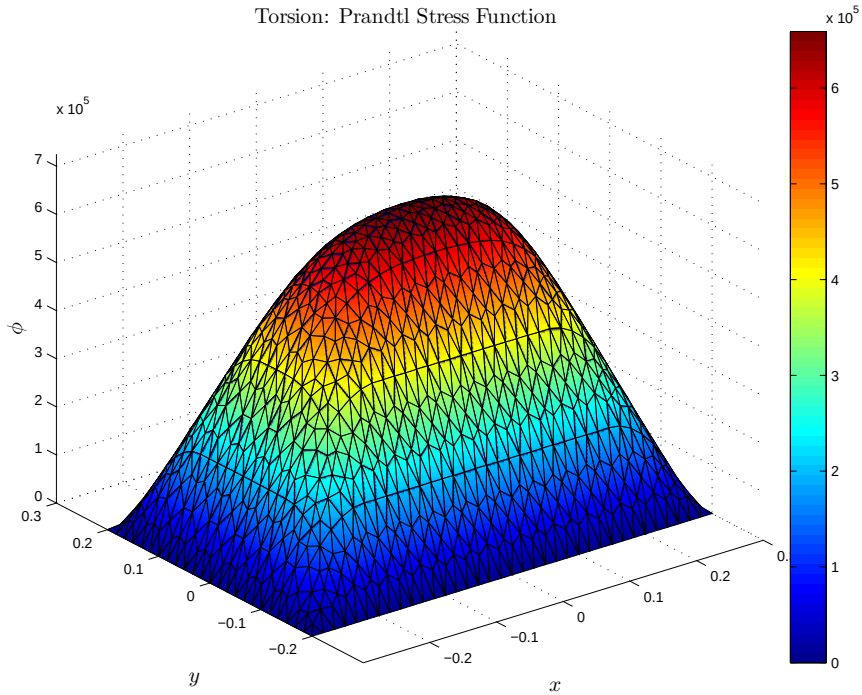
```
for e = 1:nElements
    % Get element coordinates
    xe = Coord(IEN(:,e),1);
    ye = Coord(IEN(:,e),2);
    % Define Shorthand
    x12 = xe(1) - xe(2); x23 = xe(2) - xe(3); x31 = xe(3) - xe(1);
    y12 = ye(1) - ye(2); y23 = ye(2) - ye(3); y31 = ye(3) - ye(1);
    % Area of a triangle:
    Ae(e) = (xe(1)*(ye(2)-ye(3))+xe(2)*(ye(3)-ye(1))+xe(3)*(ye(1)-ye(2)))/2;
    % Compute element jacobian
    J_e = [ -x31, x23; -y31 y23 ];
    % Calculate natural derivatives of N1,N2,N3
    N1es =  0; N2es =  1; N3es = -1;
    N1er =  1; N2er =  0; N3er = -1;
    d_Nrs = [N1er N1es; N2er N2es; N3er N3es];
    d_Nxy = d_Nrs/J_e;
    Nxe(e,IEN(:,e)) = d_Nxy(:,1)';
    Nye(e,IEN(:,e)) = d_Nxy(:,2)';
end

% compute volume contribution from each node
for ii = 1:nNodes
    [row,col] = find(IEN == ii);
    V_ave(ii)= sum(Ae(col))/3; % 3 for triangular elements
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Part 5: Compute Finite Element Solution (including plasticity)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Step 1: Solve elasticity problem
d_slash = K\F;

% Step 2: Solve plasticity problem by
%  minimize (0.5*d'*K*d - F'*d)_in_elast_region + (penalty)_in_plast_region
%  using steepest descent algorithm
%  Plast_region is defined for nodes on elements whose slope exceeds tau_max
%  Elast_region is defined for remaining nodes
%  For nodes in Plast_region, the (elastic) gradient
%  from (0.5*d'*K*d - F'*d) term is set to zero, and the gradient from
%  penalty function = (tau^2-tau_max^2)*penalty_factor is added.

% initialize d from elasticity solution (scaled)
if ~exist('d')
    d = d_slash*0.25;
end
I = (EBC == 0); u = zeros(size(g)); tau_max2 = tau_max^2;
for iter = 1:Niter,
    % grad is the residual of Poisson's equation
    grad = K*d - F;
    u(I)  = d(ID(I)); u(~I) = g(~I);
    dxe = Nxe*u; dye = Nye*u;
```

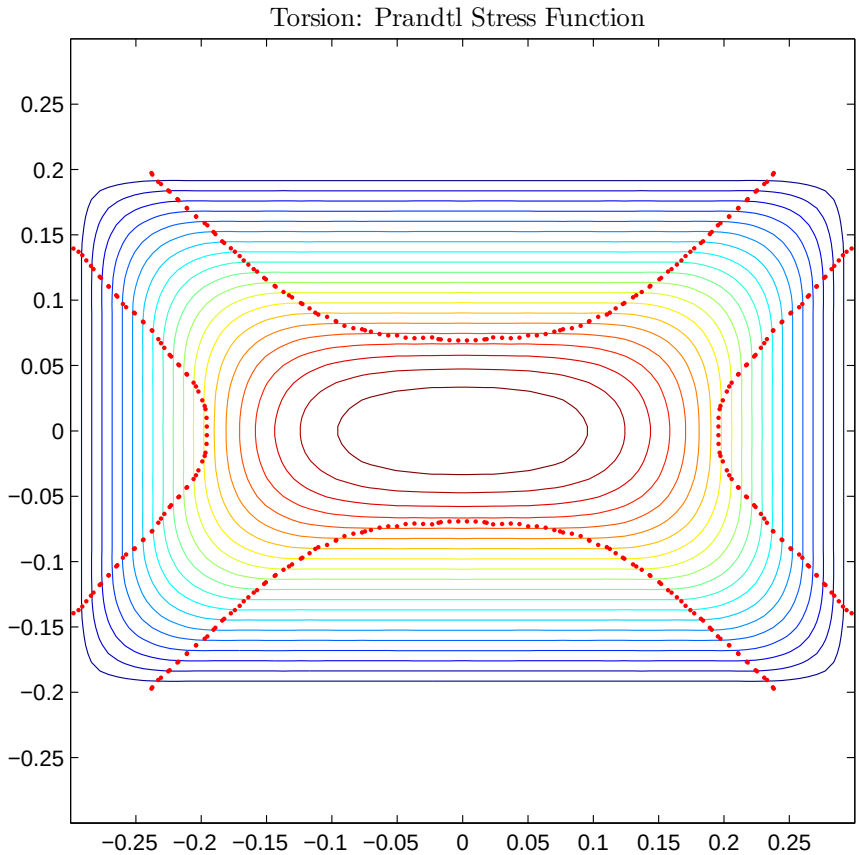
```
    % total torque
    Torque = 2*dot(V_ave, u);

    % find elements whose slope exceeds tau_max
    tau2 = (dxe).^2+(dye).^2;
    element_factor = (tau2 >= tau_max2);
    ind_e = find(element_factor);
    id_in_elements = reshape(ID(IEN(:,ind_e)),length(ind_e)*3,1);
    non_zero_entries = find(id_in_elements>0);
    % for these elements, set the residual term from Poisson's equation to zero
    grad(id_in_elements(non_zero_entries)) = 0;

    % for elements whose slope exceeds tau_max, add penalty function
    penalty = sum( (tau2-tau_max2).*element_factor ) * penalty_factor;
    dpenalty_du = zeros(size(u));  dpenalty = zeros(size(d));
    for e = 1:nElements,
        if element_factor(e)
            % derivative of slope wrt nodal value
            dpenalty_du = dpenalty_du + ...
                (2*(Nxe(e,:)*u)*Nxe(e,:)' + 2*(Nye(e,:)*u)*Nye(e,:)');
        end
    end
    dpenalty_du = dpenalty_du * penalty_factor;
    dpenalty(ID(I)) = dpenalty_du(I);
    grad = grad + dpenalty;

    % move d in the steepest descent direction
    d = d - grad*dt;

    % plot intermediate results during minimization
    if (mod(iter,plotfreq)==0)
        disp(sprintf('iter = %d/%d  Torque = %e sum(d) = %e  norm(grad) = %e', ...
            iter,Niter,Torque,sum(d),norm(grad)));
        figure(1)
        x = Coord;
        T = trisurf(IEN',x(:,1),x(:,2),u);
        xlim([-clim clim]); ylim([-clim clim]);
        figure(3)
        for ii = 1:nNodes
            [row,col] = find(IEN == ii);
            du_ave(ii,1)= mean(dxe(col,:),1);
            du_ave(ii,2)= mean(dye(col,:),1);
        end;
        T = trisurf(IEN',x(:,1),x(:,2),sqrt(du_ave(:,2).^2+du_ave(:,1).^2));
        xlim([-clim clim]); ylim([-clim clim]);
        drawnow
    end
end

% Step 3: Arrange results (d) into solution vector (u)
```

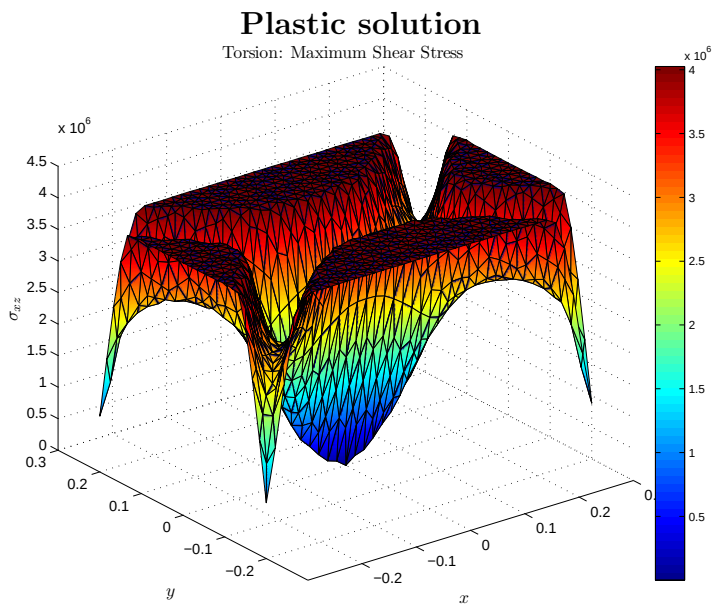
```
u = zeros(nNodes,nDoF);
I     = (EBC == 0);
u(I)  = d(ID(I));
u(~I) = g(~I);
Torque = 2*dot(V_ave, u);

% Step 4: Create coordinate array
x = Coord;

% Step 5: compute average slope on nodes
for ii = 1:nNodes
    [row,col] = find(IEN == ii);
    du_ave(ii,1)= mean(dxe(col,:),1);
    du_ave(ii,2)= mean(dye(col,:),1);
end;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Part 6: Plotting
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Step 1: Plot Prandtl's stress function (u, i.e. phi)
figure(1)
T = trisurf(IEN',x(:,1),x(:,2),u);
title('Torsion: Prandtl Stress Function',...
    'Interpreter','Latex','FontSize',12);
xlabel('$x$','Interpreter','Latex','FontSize',12);
ylabel('$y$','Interpreter','Latex','FontSize',12);
zlabel('$\phi$','Interpreter','Latex','FontSize',12);
set(gca,'FontSize',8);
axis([-clim clim -clim clim 0 max(u*1.1)])
co = colorbar;
set(co,'FontSize',8);
colormap jet
shading interp
set(T,'EdgeColor','k');

% Step 2: Plot sigma_xz and sigma_yz (gradient of u)
figure(2)
subplot(2,1,1)
T = trisurf(IEN',x(:,1),x(:,2),du_ave(:,2));
axis equal
axis([-clim clim -clim clim])
view([0 90])
title('FE Solution for $\sigma_{xz}$',...
    'Interpreter','Latex','FontSize',12);
xlabel('$x$','Interpreter','Latex','FontSize',12);
ylabel('$y$','Interpreter','Latex','FontSize',12);
zlabel('$\sigma_{xz}$','Interpreter','Latex','FontSize',12);
co = colorbar; set(co,'FontSize',8);
colormap jet; shading interp
set(T,'EdgeColor','k');
subplot(2,1,2)
```

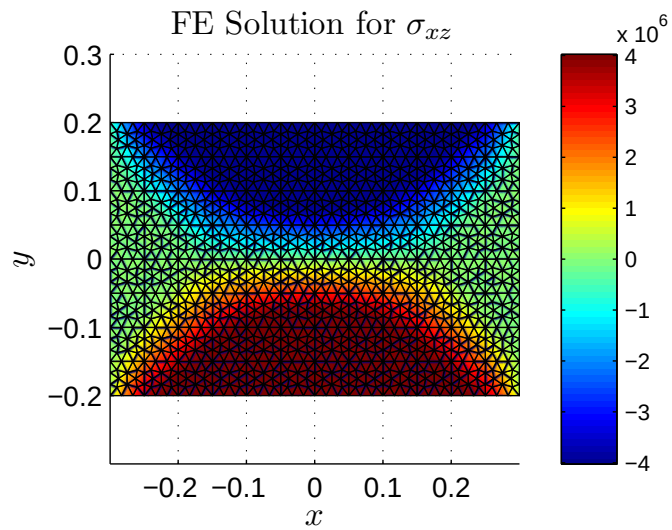
```
T = trisurf(IEN',x(:,1),x(:,2),-du_ave(:,1));
axis equal
axis([-clim clim -clim clim])
view([0 90])
title('FE Solution for $\sigma_{yz}$',...
    'Interpreter','Latex','FontSize',12);
xlabel('$x$','Interpreter','Latex','FontSize',12);
ylabel('$y$','Interpreter','Latex','FontSize',12);
zlabel('$\sigma_{yz}$','Interpreter','Latex','FontSize',12);
co = colorbar; set(co,'FontSize',8);
colormap jet; shading interp
set(T,'EdgeColor','k');

% Step 3: Plot maximum shear stress (magnitude of gradient of u)
figure(3)
T = trisurf(IEN',x(:,1),x(:,2),sqrt(du_ave(:,2).^2+du_ave(:,1).^2));
xlim([-clim clim]); ylim([-clim clim]);
title('Torsion: Maximum Shear Stress',...
    'Interpreter','Latex','FontSize',12);
xlabel('$x$','Interpreter','Latex','FontSize',12);
ylabel('$y$','Interpreter','Latex','FontSize',12);
zlabel('$\sigma_{xz}$','Interpreter','Latex','FontSize',12);
cbar = colorbar; set(cbar,'FontSize',8);
colormap jet; shading interp
set(T,'EdgeColor','k');

% Step 4: Plot contour of stress function with elast-plast boundary
figure(4);
lxi = linspace(-1,1,300);lyi = linspace(-1,1,300);[xi,yi] = meshgrid(lxi,lyi);
ui = griddata(x(:,1),x(:,2),u,xi,yi,'cubic');
dui = griddata(x(:,1),x(:,2),sqrt(du_ave(:,2).^2+du_ave(:,1).^2),xi,yi,'cubic');
[c h]=contour(xi,yi,ui,20);
hold on
c2=contourc(lxi,lyi,dui,[0.95 0.95]*tau_max,'b--');
plot(c2(1,2:end),c2(2,2:end),'r.');
hold off
title('Torsion: Prandtl Stress Function',...
    'Interpreter','Latex','FontSize',12);
axis equal; axis([-clim clim -clim clim])
figure(5);
xi = linspace(-1,1,300);yi = linspace(-1,1,300);[xi,yi] = meshgrid(xi,yi);
contour(xi,yi,dui,20);
title('Torsion: Maximum Shear Stress',...
    'Interpreter','Latex','FontSize',12);
axis equal; axis([-clim clim -clim clim])
```

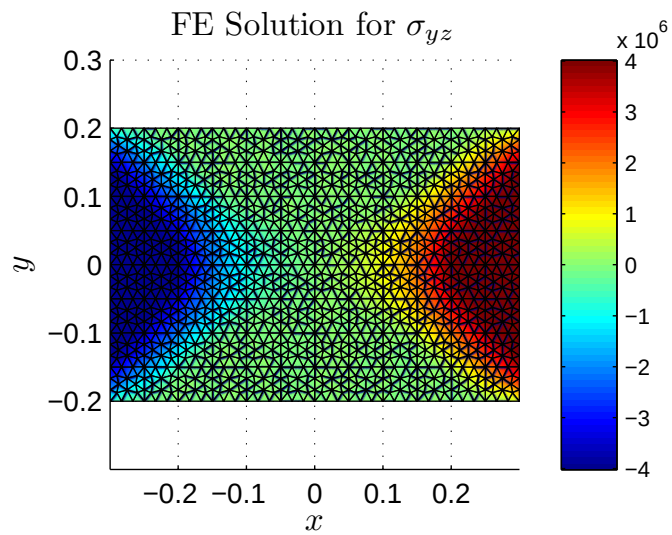
Plastic solution





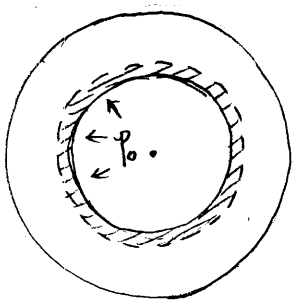

Plastic solution
Torsion: Maximum Shear Stress
x 10^6
σ_{xz}
y
x


FE Solution for σ_{xz}
x 10^6
y
x


FE Solution for σ_{yz}
x 10^6
y
x

This is our first 2D plane-strain example. The axisymmetry makes the problem simpler than the general 2D plane-strain problem, which will be discussed in the next lecture note

§1. Problem Statement



Consider a thick-walled cylindrical tube (pressure vessel) with internal (gas) pressure p_0

Determine the thickness of plastic region (shaded) with increasing p_0

Determine if plastic flow occurs if p_0 is reduced back to zero.

We expect the plastic region to appear if $p_0 > p_Y$, where p_Y is a threshold value (onset of yield)

There is also a maximum value p_{max}, at which the pressure vessle will burst!

Find $\frac{p_{max}}{p_Y}$.

§2. plane-strain Elasticity

$u_z = 0$, Solution independent of z (i.e. $\frac{\partial}{\partial z} = 0$)

Look for: $u_x(x,y)$, ε_{xx}, σ_{xx}
$u_y(x,y)$ ε_{yy} σ_{yy}
ε_{xy} σ_{zz}
σ_{xy}

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} = 0 \rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

equilibrium condition:
$$\sigma_{xx,x} + \sigma_{yx,y} + F_x = 0$$
$$\sigma_{xy,x} + \sigma_{yy,y} + F_y = 0$$

compatibility condition. $\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$

Generalized Hookes Law:
$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}$$
$$\varepsilon_{yy} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}$$
$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy} \qquad E = 2\mu(1+\nu)$$

Introduce Airy stress function: $\phi(x,y)$, such that
$$\sigma_{xx} = \phi_{,yy}, \quad \sigma_{yy} = \phi_{,xx}, \quad \sigma_{xy} = -\phi_{,xy}$$

Then in the absence of body force ($F_x = F_y = 0$),
the equilibrium condition is automatically satisfied.

The compatibility condition becomes (biharmonic equation)
$$\nabla^4\phi \equiv \phi_{,xxxx} + \phi_{,yyyy} - 2\phi_{,xxyy} = 0$$

In polar coordinates, $\phi(r,\theta) = r^m e^{in\theta}$ satisfies biharmonic equation,
as long as $m = n, -n, 2+n, 2-n$.

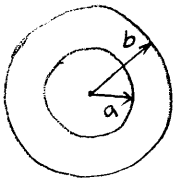
(See ME340 Winter 2013 Lecture Notes, "Polar Coordinates".)

For problems with axial symmetry, $n = 0$, (Mitchell solution)
$$\phi = A_1 r^2 + A_2 r^2 \ln r + A_3 \ln r + A_4 \theta$$
$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right), \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$

$$\sigma_{rr} = A_1 \cdot 2 + A_2 \cdot (2\ln r + 1) + A_3 \cdot \frac{1}{r^2}$$

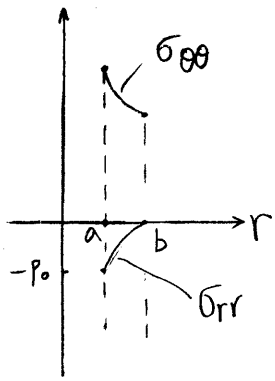
$$\sigma_{r\theta} = \qquad A_4 \cdot \frac{1}{r^2}$$

$$\sigma_{\theta\theta} = A_1 \cdot 2 + A_2 \cdot (2\ln r + 3) + A_3 \cdot \left(-\frac{1}{r^2}\right)$$



Boundary condition for pressurized tube

$$\begin{cases} \sigma_{rr}\big|_{r=a} = -p_0 \\ \sigma_{r\theta}\big|_{r=a} = 0 \end{cases} \qquad \begin{cases} \sigma_{rr}\big|_{r=b} = 0 \\ \sigma_{r\theta}\big|_{r=b} = 0 \end{cases}$$



$$A_2 = A_3 = 0, \quad A_1 = \frac{p_0}{2} \cdot \frac{a^2}{b^2-a^2}, \quad A_3 = -\frac{p_0 a^2 b^2}{b^2-a^2}$$

$$\begin{cases} \sigma_{rr} = \dfrac{p_0 a^2}{b^2-a^2}\left(1 - \dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = \dfrac{p_0 a^2}{b^2-a^2}\left(1 + \dfrac{b^2}{r^2}\right) \end{cases}$$

$$\sigma_1 = \sigma_{\theta\theta}, \quad \sigma_2 = \sigma_{rr}, \quad \sigma_3 = \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{p_0 a^2}{b^2-a^2} \cdot 2\nu$$

$$\bar{\sigma} = \frac{1}{3}(\sigma_{\theta\theta} + \sigma_{rr} + \sigma_{zz}) = \frac{1+\nu}{3}(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{2(1+\nu)}{3} \cdot \frac{p_0 a^2}{b^2-a^2}$$

$$S_1 = \sigma_1 - \bar{\sigma} = \frac{p_0 a^2}{b^2-a^2}\left(\frac{1-2\nu}{3} + \frac{b^2}{r^2}\right)$$

$$S_2 = \sigma_2 - \bar{\sigma} = \frac{p_0 a^2}{b^2-a^2}\left(\frac{1-2\nu}{3} - \frac{b^2}{r^2}\right)$$

$$S_3 = \sigma_3 - \bar{\sigma} = -\frac{p_0 a^2}{b^2-a^2} \cdot \frac{2(1-2\nu)}{3}$$

define $p' \equiv \dfrac{p_0 a^2}{b^2-a^2}$

$$J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2)$$

$$= \left(\frac{p_0 a^2}{b^2-a^2}\right)\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{r^4}\right]$$

note $K = \dfrac{2\mu(1+\nu)}{3(1-2\nu)}$ bulk modulus

$$3\left(\frac{\mu}{\mu+3K}\right)^2 = \frac{(1-2\nu)^2}{3}$$

$$J_2 = p'\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{r^4}\right]$$

$$\begin{cases} \sigma_{rr} = p' \cdot \left(1 - \dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = p' \cdot \left(1 + \dfrac{b^2}{r^2}\right) \end{cases}$$

maximum J_2 occurs at $r=a$.

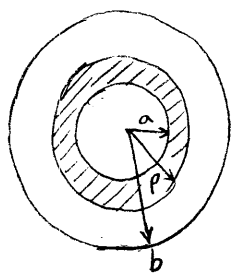
At onset of yield $J_2\Big|_{r=a} = (p_o^Y)^2\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{a^4}\right] = k^2$

$$p_o^Y = k \cdot \frac{b^2-a^2}{a^2} \cdot \left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{a^4}\right]^{-1/2}$$

§3. Elastic region after yield

For $p_o > p_o^Y$, the tube contains a plastic region.

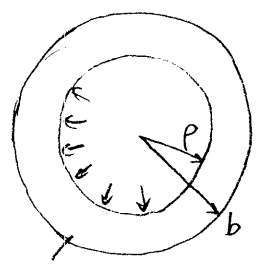
Suppose p_o increases montonically from 0 to a value greater than p_o^Y. Then the plastic region starts from the inner wall ($r=a$) and grow thicker.



Let $r=\rho$ be the elastic-plastic boundary.

The elastic region $\rho \le r \le b$ has similar boundary conditions as before.

$$\begin{cases} \sigma_{rr}\Big|_{r=\rho} = -p_\rho \\ \sigma_{r\theta}\Big|_{r=\rho} = 0 \end{cases} \qquad \begin{cases} \sigma_{rr}\Big|_{r=b} = 0 \\ \sigma_{r\theta}\Big|_{r=b} = 0 \end{cases}$$



elastic region

So the stress field in the elastic region must be:

$$\begin{cases} \sigma_{rr} = \dfrac{p_\rho \rho^2}{b^2-\rho^2}\left(1-\dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = \dfrac{p_\rho \rho^2}{b^2-\rho^2}\left(1+\dfrac{b^2}{r^2}\right) \end{cases}$$

$$J_2\Big|_{r=\rho} = \frac{p_\rho \rho^2}{b^2-\rho^2}\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{\rho^4}\right] = k^2$$

$$\therefore \quad p_\rho = k\,\frac{b^2-\rho^2}{\rho^2}\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{\rho^4}\right]^{-1/2}$$

This is the magnitude of the normal stress $|\sigma_{rr}|$ at the elastic-plastic boundary.

define $p'' \equiv \dfrac{p_\rho \rho^2}{b^2-\rho^2} = k \left[\dfrac{(1-2\nu)^2}{3} + \dfrac{b^4}{\rho^4} \right]^{-1/2}$

Then in the elastic region $\rho \le r \le b$.

$$\begin{cases} \sigma_{rr} = p'' \cdot \left(1 - \dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = p'' \cdot \left(1 + \dfrac{b^2}{r^2}\right) \end{cases}$$

(very similar to the expression in p.3)

With increasing p_o, ρ increases from a to b.

§4. Numerical Solution in Elastic Regime

Before discussing the plastic regime, let us see how the same solution in the elastic regime as derived above can be obtained numerically. This prepares us for the numerical solution in the plastic regime.

Equilibrium condition in polar coordinates:

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\theta}}{\partial \theta} + \dfrac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + F_r = 0 \\ \dfrac{1}{r}\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial \sigma_{r\theta}}{\partial r} + \dfrac{2\sigma_{r\theta}}{r} + F_\theta = 0 \end{cases}$$

In our example, $\sigma_{r\theta}=0$, $F_r = F_\theta = 0$, $\dfrac{\partial}{\partial \theta} = 0$

$$\therefore \quad \boxed{\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0} \qquad \text{(equilibrium condition)}$$

The compatibility condition can be obtained from the definition of strain (in polar coordinates)

$$\begin{cases} \varepsilon_{rr} = \dfrac{\partial}{\partial r} u_r \\ \varepsilon_{\theta\theta} = \dfrac{1}{r}\left(u_r + \dfrac{\partial u_\theta}{\partial \theta}\right) \\ \varepsilon_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) \end{cases}$$

By symmetry, in our example, we have only one non-zero displacement field: $u_r(r)$

$$\begin{cases} \varepsilon_{rr} = \dfrac{\partial}{\partial r} u_r \\ \varepsilon_{\theta\theta} = \dfrac{1}{r} u_r \\ \varepsilon_{r\theta} = 0 \end{cases}$$

hydrostatic strain:

$$\bar{\varepsilon} \equiv \frac{1}{3}(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) = \frac{1}{3}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right)$$

$$\therefore \quad \varepsilon_{rr} = \frac{\partial}{\partial r}(r\,\varepsilon_{\theta\theta}) = \varepsilon_{\theta\theta} + r\frac{\partial \varepsilon_{\theta\theta}}{\partial r}$$

$$\boxed{\frac{\partial \varepsilon_{\theta\theta}}{\partial r} = \frac{1}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta})} \qquad \text{(compatibility condition)}$$

Generalized Hooke's Law.

$$\sigma_{rr} = (\lambda+2\mu)\,\varepsilon_{rr} + \lambda\,\varepsilon_{\theta\theta}$$
$$\sigma_{\theta\theta} = \lambda\,\varepsilon_{rr} + (\lambda+2\mu)\,\varepsilon_{\theta\theta}$$
$$\sigma_{zz} = \lambda\,\varepsilon_{rr} + \lambda\,\varepsilon_{\theta\theta}$$

$$\sigma_{rr} - \sigma_{\theta\theta} = 2\mu\,(\varepsilon_{rr} - \varepsilon_{\theta\theta})$$

Plug into equilibrium condition

$$\boxed{(\lambda+2\mu)\frac{\partial \varepsilon_{rr}}{\partial r} + \lambda\frac{\partial \varepsilon_{\theta\theta}}{\partial r} + \frac{2\mu}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta}) = 0}$$

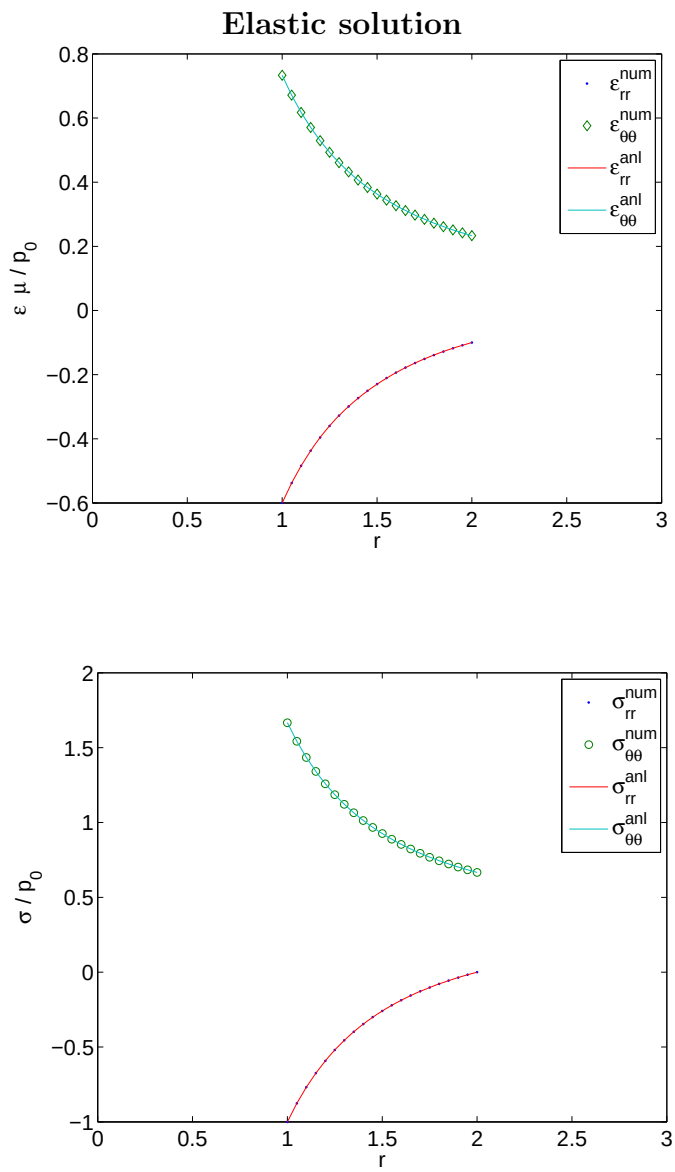
So we have two equations for the two unknowns: ε_{rr}, $\varepsilon_{\theta\theta}$.

Boundary conditions:

$$\begin{cases} \sigma_{rr}\big|_{r=a} = (\lambda+2\mu)\,\varepsilon_{rr}\big|_{r=a} + \lambda\,\varepsilon_{\theta\theta}\big|_{r=a} = -p_0 \\ \sigma_{rr}\big|_{r=b} = (\lambda+2\mu)\,\varepsilon_{rr}\big|_{r=b} + \lambda\,\varepsilon_{\theta\theta}\big|_{r=b} = 0 \end{cases}$$

A Matlab code that finds $\varepsilon_{rr}(r)$ and $\varepsilon_{\theta\theta}(r)$ that satisfy the compatibility, equilibrium, and boundary condition is written —— cylind_tube_elast.m , eqns_cylind_tube elast.m

It makes uses of Matlab's fsolve function.


Elastic solution
ε μ / p0
r
σ / p0
r

§5. Plastic Region

Our next task is to find the relation between p_o and ρ
(ρ is the radius of elastic-plastic boundary, when $p_o > p_o^Y$)
and to find the stress and strain field inside the plastic region $a \le r \le \rho$.

We shall choose our unknowns as $\underbrace{\varepsilon_{rr},\ \varepsilon_{\theta\theta}}_{\text{total strain}}$, $\underbrace{S_{rr}}_{\text{deviatoric stress}}$

and establish 3 PDE's.

All other quantities of interest can be expressed in terms of these three.

Hydrostatic strain: $\bar{\varepsilon} = \frac{1}{3}(\varepsilon_{rr} + \varepsilon_{\theta\theta})$ note $\varepsilon_{zz} = 0$

deviatoric strain:

$$e_{rr} = \varepsilon_{rr} - \bar{\varepsilon} = \frac{2}{3}\varepsilon_{rr} - \frac{1}{3}\varepsilon_{\theta\theta}$$

$$e_{\theta\theta} = \varepsilon_{\theta\theta} - \bar{\varepsilon} = -\frac{1}{3}\varepsilon_{rr} + \frac{2}{3}\varepsilon_{\theta\theta}$$

$$e_{zz} = \varepsilon_{zz} - \bar{\varepsilon} = -\frac{1}{3}\varepsilon_{rr} - \frac{1}{3}\varepsilon_{\theta\theta}$$

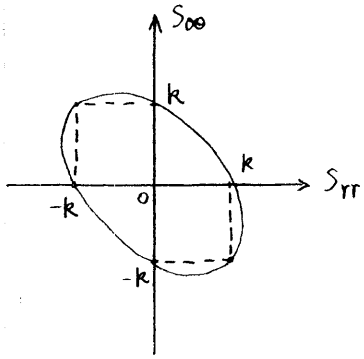
Hydrostatic stress: $\bar{\sigma} = 3K\bar{\varepsilon} = K(\varepsilon_{rr} + \varepsilon_{\theta\theta})$

deviatoric stress: $S_{rr},\ S_{\theta\theta},\ S_{zz}$

$$S_{rr} + S_{\theta\theta} + S_{zz} = 0, \quad S_{zz} = -(S_{rr} + S_{\theta\theta})$$

in the plastic region, stress must satisfy yield condition

$$J_2 = \frac{1}{2}(S_{rr}^2 + S_{\theta\theta}^2 + S_{zz}^2) = S_{rr}^2 + S_{\theta\theta}^2 + S_{rr}S_{\theta\theta} = k^2$$



$$S_{\theta\theta} = \frac{1}{2}\left(-S_{rr} \pm \sqrt{4k^2 - 3S_{rr}^2}\right)$$

<u>choose + sign, why?</u>

Stress:

$$\sigma_{rr} = S_{rr} + \bar{\sigma} = S_{rr} + K(\varepsilon_{rr} + \varepsilon_{\theta\theta})$$

$$\sigma_{\theta\theta} = S_{\theta\theta} + \bar{\sigma}$$

$$\sigma_{rr} - \sigma_{\theta\theta} = S_{rr} - S_{\theta\theta}$$

We now set up the 3 PDE's for ε_{rr}, $\varepsilon_{\theta\theta}$, S_{rr}.

<u>equilibrium condition:</u> $$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

now becomes

$$\frac{\partial S_{rr}}{\partial r} + K\left(\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right) + \frac{S_{rr} - S_{\theta\theta}}{r} = 0 \quad \cdots\cdots ①$$

<u>Compatibility condition:</u>

$$\frac{\partial \varepsilon_{\theta\theta}}{\partial r} = \frac{1}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta}) \quad \cdots\cdots ②$$

<u>Plastic flow rule</u> will lead to the following equation:

$$2\mu\left[\left(\frac{2}{3}\frac{\partial \varepsilon_{rr}}{\partial \rho} - \frac{1}{3}\frac{\partial \varepsilon_{\theta\theta}}{\partial \rho}\right) S_{\theta\theta} - \left(\frac{2}{3}\frac{\partial \varepsilon_{\theta\theta}}{\partial \rho} - \frac{1}{3}\frac{\partial \varepsilon_{rr}}{\partial \rho}\right) S_{rr}\right] = \frac{\partial S_{rr}}{\partial \rho} S_{\theta\theta} - \frac{\partial S_{\theta\theta}}{\partial \rho} S_{rr} \quad \cdots ③$$

note this is the only PDE with $\frac{\partial}{\partial \rho}$.

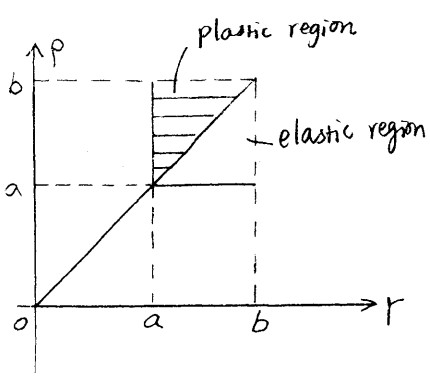
PDE ③ is derived in P.10

Compare these equations with those in the elastic regime (§4).

The compatibility condition is the same.

But the equilibrium condition here cannot be written in terms of ε_{rr}, $\varepsilon_{\theta\theta}$ alone (because of the plastic strain)

Instead, S_{rr} is introduced here as the 3rd unknown and the 3rd PDE is supplied by the plastic flow rule.



The equations now need to be solved in the 2D space of r and ρ.

The solution in the elastic region is already known.

The plastic strain <u>rate</u> is in the same direction as the deviatoric stress

$$\begin{cases} \dot{\varepsilon}^{pl}_{rr} = \frac{\tilde{\lambda}}{2\mu} S_{rr} \\ \dot{\varepsilon}^{pl}_{\theta\theta} = \frac{\tilde{\lambda}}{2\mu} S_{\theta\theta} \\ \dot{\varepsilon}^{pl}_{zz} = \frac{\tilde{\lambda}}{2\mu} S_{zz} \\ \dot{\varepsilon}^{pl}_{r\theta} = \frac{\tilde{\lambda}}{2\mu} S_{r\theta} = 0 \end{cases}$$

(plastic flow rule)

The deviatoric elastic strain is in the same direction as the deviatoric stress

$$\begin{cases} e^{el}_{rr} = \frac{1}{2\mu} S_{rr} \\ e^{el}_{\theta\theta} = \frac{1}{2\mu} S_{\theta\theta} \\ e^{el}_{zz} = \frac{1}{2\mu} S_{zz} \\ e^{el}_{r\theta} = \frac{1}{2\mu} S_{r\theta} = 0 \end{cases} \rightarrow \begin{cases} \dot{e}^{el}_{rr} = \frac{1}{2\mu} \dot{S}_{rr} \\ \dot{e}^{el}_{\theta\theta} = \frac{1}{2\mu} \dot{S}_{\theta\theta} \end{cases}$$

Total strain rate (deviatoric part)

$$\begin{cases} \dot{e}_{rr} = \dot{e}^{el}_{rr} + \dot{\varepsilon}^{pl}_{rr} = \frac{1}{2\mu}(\dot{S}_{rr} + \lambda S_{rr}) \\ \dot{e}_{\theta\theta} = \dot{e}^{el}_{\theta\theta} + \dot{\varepsilon}^{pl}_{\theta\theta} = \frac{1}{2\mu}(\dot{S}_{\theta\theta} + \lambda S_{\theta\theta}) \\ \dot{e}_{zz} = \dot{e}^{el}_{zz} + \dot{\varepsilon}^{pl}_{zz} = \frac{1}{2\mu}(\dot{S}_{zz} + \lambda S_{zz}) \\ \dot{e}_{r\theta} = \dot{e}^{el}_{r\theta} + \dot{\varepsilon}^{pl}_{r\theta} = 0 \end{cases} \xrightarrow{\text{eliminate } \tilde{\lambda}}$$

$$(2\mu \dot{e}_{rr} - \dot{S}_{rr}) S_{\theta\theta} = \tilde{\lambda} S_{rr} S_{\theta\theta} = (2\mu \dot{e}_{\theta\theta} - \dot{S}_{\theta\theta}) S_{rr}$$

$$2\mu(\dot{e}_{rr} S_{\theta\theta} - \dot{e}_{\theta\theta} S_{rr}) = \dot{S}_{rr} S_{\theta\theta} - \dot{S}_{\theta\theta} S_{rr}$$

Imagine that the elastic-plastic boundary is moving at rate $\dot{\rho}$

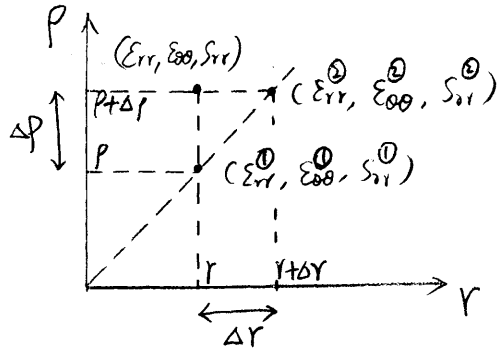
Divide the equation by $\dot{\rho}$, note $\frac{\dot{e}_{rr}}{\dot{\rho}} = \frac{de_{rr}}{dt} \Big/ \frac{d\rho}{dt} = \frac{\partial e_{rr}}{\partial \rho}$

$$2\mu \left(\frac{\partial e_{rr}}{\partial \rho} S_{\theta\theta} - \frac{\partial e_{\theta\theta}}{\partial \rho} S_{rr} \right) = \frac{\partial S_{rr}}{\partial \rho} S_{\theta\theta} - \frac{\partial S_{\theta\theta}}{\partial \rho} S_{rr}$$

plug in $e_{rr} = \frac{2}{3}\varepsilon_{rr} - \frac{1}{3}\varepsilon_{\theta\theta}$, $e_{\theta\theta} = \frac{2}{3}\varepsilon_{\theta\theta} - \frac{1}{3}\varepsilon_{rr}$

we arrive at PDE ③ on P.9.

§6. Numerical Solution in Plastic Region



Suppose the solution is already known at (r, ρ): $(\varepsilon_{rr}^{①}, \varepsilon_{\theta\theta}^{①}, S_{rr}^{①})$

and $(r+\Delta r, \rho+\Delta\rho)$: $(\varepsilon_{rr}^{②}, \varepsilon_{\theta\theta}^{②}, S_{rr}^{②})$

Here we describe an algorithm to find the solution at

$(r, \rho+\Delta\rho)$: $(\varepsilon_{rr}, \varepsilon_{\theta\theta}, S_{rr})$

The discretized PDE becomes:

equilibrium condition:

$$\frac{S_{rr}^{②}-S_{rr}}{\Delta r} + K\left[\frac{\varepsilon_{rr}^{②}-\varepsilon_{rr}}{\Delta r} + \frac{\varepsilon_{\theta\theta}^{②}-\varepsilon_{\theta\theta}}{\Delta r}\right] + \frac{1}{r+\Delta r/2}\left[\left(\frac{S_{rr}^{②}+S_{rr}}{2}\right) - \left(\frac{S_{\theta\theta}^{②}+S_{\theta\theta}}{2}\right)\right] = 0$$

compatibility condition:

$$\frac{\varepsilon_{\theta\theta}^{②}-\varepsilon_{\theta\theta}}{\Delta r} - \frac{1}{r+\Delta r/2}\left[\left(\frac{\varepsilon_{rr}^{②}+\varepsilon_{rr}}{2}\right) - \left(\frac{\varepsilon_{\theta\theta}^{②}-\varepsilon_{\theta\theta}}{2}\right)\right] = 0$$

Plastic flow rule:

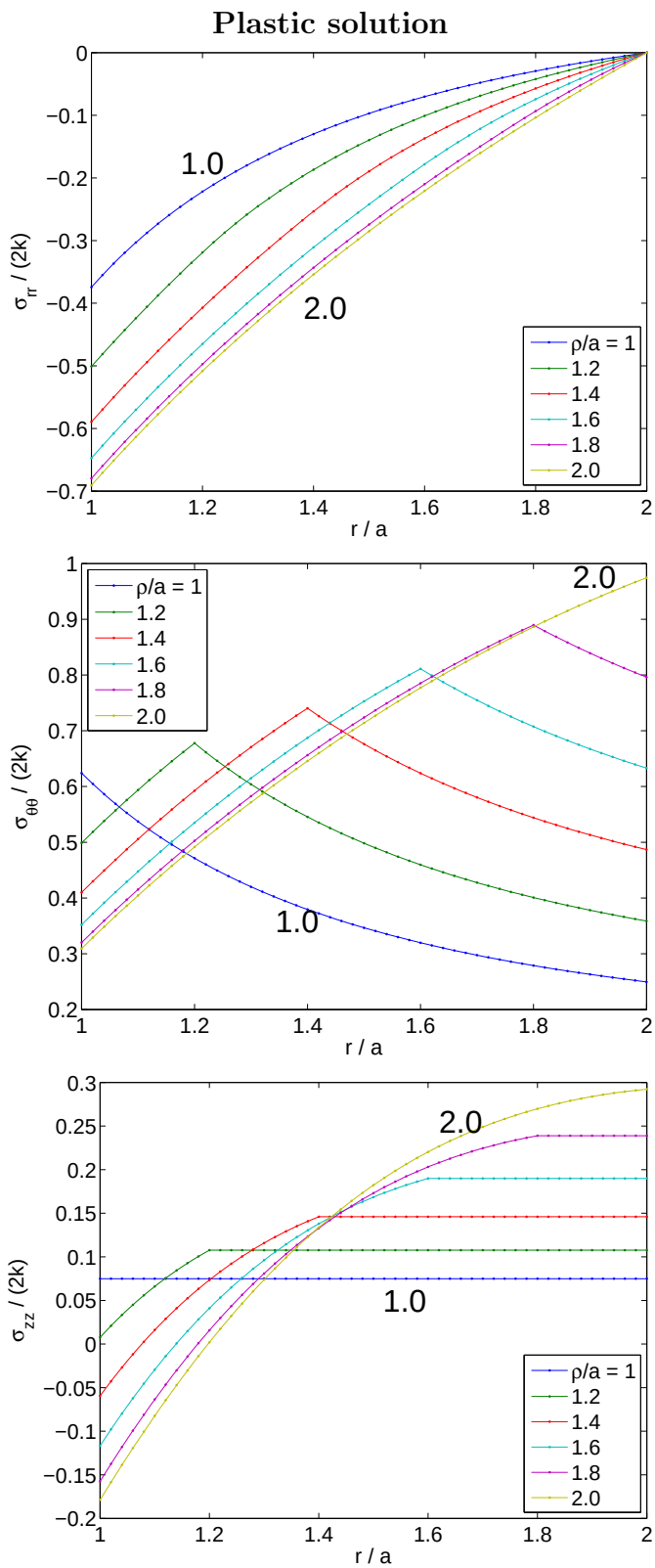
$$2\mu\left[\left(\frac{2}{3}\frac{\varepsilon_{rr}-\varepsilon_{rr}^{①}}{\Delta\rho} - \frac{1}{3}\frac{\varepsilon_{\theta\theta}-\varepsilon_{\theta\theta}^{①}}{\Delta\rho}\right)\left(\frac{S_{\theta\theta}+S_{\theta\theta}^{①}}{2}\right) - \left(\frac{2}{3}\frac{\varepsilon_{\theta\theta}-\varepsilon_{\theta\theta}^{①}}{\Delta\rho} - \frac{1}{3}\frac{\varepsilon_{rr}-\varepsilon_{rr}^{①}}{\Delta\rho}\right)\left(\frac{S_{rr}+S_{rr}^{①}}{2}\right)\right]$$

$$- \frac{S_{rr}-S_{rr}^{①}}{\Delta\rho}\cdot\frac{S_{\theta\theta}+S_{\theta\theta}^{①}}{2} + \frac{S_{\theta\theta}-S_{\theta\theta}^{①}}{\Delta\rho}\cdot\frac{S_{rr}+S_{rr}^{①}}{2} = 0$$

$$S_{\theta\theta} = \frac{1}{2}\left(-S_{rr} + \sqrt{4k^2 - 3S_{rr}^2}\right)$$

Numerical solution obtained by

cylind_tube_plast.m, eqns_cylind_tube_plast.m.


Plastic solution
σrr / (2k)
r / a
1.0
2.0
ρ/a = 1
1.2
1.4
1.6
1.8
2.0
σθθ / (2k)
r / a
2.0
1.0
σzz / (2k)
r / a
2.0
1.0

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2D Example: solve strain field in pressurized cylindrical tube
%             elastic deformation only
%
% Wei Cai caiwei@stanford.edu
% Created       03/24/2013
% Last Modified 04/29/2013
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

mu = 100;
nu = 0.3;
lambda = 2*mu*nu/(1-2*nu);

a = 1; b = 2;
dr = 0.05;
r = [a:dr:b];
p0 = 1;

eps_rr = zeros(size(r));
eps_qq = zeros(size(r));

Niter = 10000;
plotfreq = 1000; % 50;

param = [a b dr mu nu p0];

F0 = eqns_cylind_tube_elast([eps_rr, eps_qq],param);
%options = optimset('Display','iter','TolFun',1e-8);
options = optimset('Display','off','TolFun',1e-8);
sol = fsolve('eqns_cylind_tube_elast', [eps_rr, eps_qq], options, param);
eps_rr = sol(1:end/2); eps_qq = sol(end/2+1:end);
F1 = eqns_cylind_tube_elast([eps_rr, eps_qq],param);

% (Alternative method using cgrelax, require gradient evaluation)
% [U dU_depsrrqq] = grad_cylind_tube_elast([eps_rr, eps_qq],param);
% acc = 1e-6; maxfn = 40000; dfpred = 1e-5; n = length([eps_rr, eps_qq]);
% [U,eps_rrqq_rlx] = cgrelax('grad_cylind_tube_elast',n,acc,maxfn,dfpred,...
%                            [eps_rr,eps_qq],param);
% eps_rr = eps_rrqq_rlx(1:end/2); eps_qq = eps_rrqq_rlx(end/2+1:end);

% stress field
sig_rr = (lambda + 2*mu) * eps_rr + lambda * eps_qq;
sig_qq = (lambda + 2*mu) * eps_qq + lambda * eps_rr;
sig_zz = lambda * (eps_rr + eps_qq);

pp=p0*a^2/(b^2-a^2); E = 2*mu*(1+nu);
sig_rr_anl = pp*(1-b^2./r.^2);
sig_qq_anl = pp*(1+b^2./r.^2);
eps_rr_anl = (1-nu^2)/E*sig_rr_anl - nu*(1+nu)/E*sig_qq_anl;
eps_qq_anl = (1-nu^2)/E*sig_qq_anl - nu*(1+nu)/E*sig_rr_anl;
```

```
% plot solution
fs = 17;
figure(1);
plot(r, eps_rr /p0*mu, '.', r, eps_qq/p0*mu, 'd', ...
     r, eps_rr_anl/p0*mu, r, eps_qq_anl/p0*mu);
xlim([0 max(r)*1.5]);
set(gca,'FontSize',fs);
xlabel('r'); ylabel('\epsilon  \mu / p_0');
legend('\epsilon_{rr}^{num}','\epsilon_{\theta\theta}^{num}',...
       '\epsilon_{rr}^{anl}','\epsilon_{\theta\theta}^{anl}');
figure(2);
plot(r, sig_rr/p0, '.', r, sig_qq/p0, 'o', r, sig_rr_anl/p0, r, sig_qq_anl/p0);
xlim([0 max(r)*1.5]);
set(gca,'FontSize',fs);
xlabel('r'); ylabel('\sigma / p_0');
legend('\sigma_{rr}^{num}','\sigma_{\theta\theta}^{num}',...
       '\sigma_{rr}^{anl}','\sigma_{\theta\theta}^{anl}');
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2D Example: solve strain field in pressurized cylindrical tube
%              elastic deformation only
% Construct the equations to be solved by cylind_tube_elast.m
%
% Wei Cai caiwei@stanford.edu
% Created       03/24/2013
% Last Modified 04/29/2013
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function F = eqns_cylind_tube_elast(vars, param)
% F(1:N-1): equilibrium equation
% F(N:2N-2): compatibility equation
% F(2N-1:2N): boundary condition

a = param(1);
b = param(2);
dr = param(3);
mu = param(4);
nu = param(5);
p0 = param(6);

N = length(vars)/2; F = zeros(1,2*N);
eps_rr = vars(1:N); eps_qq = vars(N+1:end);

lambda = 2*mu*nu/(1-2*nu);
r = [a:dr:b]; r_avg = (r(2:end) + r(1:end-1)) / 2;

% stress field
sig_rr = (lambda + 2*mu) * eps_rr + lambda * eps_qq;
sig_qq = (lambda + 2*mu) * eps_qq + lambda * eps_rr;
sig_zz = lambda * (eps_rr + eps_qq);

dsigrrdr = (sig_rr(2:end) - sig_rr(1:end-1)) / dr;
sig_rr_avg = (sig_rr(2:end) + sig_rr(1:end-1)) / 2;
sig_qq_avg = (sig_qq(2:end) + sig_qq(1:end-1)) / 2;

% equilibrium equation
F(1:N-1) = dsigrrdr + (sig_rr_avg-sig_qq_avg)./r_avg;

% strain field
depsqqdr = (eps_qq(2:end) - eps_qq(1:end-1)) / dr;
eps_qq_avg = (eps_qq(2:end) + eps_qq(1:end-1)) / 2;
eps_rr_avg = (eps_rr(2:end) + eps_rr(1:end-1)) / 2;

% compatibility condition
F(N:2*N-2) = depsqqdr - (eps_rr_avg-eps_qq_avg)./r_avg;

% boundary condition
F(2*N-1:2*N) = [sig_rr(1) + p0, sig_rr(end)];
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2D Example: solve strain field in pressurized cylindrical tube
%              pressure large enough to cause plastic deformation
%
% Wei Cai caiwei@stanford.edu
% Created       03/24/2013
% Last Modified 04/29/2013
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

mu = 100;
nu = 0.3;
k = 4;                       % yield stress / sqrt(3)
E = 2*mu*(1+nu);             % Young's modulus
K = 2*mu*(1+nu)/3/(1-2*nu); % bulk modulus

a = 1; b = 2;
dr = 0.02;  drho = dr;
r = [a:dr:b]; rho = r;
p0 = 1;

S_rr_sol   = zeros(length(r), length(rho));
S_qq_sol   = zeros(length(r), length(rho));
sig_rr_sol = zeros(length(r), length(rho));
sig_qq_sol = zeros(length(r), length(rho));
sig_zz_sol = zeros(length(r), length(rho));
eps_rr_sol = zeros(length(r), length(rho));
eps_qq_sol = zeros(length(r), length(rho));

% analytic solution in elastic region
for i=1:length(r),
    for j=1:length(rho),
        % limit to elastic region
        if r(i) >= rho(j),
            ppp=k*((1-2*nu)^2/3 + b^4/rho(j)^4)^(-1/2);
            sig_rr_1 = ppp*(1-b^2./r(i).^2);
            sig_qq_1 = ppp*(1+b^2./r(i).^2);
            sig_zz_1 = nu*(sig_rr_1 + sig_qq_1);
            S_rr_1 = sig_rr_1 - (sig_rr_1+sig_qq_1+sig_zz_1)/3;
            S_qq_1 = sig_qq_1 - (sig_rr_1+sig_qq_1+sig_zz_1)/3;
            eps_rr_1 = (1-nu^2)/E*sig_rr_1 - nu*(1+nu)/E*sig_qq_1;
            eps_qq_1 = (1-nu^2)/E*sig_qq_1 - nu*(1+nu)/E*sig_rr_1;
            S_rr_sol(i,j)   = S_rr_1;
            S_qq_sol(i,j)   = S_qq_1;
            sig_rr_sol(i,j) = sig_rr_1;
            sig_qq_sol(i,j) = sig_qq_1;
            sig_zz_sol(i,j) = sig_zz_1;
            eps_rr_sol(i,j) = eps_rr_1;
            eps_qq_sol(i,j) = eps_qq_1;
        end
    end
end
```

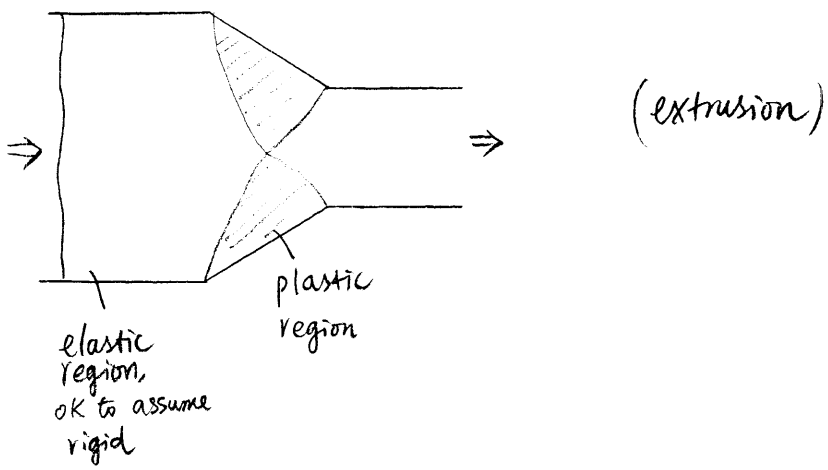
```
% find solution in plastic region
disp('find solution in plastic region...');
for j=2:length(rho),
    disp(sprintf('j = %d / %d',j, length(rho)));
    for i=j-1:-1:1,
        % find solution at (r, rho+drho)
        eps_rr_1 = eps_rr_sol(i,j-1); eps_qq_1 = eps_qq_sol(i,j-1);
        S_rr_1   = S_rr_sol(i,j-1);
        eps_rr_2 = eps_rr_sol(i+1,j); eps_qq_2 = eps_qq_sol(i+1,j);
        S_rr_2   = S_rr_sol(i+1,j);
        param = [r(i), dr, mu, nu, k, eps_rr_1, eps_qq_1, ...
                 S_rr_1, eps_rr_2, eps_qq_2, S_rr_2];
        %options = optimset('Display','iter','TolFun',1e-8);
        options = optimset('Display','off','TolFun',1e-8);
        %F0 = eqns_cylind_tube_plast([eps_rr_1, eps_qq_1, S_rr_1], param);
        %sol = fsolve('eqns_cylind_tube_plast', [eps_rr_1, eps_qq_1, S_rr_1], ...
        %             options, param);
        [sol,Fval,exitflag] = fsolve('eqns_cylind_tube_plast', ...
                      [eps_rr_1, eps_qq_1, S_rr_1], options, param);
        switch exitflag
            case 0,  fprintf(['Solution incorrect: ' ...
                 'Maximum number of function evaluations or iterations reached.\n']);
            case -1, fprintf(['Solution incorrect: ' ...
                 'Algorithm terminated by the output function.\n']);
            case -2, fprintf(['Solution incorrect: ', ...
                 'Algorithm seems to be converging to a point that is not a root.\n']);
            case -3, fprintf(['Solution incorrect: ', ...
                 'Trust region radius became too small.\n']);
            case -4, fprintf(['Solution incorrect: ' ...
                 'Line search cannot sufficiently decrease the residual '...
                 'along the current search direction.\n']);
        end
        %F1 = eqns_cylind_tube_plast(sol, param);
        eps_rr_sol(i,j) = sol(1); eps_qq_sol(i,j) = sol(2); S_rr_sol(i,j) = sol(3);
        S_qq_sol(i,j) = (-S_rr_sol(i,j) + sqrt(4*k^2-3*S_rr_sol(i,j)^2))/2;
        sig_rr_sol(i,j) = S_rr_sol(i,j) + K*(eps_rr_sol(i,j)+eps_qq_sol(i,j));
        sig_qq_sol(i,j) = S_qq_sol(i,j) + K*(eps_rr_sol(i,j)+eps_qq_sol(i,j));
        sig_zz_sol(i,j) = -S_rr_sol(i,j)-S_qq_sol(i,j) ...
                        + K*(eps_rr_sol(i,j)+eps_qq_sol(i,j));
    end
end

% verify solution
disp('verify solution in plastic region...');
for j=2:length(rho),
    for i=j-1:-1:1,
        % find solution at (r, rho+drho)
        eps_rr_1 = eps_rr_sol(i,j-1); eps_qq_1 = eps_qq_sol(i,j-1);
        S_rr_1   = S_rr_sol(i,j-1);
        eps_rr_2 = eps_rr_sol(i+1,j); eps_qq_2 = eps_qq_sol(i+1,j);
```

```
        S_rr_2   = S_rr_sol(i+1,j);
        param = [r(i), dr, mu, nu, k, eps_rr_1, eps_qq_1, ...
                 S_rr_1, eps_rr_2, eps_qq_2, S_rr_2];
        F = eqns_cylind_tube_plast([eps_rr_sol(i,j), eps_qq_sol(i,j), ...
                 S_rr_sol(i,j)], param);
        disp(sprintf('i = %d  j = %d  F = %e %e %e',i,j,F(1),F(2),F(3)));
    end
end

% plot solution
fs = 17;
figure(1);
skip = find(r/r(1)==1.2,1,'First')-1;
plot(r/a, sig_rr_sol(:,1:skip:end)/(2*k), '.-');
set(gca,'FontSize',fs);
xlabel('r / a');
ylabel('\sigma_{rr} / (2k)');
legend('\rho/a = 1','1.2','1.4','1.6','1.8','2.0','Location','SouthEast');

figure(2);
plot(r, sig_qq_sol(:,1:skip:end)/(2*k), '.-');
set(gca,'FontSize',fs);
xlabel('r / a');
ylabel('\sigma_{\theta\theta} / (2k)');
legend('\rho/a = 1','1.2','1.4','1.6','1.8','2.0','Location','NorthWest');

figure(3);
plot(r, sig_zz_sol(:,1:skip:end)/(2*k), '.-');
set(gca,'FontSize',fs);
xlabel('r / a');
ylabel('\sigma_{zz} / (2k)');
legend('\rho/a = 1','1.2','1.4','1.6','1.8','2.0','Location','SouthEast');
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2D Example: solve strain field in pressurized cylindrical tube
%             pressure large enough to cause plastic deformation
% Construct the equations to be solved by cylind_tube_plast.m
%
% Wei Cai caiwei@stanford.edu
% Created       03/24/2013
% Last Modified 04/29/2013
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function F = eqns_cylind_tube_plast(vars, param)
% F(1): equilibrium equation
% F(2): compatibility equation
% F(3): plastic flow rule

eps_rr = vars(1);
eps_qq = vars(2);
S_rr   = vars(3);

r  = param(1);
dr = param(2); drho = dr;
mu = param(3);
nu = param(4);
k  = param(5);
eps_rr_1 = param(6);
eps_qq_1 = param(7);
S_rr_1   = param(8);
eps_rr_2 = param(9);
eps_qq_2 = param(10);
S_rr_2   = param(11);

K = 2*mu*(1+nu)/3/(1-2*nu); % bulk modulus

F = [0 0 0]';

S_qq   = (-S_rr   + sqrt(4*k^2-3*S_rr^2  ))/2;
S_qq_1 = (-S_rr_1 + sqrt(4*k^2-3*S_rr_1^2))/2;
S_qq_2 = (-S_rr_2 + sqrt(4*k^2-3*S_rr_2^2))/2;

dSrr_dr = (S_rr_2-S_rr)/dr;
dSqq_dr = (S_qq_2-S_qq)/dr;
depsrr_dr = (eps_rr_2-eps_rr)/dr;
depsqq_dr = (eps_qq_2-eps_qq)/dr;

depsrr_drho = (eps_rr - eps_rr_1)/drho;
depsqq_drho = (eps_qq - eps_qq_1)/drho;
dSrr_drho   = (S_rr - S_rr_1)/drho;
dSqq_drho   = (S_qq - S_qq_1)/drho;
```

```
% equilibrium condition
F(1) = dSrr_dr + K*( depsrr_dr + depsqq_dr  ) ...
       + ( (S_rr_2+S_rr)/2 - (S_qq_2+S_qq)/2 ) / (r+dr/2);

% compatibility condition
F(2) = depsqq_dr - ( (eps_rr_2+eps_rr)/2 - (eps_qq_2+eps_qq)/2 ) / (r+dr/2);

% plastic flow rule
F(3) = 2*mu*( (2/3*depsrr_drho - 1/3*depsqq_drho)*(S_qq+S_qq_1)/2   ...
             -(2/3*depsqq_drho - 1/3*depsrr_drho)*(S_rr+S_rr_1)/2 ) ...
         - dSrr_drho*(S_qq+S_qq_1)/2 + dSqq_drho*(S_rr+S_rr_1)/2 ;
```

To perform theoretical analysis for general plane strain problems, we shall have to assume the <u>plastic-rigid</u> model for simplicity. This means we shall ignore elastic strain. This is a good approximation when plastic strain is large, e.g. in metal forming.



We will introduce the <u>slip line method</u> to construct solutions to this class of problems.

Due to time constraints, we limit our scope to the interpretation of existing (classical) slip line solutions.

§1. Equations in Plane Strain Plastic-rigid model

plane strain : $u_z = 0$, $\varepsilon_{zz} = 0$

plastic-rigid : $\varepsilon_{ij}^{el} = 0$, $\xrightarrow{\text{implies}}$ incompressibility : $\nu = \frac{1}{2}$

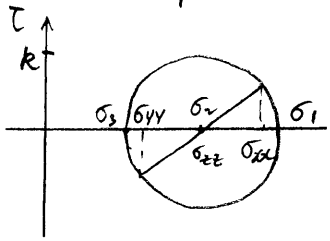
$$\therefore \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$$

non-zeros: $\begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{cases}$ $\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{zz} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \end{cases}$ $\begin{cases} v_x \\ v_y \end{cases}$ (velocity field) $\begin{cases} \dot{\varepsilon}_{xx} = v_{x,x} \\ \dot{\varepsilon}_{yy} = v_{y,y} \\ \dot{\varepsilon}_{xy} = \frac{1}{2}(v_{x,y} + v_{y,x}) \end{cases}$ (plastic) strain rate

In both elastic (rigid) and plastic regions, stress field satisfies <u>equilibrium</u>: (assume zero body force)

$$\begin{cases} \sigma_{xx,x} + \sigma_{yx,y} = 0 & \text{------- (1)} \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 & \text{------- (2)} \end{cases}$$

In the plastic region, the yield condition must be satisfied



note $\sigma_{zz} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) = \frac{1}{2}(\sigma_1 + \sigma_3) = \sigma_2$

$$\sigma_1 > \sigma_2 > \sigma_3 ,$$

$$\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{2}(\sigma_1 + \sigma_3) = \sigma_2$$

$$\therefore \quad S_1 = \sigma_1 - \bar{\sigma} = \frac{1}{2}(\sigma_1 - \sigma_3)$$
$$S_2 = \sigma_2 - \bar{\sigma} = 0$$
$$S_3 = \sigma_3 - \bar{\sigma} = -\frac{1}{2}(\sigma_1 - \sigma_3)$$

$$J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2) = \left[\frac{1}{2}(\sigma_1 - \sigma_3)\right]^2 = k^2$$

yield condition: $\sigma_1 - \sigma_3 = 2k$

Hence the <u>von Mises</u> yield condition <u>coincides</u> with the <u>Tresca</u> yield condition, due to incompressibility.

In terms of σ_{xx}, σ_{yy}, σ_{xy}, the <u>yield condition</u> is:

$$\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2 = k^2 \quad \text{----- (3)}$$

Because the elastic region is rigid, all strain rates are plastic strain rates, which follow the flow rule:

$$\dot{\varepsilon}_{xx} = \tilde{\lambda} S_{xx}$$
$$\dot{\varepsilon}_{yy} = \tilde{\lambda} S_{yy}$$
$$\dot{\varepsilon}_{xy} = \tilde{\lambda} S_{xy}$$

(notice we have changed $\frac{\tilde{\lambda}}{2\mu}$ to $\tilde{\lambda}$ since $\mu \to \infty$ here)

$$\bar{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_3) = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$$

$$S_{xx} = \sigma_{xx} - \bar{\sigma} = \frac{1}{2}(\sigma_{xx} - \sigma_{yy})$$

$$S_{yy} = \sigma_{yy} - \bar{\sigma} = \frac{1}{2}(\sigma_{yy} - \sigma_{xx})$$

$$S_{xy} = \sigma_{xy}$$

$\therefore$ <u>flow rule</u> can be written as

$$\dot{\varepsilon}_{xx} = \tilde{\lambda} \frac{\sigma_{xx} - \sigma_{yy}}{2}$$

$$\dot{\varepsilon}_{yy} = \tilde{\lambda} \frac{\sigma_{yy} - \sigma_{xx}}{2}$$

$$\dot{\varepsilon}_{xy} = \tilde{\lambda}\, \sigma_{xy}$$

Combined with the relation between $\dot{\varepsilon}_{ij}$ and v_i, we have

$$v_{x,x} = \tilde{\lambda} \frac{\sigma_{xx} - \sigma_{yy}}{2}$$

$$v_{y,y} = \tilde{\lambda} \frac{\sigma_{yy} - \sigma_{xx}}{2}$$

$$\frac{v_{x,y} + v_{y,x}}{2} = \tilde{\lambda}\, \sigma_{xy}$$

Adding the first two equations above, we get

$$v_{x,x} + v_{y,y} = 0 \quad \text{-----} \quad (4) \qquad \text{incompressibility condition}$$

Subtracting the first two equations, we get

$$v_{x,x} - v_{y,y} = \tilde{\lambda}(\sigma_{xx} - \sigma_{yy})$$

Dividing by the third equation, we eliminate $\tilde{\lambda}$

$$\frac{v_{x,x} - v_{y,y}}{v_{x,y} + v_{y,x}} = \frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}} \quad \text{-----} (5)$$

Consider 5 unknowns: σ_{xx}, σ_{yy}, σ_{xy}, v_x, v_y

there are 5 equations:

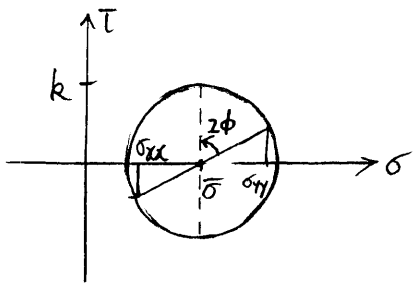
(1), (2)	equilibrium
(3)	yield condition
(4)	incompressibility
(5)	flow rule

§2. Slip line fields

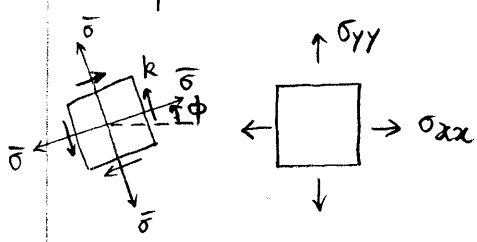
For the 5 unknowns, if the boundary conditions are sufficient to determine the stress field, the problem is called <u>statically determinate</u> (and easier to solve), and the velocity fields can be obtained after the stress fields have been obtained.

Otherwise, the problem is called statically indeterminate and stress fields and velocity fields must be solved together.

We will focus on statically determinate problems here.



In the plastic region, the stress state must lie on the Mohr's circle with center $\bar{\sigma}$ and radius k (maximum shear)



Let ϕ be the angle of rotation to the orientation of maximum shear, where all normal stresses equal to $\bar{\sigma}$

$$\begin{cases} \sigma_{xx} = \bar{\sigma} - k\sin 2\phi \\ \sigma_{yy} = \bar{\sigma} + k\sin 2\phi \\ \sigma_{xy} = k\cos 2\phi \end{cases}$$

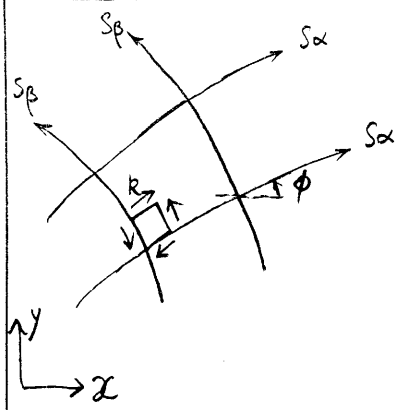
$\bar{\sigma}$ and ϕ are functions of (x, y)

The yield condition, Eq (3), is automatically satisfied.

The <u>equilibrium condition</u> can be written in terms of $\bar{\sigma}$ and ϕ

$$\begin{cases} \dfrac{\partial \bar{\sigma}}{\partial x} + k\left(-\dfrac{\partial}{\partial x}\sin 2\phi + \dfrac{\partial}{\partial y}\cos 2\phi\right) = 0 \\ \dfrac{\partial \bar{\sigma}}{\partial y} + k\left(\dfrac{\partial}{\partial x}\cos 2\phi + \dfrac{\partial}{\partial y}\sin 2\phi\right) = 0 \end{cases}$$

The equations are hyperbolic with characteristics coinciding with slip lines (introduced below).



Consider a curvilinear, locally orthogonal family of curves S_α, S_β covering the plastic region

(S_α, S_β) can be considered as a curvilinear coordinate system.

The tangent of α-curves always have the orientation $\phi(x,y)$

Hence both α-curves and β-curves are along the direction of maximum shear.

In terms of the curvilinear coordinates, the equilibrium conditions look particularly simple:

$$\begin{cases} \dfrac{\partial}{\partial S_\alpha}(\bar{\sigma} - 2k\phi) = 0 \\[2ex] \dfrac{\partial}{\partial S_\beta}(\bar{\sigma} + 2k\phi) = 0 \end{cases}$$

In other words,

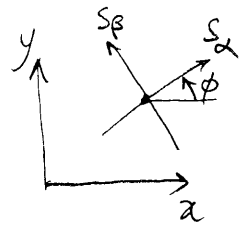
$$\begin{cases} \bar{\sigma} - 2k\phi = \text{constant} & \text{along } \alpha\text{-line} \\ \bar{\sigma} + 2k\phi = \text{constant} & \text{along } \beta\text{-line} \end{cases}$$

(These are typical properties of characteristic curves of PDEs.)

If we know the shapes of the α- and β-curves, and the $(\bar{\sigma}, \phi)$ value at some point, it is easy to find $(\bar{\sigma}, \phi)$ values at the entire region covered by these curves.

The challenge is to construct these curves in the plastic region.

§3. Geometrical Properties of Slip Lines



If we know the stress state ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$) at a point (x, y), we know the local orientation of the slip lines, because the α-line must make angle ϕ with the x-axis.

This means that the α-line satisfy the condition

$$\frac{dy}{dx} = \tan\phi \,, \qquad \frac{\bar{\sigma}}{2k} - \phi = \xi \quad \text{(constant)}$$

and the β-line satisfy the condition

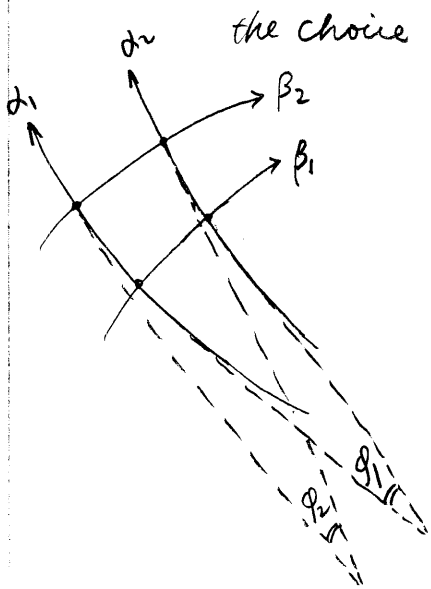
$$\frac{dy}{dx} = -\cot\phi \,, \qquad \frac{\bar{\sigma}}{2k} + \phi = \eta \quad \text{(constant)}$$

So that

$$\bar{\sigma} = k(\xi + \eta)$$

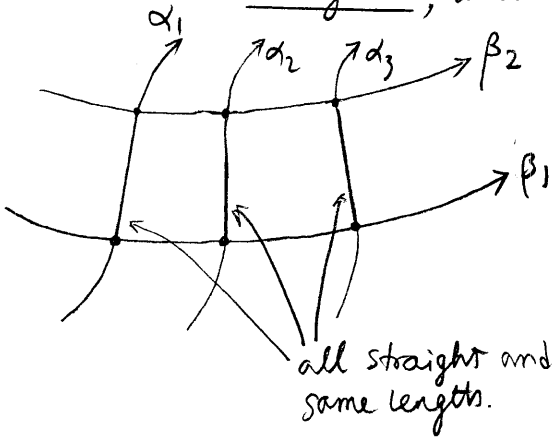
$$\theta = \frac{1}{2}(\eta - \xi)$$

<u>Property 1</u>: The change of orientation of two α-lines at intersections with another β line is independent of the choice of the β-line

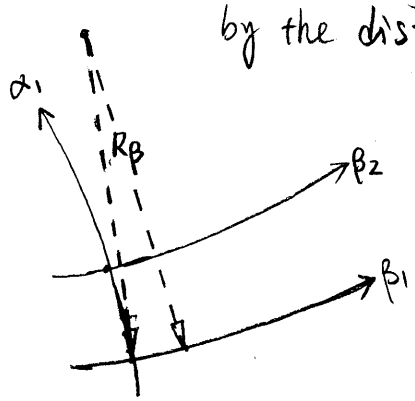


$$\underline{\varphi_1 = \varphi_2}$$

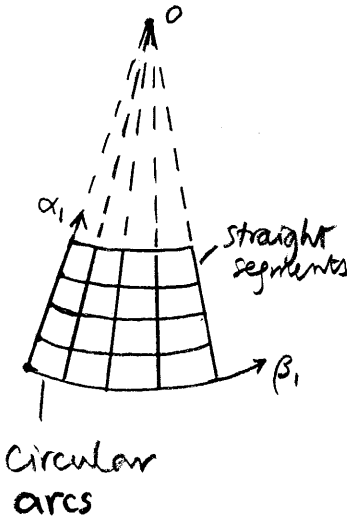
<u>Property 2</u>: If a segment of slip line (e.g. α-line) is <u>straight</u>, then $\bar{\sigma}$, θ, ξ, η, σ_{xx}, σ_{yy}, σ_{xy} are <u>constant</u> along it. and <u>all α lines</u> cut off by the same β lines are <u>straight</u>, and all have the <u>same lengths</u>.



<u>Property 3</u>: Travelling along an α-line, the radius of curvature of the β-line at the intersection changes by the distance traveled.



$$\frac{\partial R_\beta}{\partial s_\alpha} = -1, \qquad \frac{\partial R_\alpha}{\partial s_\beta} = -1$$



An example that illustrates both Property 2 and Property 3.

This region corresponds to a "simple stress state" (see below).

§4. Velocity field

For statically determinate problems, the velocity fields $V_x(x,y)$, $V_y(x,y)$ are found after the slip line fields and stress fields have been obtained.

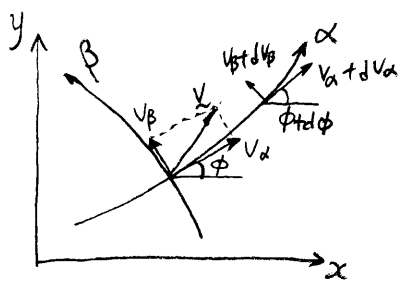
$$\frac{V_{x,x} - V_{y,y}}{V_{x,y} + V_{y,x}} = \frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}} = \frac{-2k\sin 2\phi}{2k\cos 2\phi} = -\tan 2\phi \quad \text{----from (4)}$$

Hence

(already solved)

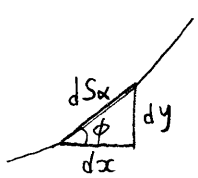
$$\begin{cases} (V_{x,y} + V_{y,x})\tan[2\phi(x,y)] + (V_{x,x} - V_{y,y}) = 0 \\ V_{x,x} + V_{y,y} = 0 \qquad \text{------(5)} \quad \text{incompressibility} \end{cases}$$

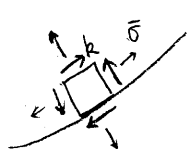
This system of equations is (again) of hyperbolic type, and its characteristics coincide with the slip lines.



Let V_α, V_β be the component of $\underset{\sim}{v}$ along α-line and β-line respectively.

$$\begin{cases} V_x = V_\alpha\cos\phi - V_\beta\sin\phi \\ V_y = V_\alpha\sin\phi + V_\beta\cos\phi \end{cases}$$





note the stress state along slip lines is pure shear (k) plus hydrostatic stress ($\bar{\sigma}$)

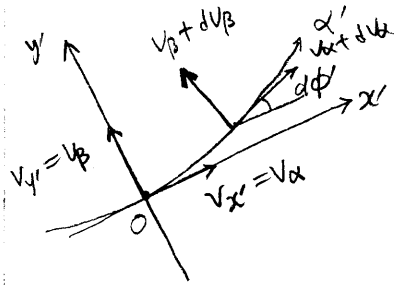
So the normal strain rate along slip line is zero (ignore elastic strain)

The rate of <u>relative elongation along slip lines are zero</u>.

The condition of zero elongation along slip lines can be expressed as

$$\boxed{\begin{array}{ll} dV_\alpha - V_\beta\, d\phi = 0 & \text{along } \alpha\text{-line} \\ dV_\beta + V_\alpha\, d\phi = 0 & \text{along } \beta\text{-line} \end{array}}$$

These are compatibility conditions for the velocities. Geiringer (1930)



To prove Geiringer's relations, we choose a local coordinate system $x'-y'$ so that the x'-axis is tangent to the α-line at point O

In the neighborhood of point O,

$$\begin{cases} V_{x'} = V_\alpha \cos\phi' - V_\beta \sin\phi' \\ V_{y'} = V_\alpha \sin\phi' + V_\beta \cos\phi' \end{cases}$$

Right at point O, $\phi' = 0$, so that $V_{x'} = V_\alpha$, $V_{y'} = V_\beta$

The condition of zero elongation along α-line can be written as

$$\dot{\varepsilon}_{x'x'} = \frac{\partial V_{x'}}{\partial x'} = 0$$

$$\frac{\partial V_\alpha}{\partial x'} \cos\phi' + V_\alpha(-\sin\phi')\frac{\partial \phi'}{\partial x'} - \frac{\partial V_\beta}{\partial x'} \sin\phi' - V_\beta \cos\phi' \frac{\partial \phi'}{\partial x'} = 0$$

At point O, $\phi' = 0$, $\cos\phi' = 1$, $\sin\phi' = 0$

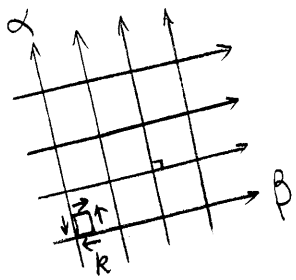
$$\frac{\partial V_\alpha}{\partial x'} - V_\beta \frac{\partial \phi'}{\partial x'} = 0$$

$$\therefore \quad dV_\alpha - V_\beta\, d\phi' = 0 \qquad \text{along } x'$$

Rotate to x-y coordinate system $\phi = \phi' + \phi|_0$

$$dV_\alpha - V_\beta\, d\phi = 0 \qquad \text{along } \alpha\text{-line}$$

§5. Simple stress states



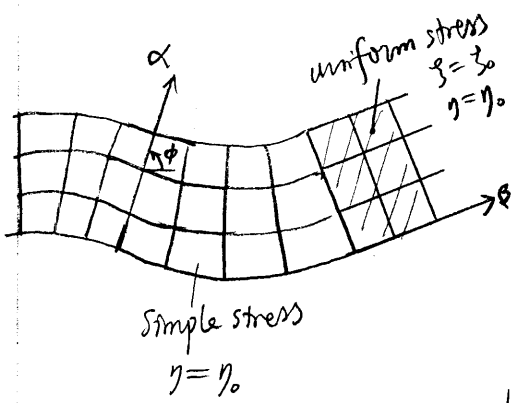
uniform stress state

both α-lines and β-lines are straight lines.

both ξ and η are constants. (ξ_0, η_0)

All stress components are uniform in this region.

$$\bar{\sigma} = k(\xi_0 + \eta_0), \quad \phi = \frac{1}{2}(\eta_0 - \xi_0)$$



Simple stress state

One family (α-line here) is straight.
the other family (β-line here) are generated by orthogonal curves.

Here α-lines are straight, so $\eta = \eta_0$ is a constant.

(How to prove?
First, η = const along each α-line if it is straight
Second, η = const along β-line by definition
Therefore, η = const in the entire region of simple stress)

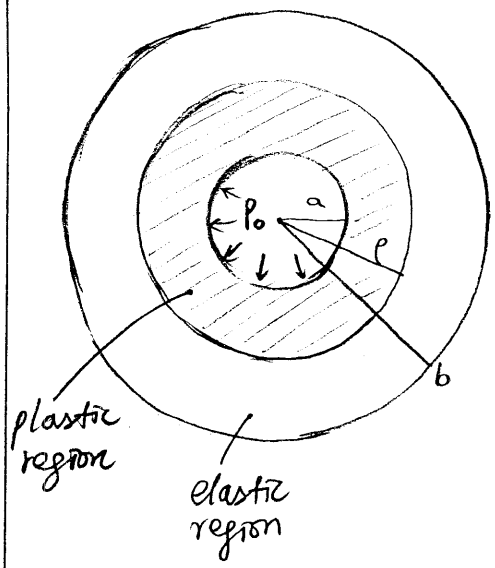
A region adjoining a region of uniform stress is always in a state of simple stress.

Simple stress state:

$$\frac{\bar{\sigma}}{2k} + \phi = \eta_0$$

$$\bar{\sigma} = 2k(\eta_0 - \phi)$$

§5. Axisymmetric field

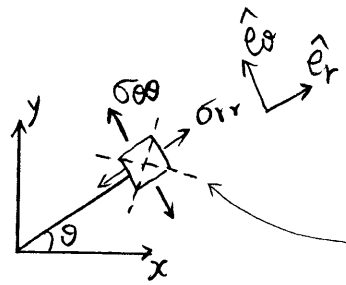


We return to the pressurized cylindrical tube problem ($p_o > p_o^Y$), and try to construct a slip line field solution in the plastic region ($a \le r \le \rho$).

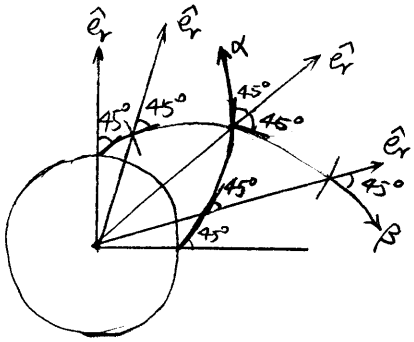
By axi-symmetry, $\sigma_{r\theta} = 0$

Hence σ_{rr} and $\sigma_{\theta\theta}$ are principal stresses, i.e.

$\hat{e}_r$ and $\hat{e}_\theta$ are principal stress directions.



The maximum shear stress direction must be at 45° with $\hat{e}_r$ and $\hat{e}_\theta$.



This means that both α-lines and β-lines must always be at 45° with $\hat{e}_r$ and $\hat{e}_\theta$

A curve that always makes the <u>same angle</u>(γ) with the <u>radial direction</u> is the <u>Logarithmic-spiral</u>.

$$r = a e^{b\theta}$$

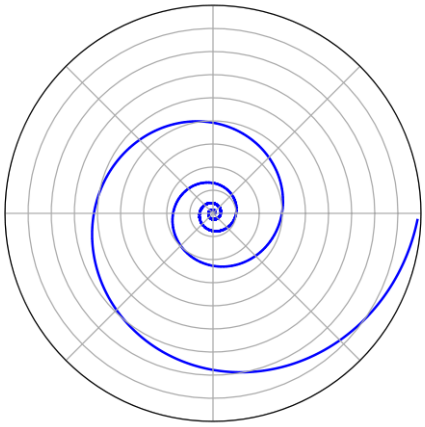
or $$\theta = \frac{1}{b} \ln \frac{r}{a}$$

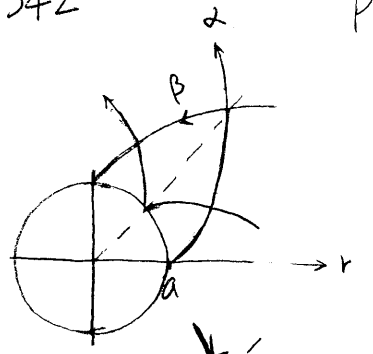
$$\arctan \frac{1}{b} = \gamma = 45^\circ, \quad \frac{1}{b} = \tan 45^\circ = 1 \rightarrow b = 1$$

Logarithmic Spiral

http://en.wikipedia.org/wiki/Logarithmic_spiral

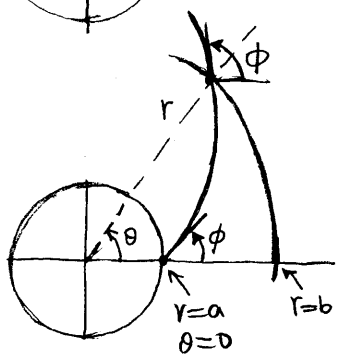
$$r = ae^{b\theta}$$
$$\theta = \frac{1}{b}\ln(r/a),$$





α-lines: $r = a\, e^{\theta-\theta_0}$, $\theta = \theta_0 + \ln\frac{r}{a}$

β-lines: $r = a\, e^{-(\theta-\theta_0)}$, $\theta = \theta_0 - \ln\frac{r}{a}$



consider an α-line, $\theta = \ln\frac{r}{a}$

note $\phi = \frac{\pi}{4} + \theta$

$\therefore\ \Delta\phi = \Delta\theta$ as we travel along the α-line

Recall $\frac{\bar\sigma}{2k} - \phi = \text{const}$ along α-line

$\therefore\ \Delta\bar\sigma = 2k\,\Delta\theta$ along the α-line

$$\bar\sigma = \bar\sigma\big|_{r=a} + 2k\ln\frac{r}{a}$$

Since $\sigma_{\theta\theta} - \sigma_{rr} = 2k$ (yield condition)

$$\sigma_{rr}\big|_{r=a} = -p_0$$

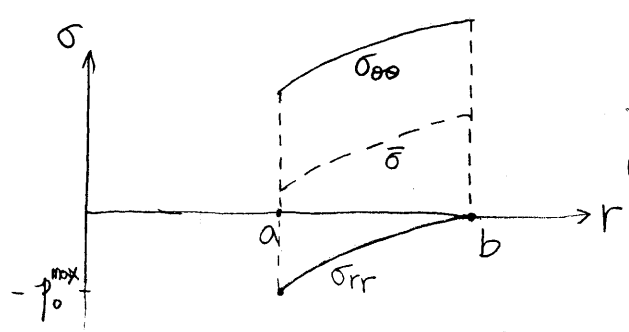
$$\sigma_{\theta\theta}\big|_{r=a} = -p_0 + 2k$$

$$\bar\sigma\big|_{r=a} = \frac{1}{2}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)_{r=a} = -p_0 + k$$

$$\therefore\ \bar\sigma = -p_0 + k + 2k\ln\frac{r}{a}$$

$$\sigma_{rr} = \bar\sigma - k = -p_0 + 2k\ln\frac{r}{a}$$

$$\sigma_{\theta\theta} = \bar\sigma + k = -p_0 + 2k + 2k\ln\frac{r}{a}$$



Applying to full plastic state ($p_0 = p_0^{max}$)

$$0 = \sigma_{rr}\big|_{r=b} = -p_0^{max} + 2k\ln\frac{b}{a}$$

$$p_0^{max} = 2k\ln\frac{b}{a}$$

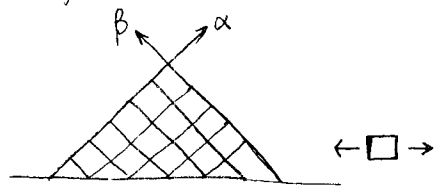
This is the solution in the rigid-plastic limit $\nu \to 1/2$, $\mu \to \infty$

(compare with numerical solution)

A lot of solutions can be constructed by combining a few slip line fields obtained before.

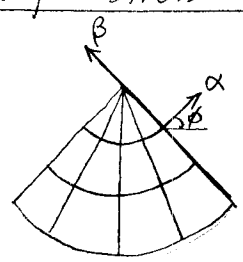
§1. Elementary Solutions

1. Uniform stress state



especially useful near flat free surfaces where the only non-zero stress is normal stress parallel to surface.

2. Simple stress state, especially with one family as concentric arcs.

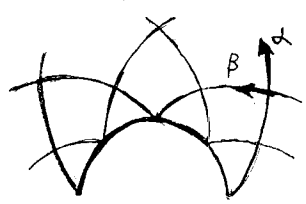


Here β-lines are straight, so

$$\frac{\bar{\sigma}}{2k} - \phi = \xi = \xi_0 \quad (\text{const})$$

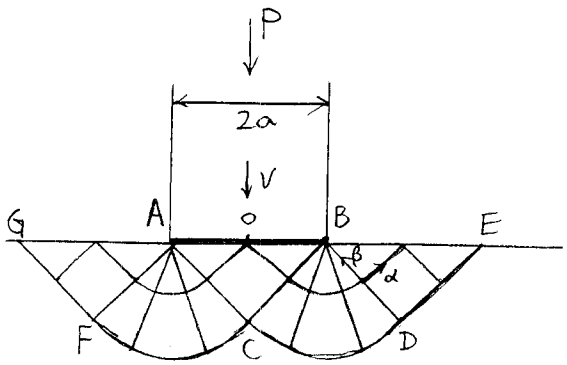
$$\bar{\sigma} = 2k(\phi + \xi_0)$$

3. Logarithmic spiral



especially useful near cylindrical free surfaces

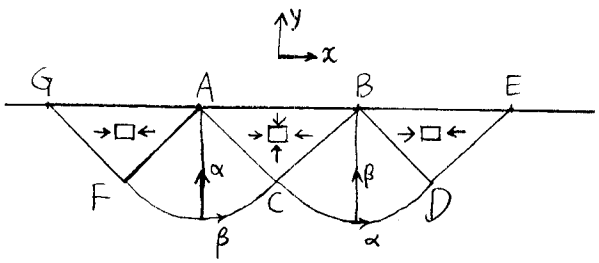
§2. Flat indentor (rigid)



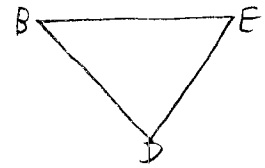
uniform stress state: BDE, GFA, ACB

simple stress state: AFC, BCD

Prandtl's solution
(Kachanov, P. 218)



Region BDE

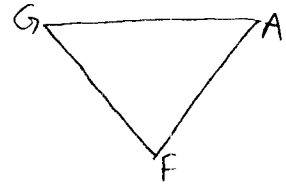


$\sigma_{yy} = 0$

$\sigma_{xx} = -2k$

$\bar{\sigma} = -k$

Region GFA

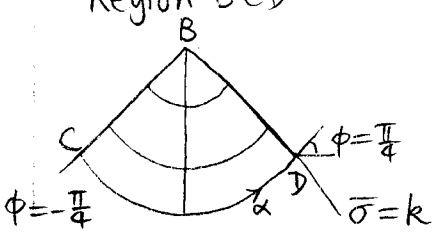


$\sigma_{yy} = 0$

$\sigma_{xx} = -2k$

$\bar{\sigma} = -k$

Region BCD



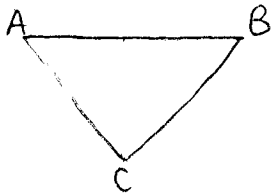
Along BD, $\phi = \frac{\pi}{4}$, $\bar{\sigma} = -k$

Along α-line, $\bar{\sigma} - 2k\phi = \text{const}$

Hence along BC $\bar{\sigma} = -k + 2k \cdot (-\frac{\pi}{2})$

$= -k(\pi + 1)$

Region ABC



$\bar{\sigma} = -k(\pi+1)$

$\sigma_{yy} = \bar{\sigma} - k = -k(\pi+2)$

$\sigma_{xx} = \bar{\sigma} + k = -k\pi$

(assume lubricated contact, no shear stress on AB)

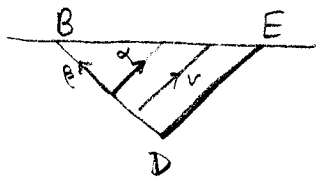
So the limit load is $P = -2a \cdot \sigma_{yy}$

$$\underline{P = 2ak(\pi+2)}$$

We now determine the velocity field using the compatibility condition $\begin{cases} dV_\alpha - V_\beta d\phi = 0 & \text{along } \alpha\text{-line} \\ dV_\beta + V_\alpha d\phi = 0 & \text{along } \beta\text{-line} \end{cases}$

and boundary condition.

Region BDE



DE is elastic-plastic boundary.

Because we assume elastic region is rigid.

$V_\beta = 0$ on DE

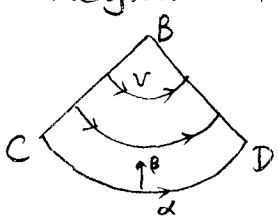
Because $d\phi = 0$ on all β-lines (straight)

$V_\beta = 0$ on entire BDE region.

So the velocity is in the DE direction

(Still need to find magnitude distribution)

Region BCD



CD is elastic-plastic boundary

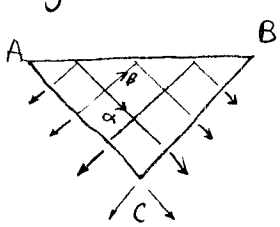
$V_\beta = 0$ on CD

Because β lines are straight,

$V_\beta = 0$ on the entire BCD region

So the velocity direction follows circular arcs

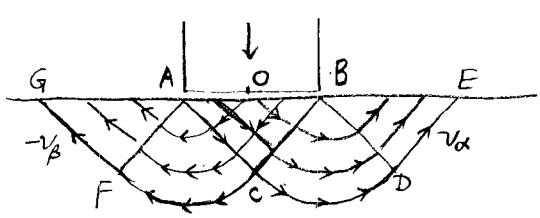
Region ABC



Because both α-lines and β-lines are straight,

V_α remain constant on α-lines

V_β remain constant on β-lines



Arrows indicate velocity stream lines

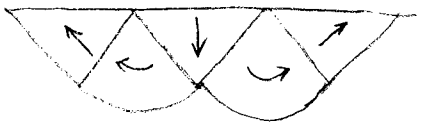
Velocities are discontinuous at AC and CB.

However, the magnitude of the velocity must be conserved along each stream line, but not necessarily the same everywhere.

All that is required is $V_y = \text{const}$ (negative) on AB

That means $-\frac{V_\alpha}{\sqrt{2}} + \frac{V_\beta}{\sqrt{2}} = V_y$ on AB.

Prandtl proposed $v_\alpha = -\frac{v_y}{\sqrt{2}}$, $v_\beta = \frac{v_y}{\sqrt{2}}$ everywhere,

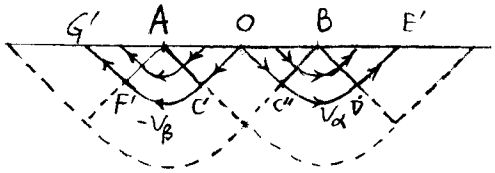


This means all triangular blocks move as rigid bodies.

But this solution is not unique.

Hill proposed an alternative solution, which essentially corresponds to letting

$$v_\alpha = -\sqrt{2}\, v_y \quad \text{on } OB$$
$$v_\beta = \sqrt{2}\, v_y \quad \text{on } AO$$



Hill's solution leads to similar stress fields:

$BD'E'$, $G'F'A$: $\sigma_{yy} = 0$
$\sigma_{xx} = -2k$
$\bar{\sigma} = -k$

$BC''D'$: $\bar{\sigma} = -k$ on BD'
$\bar{\sigma} = -k(\pi+1)$ on BC''

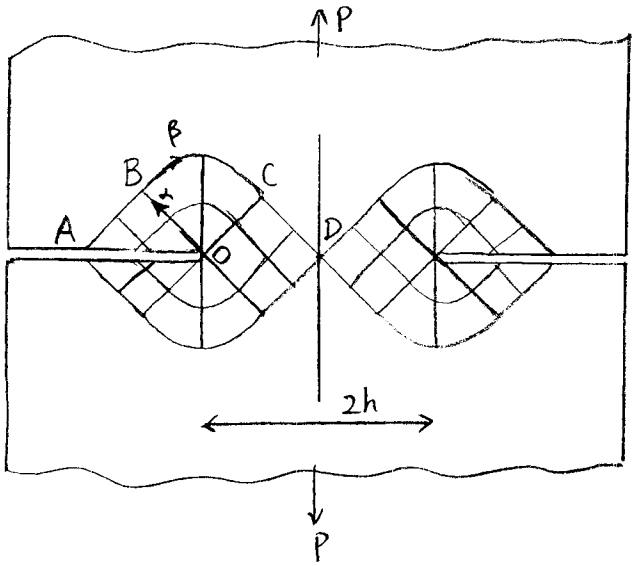
$OC''B'$, $AC'O$: $\bar{\sigma} = -k(\pi+1)$
$\sigma_{yy} = -k(\pi+2)$
$\sigma_{xx} = -k\pi$

So Hill's solution leads to the same limit load $p = 2ak(\pi+2)$

But the plastic zone is smaller, and the rigid blocks are moving at twice the speed of Prandtl's solution

Hill argued that plasticity should start to develop near points A, B (singular stress field points in elasticity), so the <u>Hill's solution is reached first</u> and Prandtl's solution may never be reached. We shall make further use of Hill's solution in the following.

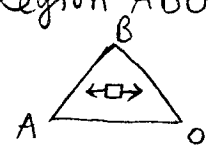
§3. Double Cracks



Turning the flat indentor in tension (with glue applied) and then replace the indentor by the mirror reflection of the sample itself, we get the double crack configuration.

Not surprisingly, the solution look a lot like Hill's solution (plus mirror reflection)

Region ABO

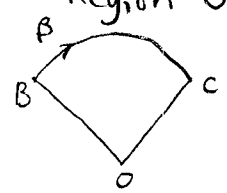


$\sigma_{xx} = 2k$

$\sigma_{yy} = 0$

$\bar{\sigma} = k$

Region OBC

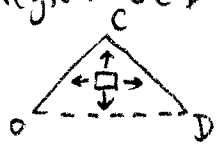


$\bar{\sigma} = k$ on OB

$\bar{\sigma} + 2k\phi = \text{const}$ on β-line

$\bar{\sigma} = (\pi+1)k$ on OC

Region OCD

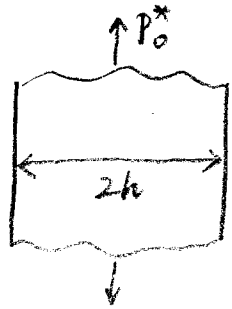


$\bar{\sigma} = (\pi+1)k$

$\sigma_{yy} = (\pi+2)k$

$\sigma_{xx} = \pi k$

The limit load is $P^* = 2(\pi+2)hk$



compared with uniform strip with width 2h whose limit load is $P_0^* = 2h \cdot 2k$

$$\frac{P^*}{P_0^*} = 1 + \frac{\pi}{2}$$

(This solution is only valid for sufficiently deep notches.)

Other Slip Line Field Solutions for Notches

L. M. Kachanov, Fundamentals of the Theory of Plasticity, Dover 2004

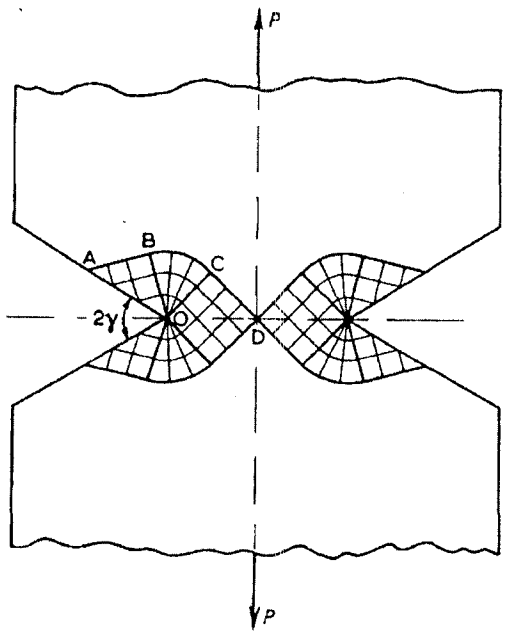


Fig. 106.

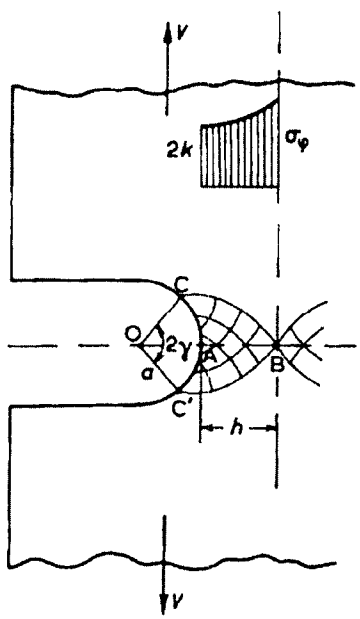


Fig. 107.

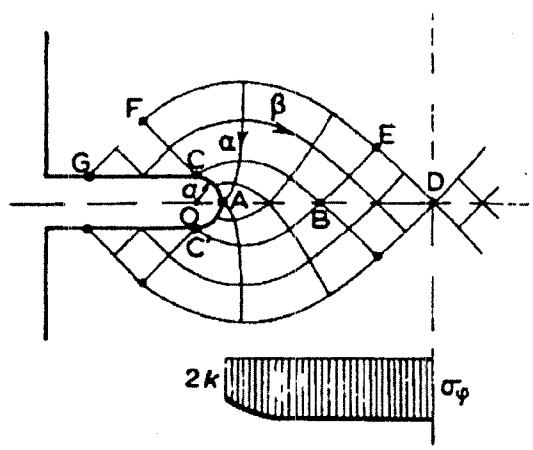
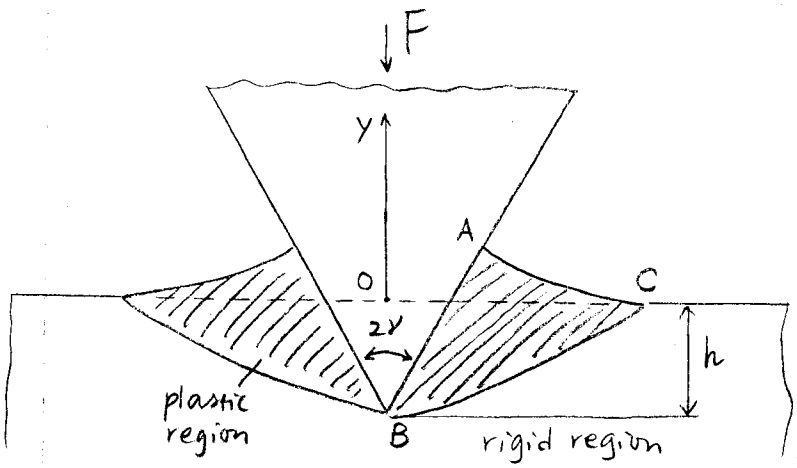


Fig. 108.

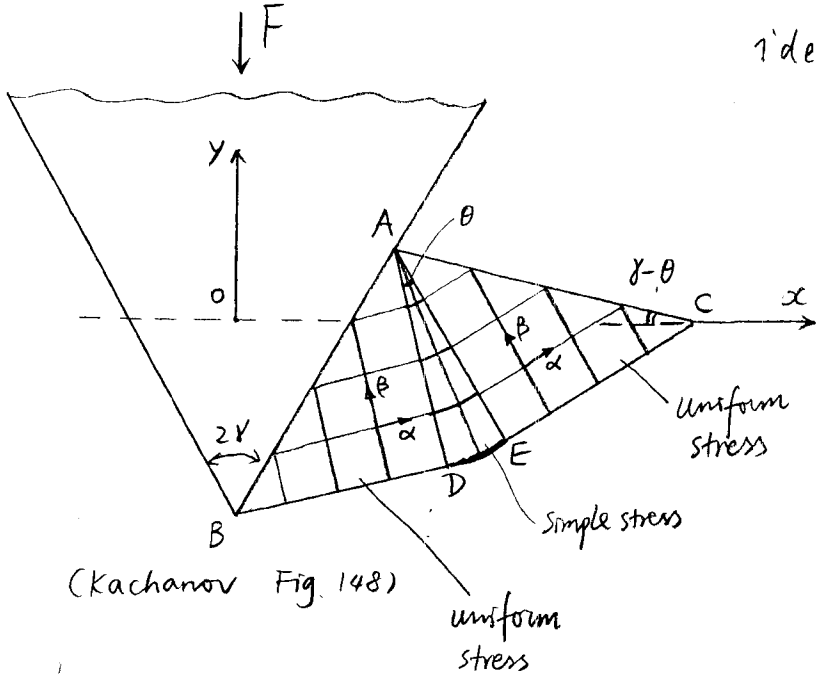
§4. Wedge Indentation

(Kachanov, p.246)



(Kachanov Fig. 147)

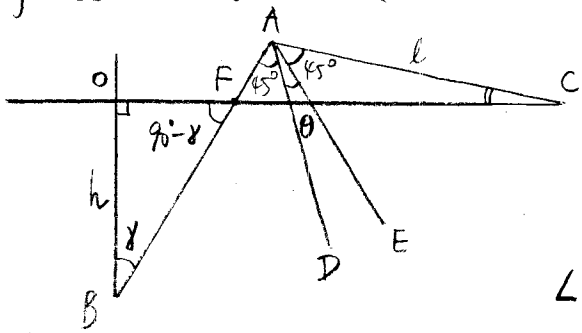
idealization



(Kachanov Fig. 148)

---

proof of $\angle OCA = \gamma - \theta$:



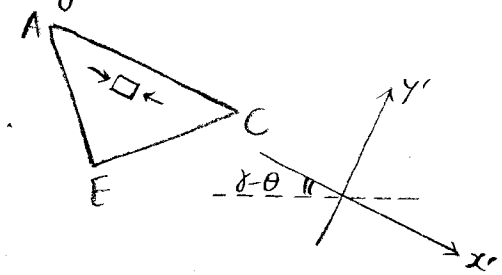
$AB = AC = \ell$
$OB = h$

$\angle OBA = \gamma \rightarrow \angle OFB = 90° - \gamma$
$\angle AFC = 90° - \gamma$

$$\left.\begin{array}{l} \angle BDA = 45° \\ \angle DAE = \theta \\ \angle EAC = 45° \end{array}\right\} \rightarrow \angle CAF = 90° + \theta$$

$$\begin{aligned} \angle OCA &= 180° - \angle CAF - \angle AFC \\ &= 180° - (90° + \theta) - (90° - \gamma) \\ &= \gamma - \theta \end{aligned}$$

Region AEC



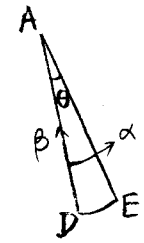
$\sigma_{x'x'} = -2k$

$\sigma_{y'y'} = 0$

$\sigma_{x'y'} = 0$

$\bar{\sigma} = -k$

Region DAE



β lines are straight, so

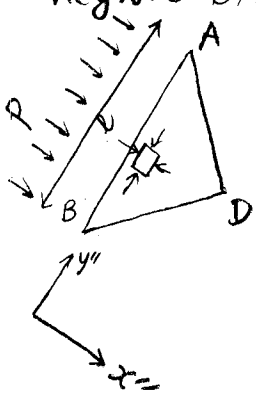
$$\frac{\bar{\sigma}}{2k} - \phi = \xi = \xi_0 \quad (\text{const})$$

Along AE, $\bar{\sigma} = -k$

Along AD, $\bar{\sigma} = -k + 2k(-\theta)$

$= -k(2\theta + 1)$

Region BAD



$\bar{\sigma} = -k(2\theta+1)$

$\sigma_{x''x''} = \bar{\sigma} - k = -2k(\theta+1)$

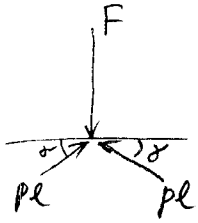
$\sigma_{y''y''} = \bar{\sigma} + k = -2k\theta$

Let: length of AB $= \ell$, pressure on AB $= p$

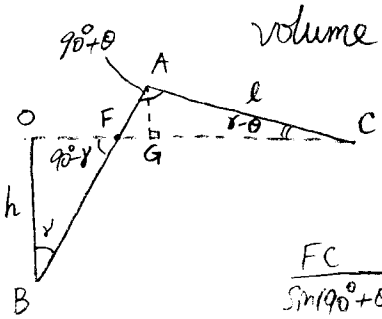
$$p = 2k(\theta+1)$$

Total indentor force

$$F = 2p\ell\sin\gamma = 4k\ell(\theta+1)\sin\gamma$$



We still need to determine θ given γ.



volume conservation: area(OFB) = area(AFC)

area(OFB) $= \frac{1}{2}(OB)\cdot(OF)$

area(AFC) $= \frac{1}{2}(AG)\cdot(FC)$

$$\frac{FC}{\sin(90^\circ+\theta)} = \frac{AF}{\sin(\gamma-\theta)} = \frac{\ell}{\sin(90^\circ-\gamma)}$$

$$\frac{FC}{\cos\theta} = \frac{AF}{\sin(\gamma-\theta)} = \frac{\ell}{\cos\gamma}$$

$$FC = \frac{\ell \cos\theta}{\cos\gamma}$$

$$AG = \ell \sin(\gamma-\theta)$$

$$\text{area}(AFC) = \frac{1}{2}\,\ell \sin(\gamma-\theta)\,\frac{\ell\cos\theta}{\cos\gamma} = \frac{\ell^2 \sin(\gamma-\theta)\cos\theta}{2\cos\gamma}$$

$$AF = \frac{\ell \sin(\gamma-\theta)}{\cos\gamma}$$

$$FB = AB - AF = \ell - \frac{\ell \sin(\gamma-\theta)}{\cos\gamma}$$

$$OF = FB\cdot\sin\gamma = \ell\left(1-\frac{\sin(\gamma-\theta)}{\cos\gamma}\right)\sin\gamma$$

$$\boxed{h = OB = FB\cdot\cos\gamma = \ell\left(\cos\gamma - \sin(\gamma-\theta)\right)}$$

$$\text{area}(OFB) = \frac{1}{2}\,\ell\left(1-\frac{\sin(\gamma-\theta)}{\cos\gamma}\right)\sin\gamma\cdot\ell\left(\cos\gamma-\sin(\gamma-\theta)\right)$$

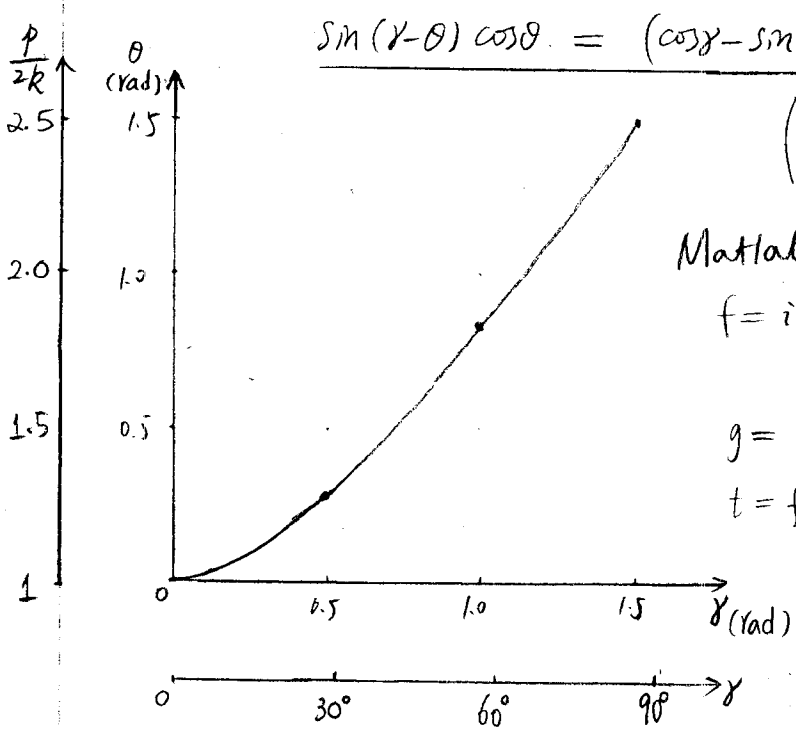
$$= \frac{\ell^2}{2}\,\frac{(\cos\gamma-\sin(\gamma-\theta))^2\sin\gamma}{\cos\gamma}$$

$$\therefore\ \frac{\ell^2\sin(\gamma-\theta)\cos\theta}{2\cos\gamma} = \frac{\ell^2(\cos\gamma-\sin(\gamma-\theta))^2\sin\gamma}{2\cos\gamma}$$

$$\underline{\sin(\gamma-\theta)\cos\theta = (\cos\gamma-\sin(\gamma-\theta))^2\sin\gamma}$$

$$\left(2\gamma = \theta + \operatorname{acos}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)$$

Kachanov, p. 247



Matlab:

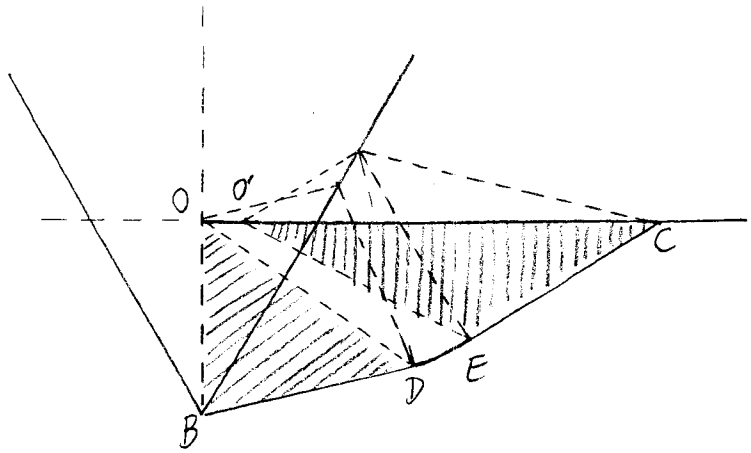
```
f= inline('sin(g-t).*cos(t)-(cos(g)-sin(g-t)).^2
          .*sin(g)', 't', 'g');
g= [0:0.02:1.5];
t= fsolve(@(x) f(x,g), g);
```

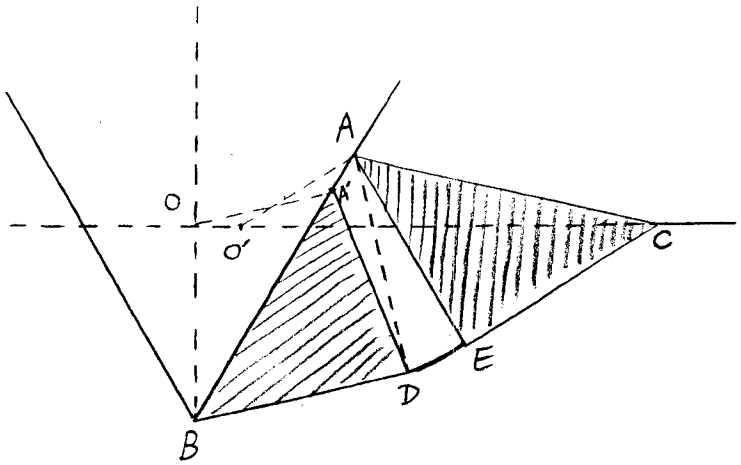
pressure at indentor face: p

$$\boxed{\frac{p}{2k} = \theta + 1}$$

total indenting force: F

$$\frac{F}{k\ell} = 4(\theta+1)\sin\gamma$$



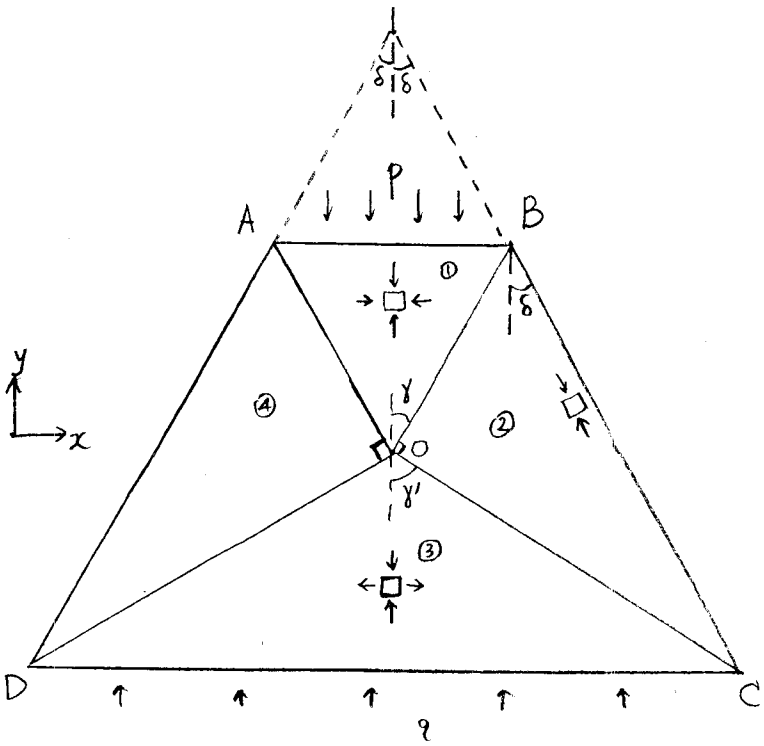


deformation pattern:

original domain		final domain	
OBD	→	A'BD	simple shear O→A'
O'EC	→	AEC	simple shear O'→A
ODEO'	→	A'DEA	complex deformation

§5 Truncated Wedge in compression

(Prager & Hodge, p.161)



all regions ①, ②, ③, ④ are in uniform stress.

lines OA, OB, OC, OD are <u>lines of discontinuity</u>

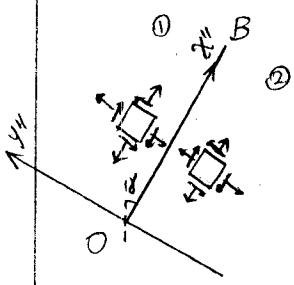
Geometrical relations:

$\gamma = \frac{\pi}{4} - \frac{\delta}{2}$ (to be proved below)

$\gamma' = \frac{\pi}{4} + \frac{\delta}{2}$

$\gamma + \gamma' = \frac{\pi}{2}$

$\therefore \underline{\angle AOD = \angle BOC = \frac{\pi}{2}}$

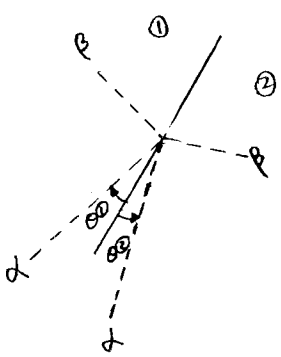


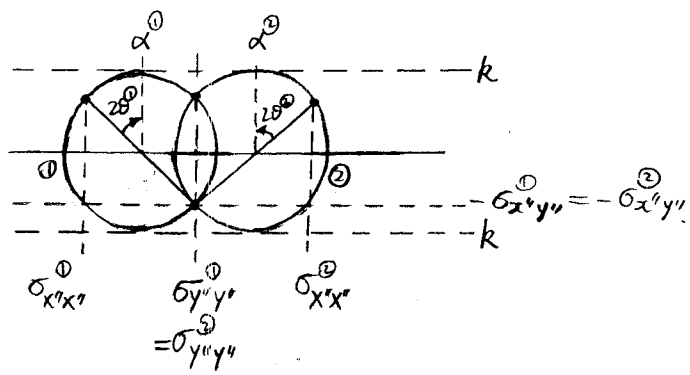
Across line of discontinuity, e.g. OB

$\sigma_{y''y''}$ $\sigma_{x''y''}$ still have to be continuous

only $\sigma_{x''x''}$ can be discontinuous, so $\bar{\sigma}$ may also be discontinuous

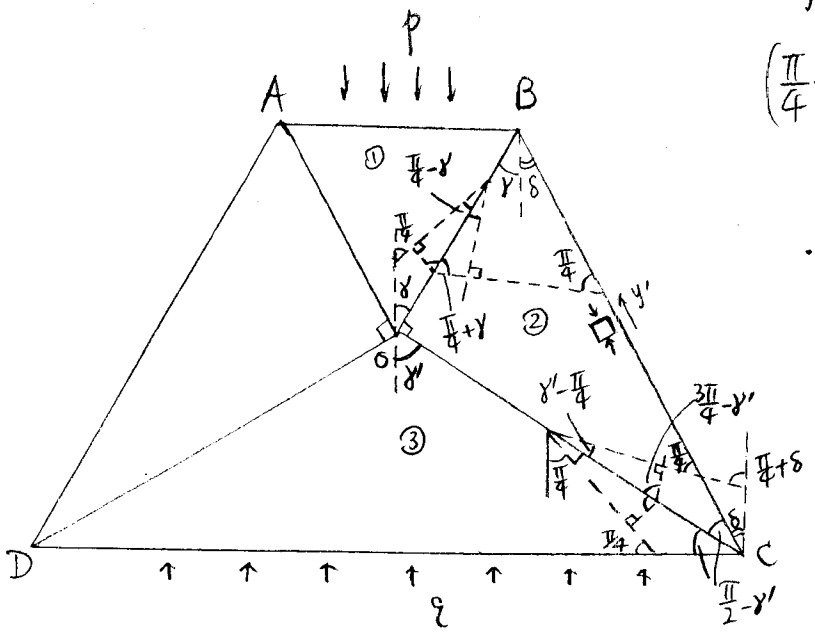
At the same time, $\underset{\approx}{\sigma}^{①}$ have to be on Mohr's circle of radius k. so is $\underset{\approx}{\sigma}^{②}$





The angle of rotation to maximum shear orientation $\theta^{①}, \theta^{②}$ must be equal in magnitude and opposite in sign.

Hence the <u>line of discontinuity must bisect slip lines of the same kind.</u>



Proof of Geometrical relations:

$$\left(\frac{\pi}{4}+\gamma\right)+(\gamma+\delta)+\frac{\pi}{4}=\pi$$

$$2\gamma+\delta=\frac{\pi}{2}$$

$$\therefore\ \gamma=\frac{\pi}{4}-\frac{\delta}{2}$$

$$\left(\frac{\pi}{2}-\gamma'\right)+\left(\frac{\pi}{2}-\gamma'\right)+\delta=\frac{\pi}{2}$$

$$-2\gamma'+\delta+\frac{\pi}{2}=0$$

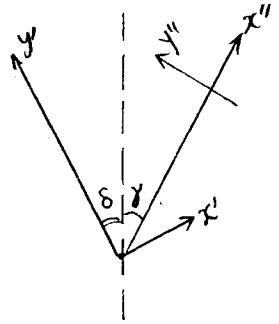
$$\therefore\ \gamma'=\frac{\pi}{4}+\frac{\delta}{2}$$

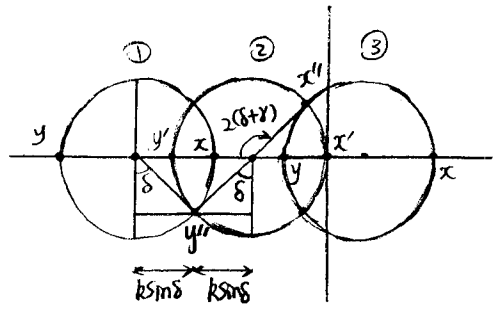
$$OB=\frac{AB}{2\sin\gamma},\quad OC=\frac{CD}{2\sin\gamma'}$$

$$\frac{OB}{OC}=\tan\left(\frac{\pi}{2}-\gamma'\right)=\frac{\cos\gamma'}{\sin\gamma'}=\frac{\sin\gamma}{\sin\gamma'}$$

$$\Rightarrow\ \boxed{\frac{AB}{CD}=\frac{\sin\gamma}{\sin\gamma'}\frac{OB}{OC}=\frac{\sin^2\gamma}{\sin^2\gamma'}=\frac{1-\sin\delta}{1+\sin\delta}}$$

(Q: what if $\frac{CD}{AB}>\frac{1+\sin\delta}{1-\sin\delta}$?)





Region ② BOC

$\sigma_{y'y'}=-2k$

$\sigma_{x'x'}=0$

$\sigma_{x'y'}=0$

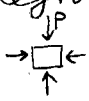
$\bar{\sigma}=-k$

rotation angle from y' to x'' on Mohr's circle $2(\delta+\gamma)=\frac{\pi}{2}+\delta$

$\sigma_{y''y''}=-k-k\sin\delta$

$\sigma_{x''y''}=k\cos\delta$

Region ①



$\sigma_{y''y''}=-k-k\sin\delta$

$\sigma_{x''y''}=k\cos\delta$

$\bar{\sigma}=-k-2k\sin\delta$

$\sigma_{xx}=\bar{\sigma}+k=-2k\sin\delta$

$\sigma_{yy}=\bar{\sigma}-k=-2k(1+\sin\delta)=-p$

$$\boxed{p=2k(1+\sin\delta)}$$

Region ③

$\bar{\sigma}=-k+2k\sin\delta$

$\sigma_{xx}=\bar{\sigma}+k=2k\sin\delta$

$\sigma_{yy}=\bar{\sigma}-k=-2k(1-\sin\delta)=-q$

$$\boxed{q=2k(1-\sin\delta)}$$

The solution to the rigid-plastic model can be used to construct upper and lower bounds for the load on a elastic-perfectly plastic material reaching a critical state.

§1. Theorems of Limit Analysis (Prager-Hodge, §33, p.213)

Assume surface traction $T_j(\underset{\sim}{x})$ increases monotonically with time by an overall multiplicative constant. A state of impending plastic flow will eventually be reached, where plastic strain increases without increasing $T_j(\underset{\sim}{x})$ (i.e. unrestricted plastic flow).

Vanishing of stress rate at impending plastic flow: $\dot{\sigma}_{ij} = 0$ everywhere

This can be proved using Druker's inequality postulate

$$\dot{\sigma}_{ij}\,\dot{\varepsilon}_{ij}^{p} \geq 0 \qquad \text{(to be discussed later)}$$

which is valid for both work-hardening and perfectly plastic (here) materials.

The statement $\dot{\sigma}_{ij} = 0$ is needed to prove the two theorems below.

Lower Bound Theorem

Any statically admissible (stress field) solution provides a lower bound of the load at incipient plasticity.

A statically admissible stress field σ_{ij}^{0} satisfies the following conditions:

① equilibrium condition everywhere: $\sigma_{ij,i}^{0} + F_j = 0$

② boundary condition: $\sigma_{ij}^{0} n_i = T_j$

(assume only two types of B.C.: S_T where T_j is specified and S_u where $u_i = 0$)

③ yield inequality everywhere: $(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 - 4k^2 \leq 0$

Since $T_j(x)$ on S_T is specified up to an overall constant, the largest constant for which a statically admissible stress field σ_{ij}^{0} can still be found is still a lower bound to this constant at impending plastic flow.

Upper Bound Theorem

Any kinematically admissible (velocity field) solution provides an upper bound of the load at incipient plasticity.

A kinematically admissible velocity field v_i^* satisfies the following conditions:

① incompressibility $v^*_{x,x} + v^*_{y,y} = 0$ everywhere, or throughout each finite subregions separated by lines of discontinuity (where only tangential velocity can be discontinuous)

② boundary condition: $v_i^* = 0$ at S_u where $u = 0$.

③ positive work rate. $\int_{S_T} T_j(x)\, v_j^*(x)\, dS > 0$

The overall multiplicative constant in $T_j(x)$ is obtained from the condition (to give an upper bound)

$$\dot{W} \equiv \int_{S_T} T_j(x)\, v_j^*(x)\, dS = k\left[\int_V \sqrt{\dot{\varepsilon}_{ij}^*\, \dot{\varepsilon}_{ij}^*}\, dV + \int_{S_J} |\Delta v_T|\, dS\right] \equiv \dot{D}$$

where $\dot{\varepsilon}_{ij}^* = \frac{1}{2}\left(v^*_{i,j} + v^*_{j,i}\right)$

lines of discontinuity

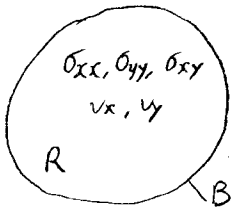
jump in tangential velocity

v_i^* provides a possible scenario of failure (incipient plastic flow). The above equation states that the work done by the external load must be sufficient to balance the internal dissipation of the failure scenario

Note that a true solution to the elastic-plastic material at incipient plasticity must has a stress field that is statically admissible and a velocity field that is kinematically admissible, simultaneously.

§2 Principle of virtual work (Prager-Hodge, §32 p. 209)

The derivation of the lower and upper bound theorems require the principle of virtual work, which is stated here (in 2D plane strain), even though we are not going to derive the lower and upper bound theorems.



consider a stress field $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ and a velocity field v_x, v_y in region R that are continuous and have continuous first derivatives.

The stress field and the velocity field can be chosen <u>independently of each other</u>

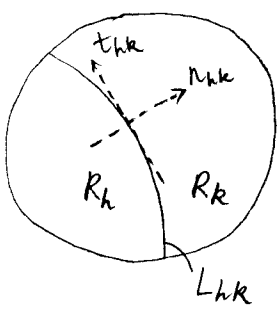
Define: traction force at boundary: $T_j = \sigma_{ij} n_i$

strain rate: $\dot{\varepsilon}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$

<u>Principle of virtual work:</u>

$$\int_R (\sigma_{xx}\dot{\varepsilon}_{xx} + \sigma_{yy}\dot{\varepsilon}_{yy} + 2\sigma_{xy}\dot{\varepsilon}_{xy})\, dA = \int_B (T_x v_x + T_y v_y)\, dS$$

as long as σ_{ij} satisfies equilibrium condition in R.



Now consider R containing lines of discontinuity that divide R into a finite number of subregions: $R_1, R_2, \cdots, R_n$ such that $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are continuous and have continuous first derivatives in each subregion.

We also allow tangential velocity to be discontinuous at line of discontinuity

<u>Principle of virtual work:</u>

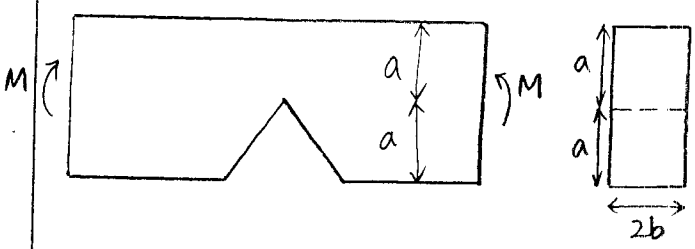
$$\int_R (\sigma_{xx}\dot{\varepsilon}_{xx} + \sigma_{yy}\dot{\varepsilon}_{yy} + 2\sigma_{xy}\dot{\varepsilon}_{xy})\, dA = \int_B (T_x v_x + T_y v_y)\, dS + \sum \int_{L_{hk}} \overbrace{T^{(hk)}}^{\text{negative}} (v_T^h - v_T^k)\, d\ell_{hk}$$

as long as σ_{ij} satisfies equilibrium condition in R.

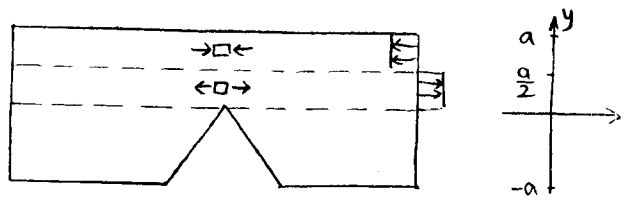
$T^{(hk)}$ is shear stress transmitted across L_{hk} from R_k to R_h

$(v_T^h - v_T^k)$ is the discontinuity in tangential velocity across L_{hk}.

§3. Notched Bar in Bending



Find M at impending plasticity



lower bound estimate

Construct a statically admissible stress field

$$\overset{\circ}{\sigma}_{xx} = -2k \qquad \frac{a}{2} < x \leq a$$

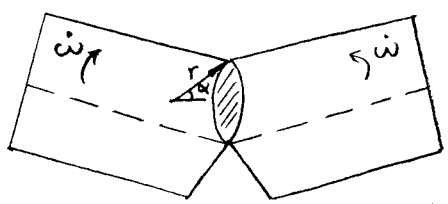
$$\overset{\circ}{\sigma}_{xx} = 2k \qquad 0 < x < \frac{a}{2}$$

$$\overset{\circ}{\sigma}_{xx} = 0 \qquad -a \leq x < 0$$

all other stress components zero

$$\left(\therefore \ (\overset{\circ}{\sigma}_{xx} - \overset{\circ}{\sigma}_{yy})^2 + 4\overset{\circ}{\sigma}_{xy}^2 - 4k^2 \leq 0 \right)$$

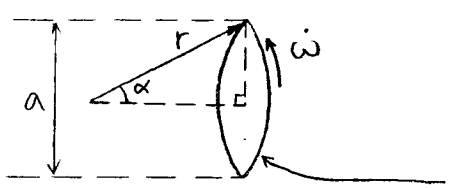
$$M = 2 \cdot 2k \cdot \frac{a}{2} \cdot \frac{a}{4} \cdot 2b = k a^2 b$$



Upper bound estimate

Construct a kinematically admissible velocity field

assume a "mechanism" in which two rigid segment rotate around a hinge



rate of external work: $\dot{W} = 2M\dot{\omega}$

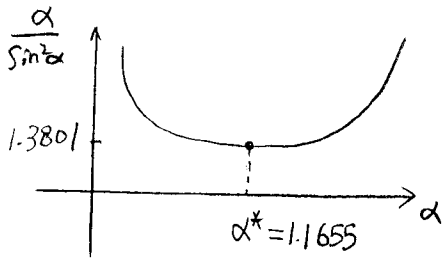
tangential velocity jump at interface

$$|\Delta v_T| = r\dot{\omega} = \frac{a/2}{\sin\alpha}\dot{\omega} \qquad \left(r\sin\alpha = \frac{a}{2}\right)$$

length of discontinuity line: $2 \cdot 2\alpha \cdot r = \frac{2\alpha a}{\sin\alpha}$

total internal dissipation:

$$\dot{D} \equiv k\left(\frac{a/2}{\sin\alpha}\right)\dot{\omega}\,\frac{2\alpha a}{\sin\alpha} \cdot 2b = 2ka^2b\dot{\omega}\frac{\alpha}{\sin^2\alpha}$$



To obtain the tightest upper bound, we choose α^* that minimizes $\frac{\alpha}{\sin^2\alpha}$

$$\min_{0<\alpha<\pi} \frac{\alpha}{\sin^2\alpha} = 1.3801 \quad \text{at} \quad \alpha^* = 1.1655$$

Matlab:
```
f=inline('x./sin(x).^2');
X0 = fminsearch(f, 1)
f(X0)
```

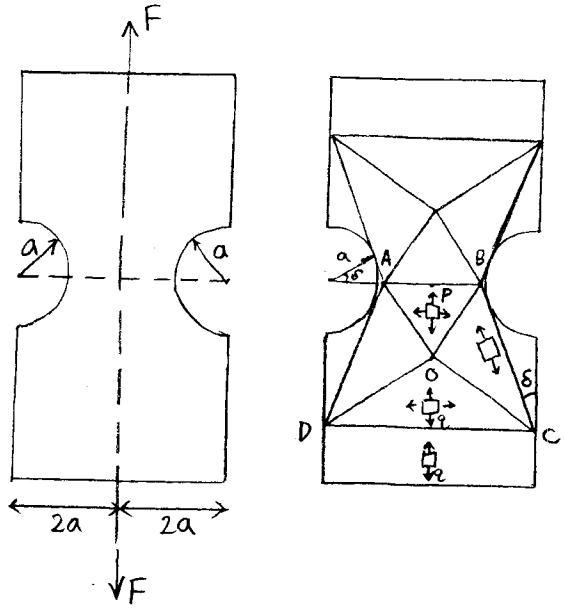
$$\dot{W} = 2M\dot{\omega} = 2ka^2b\,\dot{\omega}\frac{\alpha^*}{\sin^2\alpha^*}$$

$$M = \frac{\alpha^*}{\sin^2\alpha^*}\,ka^2b = 1.38\,ka^2b$$

Therefore, the moment M at impending plastic flow:

$$1.0ka^2b \leqslant M \leqslant 1.38\,ka^2b$$

§4. Grooved Rectangular Block



Find F at impending plasticity

Lower bound estimate

construct a statically admissible stress field

recall the geometric condition

$$\frac{AB}{CD} = \frac{1-\sin\delta}{1+\sin\delta}$$

$$AB = 2\left(2a - \frac{a}{\cos\delta}\right)$$

$$CD = 4a$$

$$\therefore\ 1 - \frac{1}{2\cos\delta} = \frac{1-\sin\delta}{1+\sin\delta} \rightarrow 2\sin 2\delta - \sin\delta - 1 = 0$$

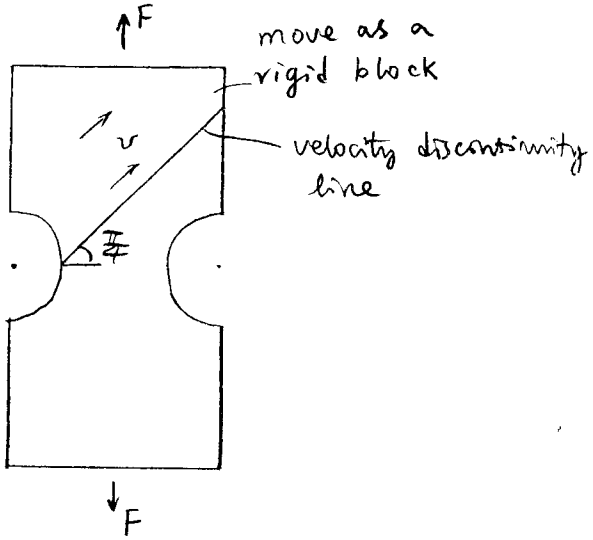
$$\delta = 0.3765, \quad \frac{AB}{CD} = \frac{1-\sin\delta}{1+\sin\delta} = 0.4623$$

$$q = 2k(1-\sin\delta)$$

$$F = 4a\cdot q = 8(1-\sin\delta)\,ka$$

$$F = 5.06\,ka$$

[Matlab: f=inline('2*sin(2*x)-sin(x)-1');
fzero(f, 0.1)]



Upper bound estimate

construct a kinematically admissible velocity field

assume a rigid block sliding off

rate of external work: $\dot{W} = F\cdot V\cdot\frac{\sqrt{2}}{2}$

tangential velocity jump: $|\Delta V_T| = V$

length of discontinuity line: $3a\cdot\sqrt{2}$

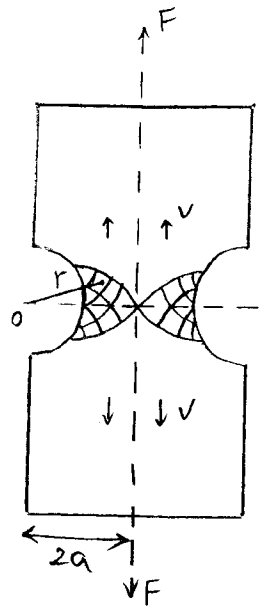
total internal dissipation:

$$\dot{D} = k\cdot V\cdot 3a\sqrt{2}$$

$$\dot{W} = F\cdot V\cdot\frac{\sqrt{2}}{2} = k\cdot V\cdot 3a\sqrt{2} = \dot{D}$$

$$F = 6\,ka$$

Therefore, the force at impending plastic flow: $5.06\,ka \le F \le 6\,ka$



(Image on Coursework front page)

a tighter upper bound can be obtained using the <u>slip line solution</u>

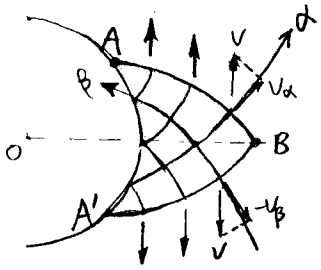
Logarithmic spirals

Stress field: $\sigma_{rr} = 0$ at $r = a$

$\sigma_{rr} = 2k \ln \frac{r}{a}$ $\quad a \le r \le 2a$

$\sigma_{\theta\theta} = 2k + 2k \ln \frac{r}{a}$

velocity field:



The constant velocity v of the rigid regions specifies boundary conditions for

v_α along spiral AB

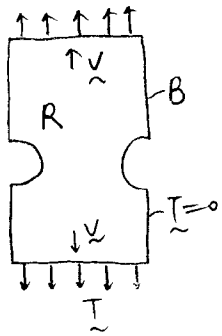
and for

v_β along spiral A'B

(tangential velocity can be discontinuous at the rigid-plastic boundary)

using: $dv_\alpha - v_\beta d\phi = 0$ along α-line

$dv_\beta - v_\alpha d\phi = 0$ along β-line



The entire velocity field v_x, v_y can be constructed for region ABA' (left as exercise)

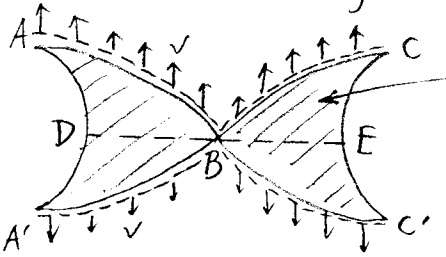
AB and A'B are lines of discontinuity

The <u>slip line velocity field</u> is <u>kinematically admissible</u> because: ① incompressibility $v_{x,x} + v_{y,y} = 0$

② there are no boundary where $\underset{\sim}{u} = 0$

③ $\int_B T_j v_j \, ds > 0$

To compute the dissipation rate $\dot{D}$ for this velocity field, we use the principle of virtual work, with the slip line stress field σ_{ij} (any stress field satisfying equilibrium and is ok)



in region $R = ABA' + CBC'$

lines of discontinuity $S_J = AB + A'B + CB + C'B$

$$F \cdot V = \dot{D} \equiv k\left[\int_R \sqrt{\dot{\varepsilon}_{ij}^* \dot{\varepsilon}_{ij}^*}\, dA + \int_{S_J} |\Delta V_T|\, dS\right]$$

$$= \int_{\text{Boundary}} (T_x V_x + T_y V_y)\, dS$$

dashed line in figure above → Boundary

where $T_j = \sigma_{ij} n_i$

σ_{ij} is the slip line field solution

$$= 2V \int_{AB+BC} T_y\, dS$$

because σ_{ij} satisfies equilibrium

$$= 2V \int_{DE} T_y\, dS$$

$$= 2V \int_{DE} \sigma_{yy}\, dS$$

$$= 4V \int_{DB} \sigma_{\theta\theta}\, dS$$

$$= 4V \int_a^{2a} 2k\left(1 + \ln\frac{r}{a}\right) dr$$

$$= 4k \cdot V \cdot \left[r \ln\frac{r}{a}\right]_a^{2a}$$

$$= 4kV \cdot (2a \ln 2)$$

$$F = 8 \ln 2 \cdot ka = 5.55\, ka \quad \text{(upper bound)}$$

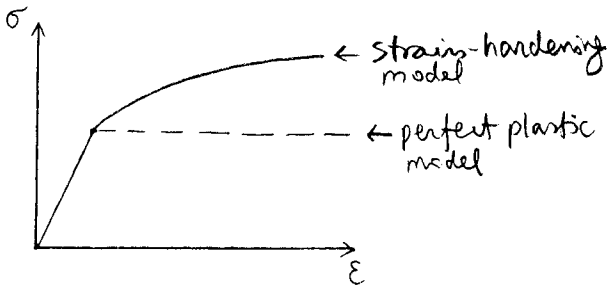
$$\therefore \quad 5.06\, ka \le F \le 5.55\, ka$$

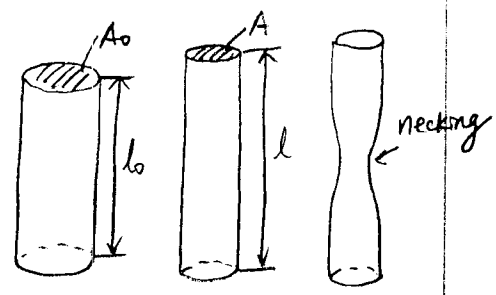
↑ a tighter upper bound than before (6ka).

So far we have considered elastic-perfect plastic models in which the yield surface remains unchanged, i.e. no strain hardening.

We now consider various ways to include strain hardening into the constitutive equation.

§1. Plastic Instability in Tension (R. Hill, p.11)





We will show that a perfect plastic material (no hardening) is unstable in tension, i.e. necking will immediately develop after yielding.

In fact, strain hardening ($\frac{d\sigma}{d\varepsilon}$) need to be sufficiently strong for stability, i.e. to prevent necking.

Most ductile materials loaded in tension eventually fail by necking, due to violation of stability criterion (i.e. insufficient hardening).

Let A_0, l_0 be the cross sectional area and length of the bar prior to deformation (i.e. reference state).

Let A, l be the values at current state (before necking)

Neglect elastic strain, because plastic strain conserves volume, we have

$$A \cdot l = A_0 \cdot l_0$$

$$d(A \cdot l) = A\, dl + l\, dA = 0$$

$$\frac{dA}{A} = -\frac{dl}{l}$$

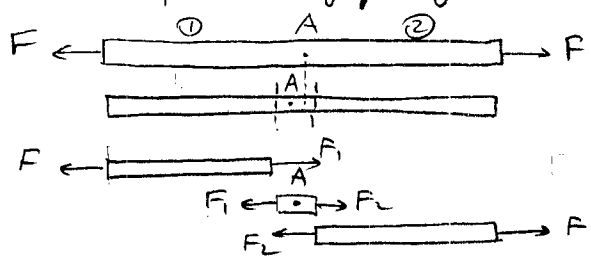
define: ε: <u>engineering strain</u> $d\varepsilon = \frac{dl}{l_0}$, $\varepsilon = \frac{l - l_0}{l_0}$, $l = l_0(1+\varepsilon)$

ε^t: <u>true strain</u> $d\varepsilon^t = \frac{dl}{l}$, $\varepsilon^t = \ln\frac{l}{l_0} = \ln(1+\varepsilon)$

(note: if $l=0$, $\varepsilon = -1$, $\varepsilon^t = -\infty$
if $l = 2l_0$, $\varepsilon = 1$, $\varepsilon^t = \ln 2$)

Stability requires that the total force F on the cross section increases with positive dl, i.e. $\frac{dF}{dl} > 0$

If this is not the case, the system can spontaneously lower the free energy by strain <u>localization</u>.



Imagine two sections of the rod.

Imagine a virtual deformation where point A moves left

i.e. $dl^{①} < 0$, $dl^{②} = -dl^{①} > 0$

Suppose $\frac{dF}{dl} < 0$, then $F_1 > F$, $F_2 < F$

then point A will move further to the left.

$\rightarrow$ unstable.

$$F = A \cdot \sigma$$

$$dF = d(A\cdot\sigma) = A\,d\sigma + \sigma\,dA$$

$$\frac{dl}{l_0} = \frac{l_0\,d\varepsilon}{l_0(1+\varepsilon)} = \frac{d\varepsilon}{1+\varepsilon}$$

$$= A\left(d\sigma + \frac{dA}{A}\sigma\right) = A\left(d\sigma - \frac{dl}{l}\sigma\right)$$

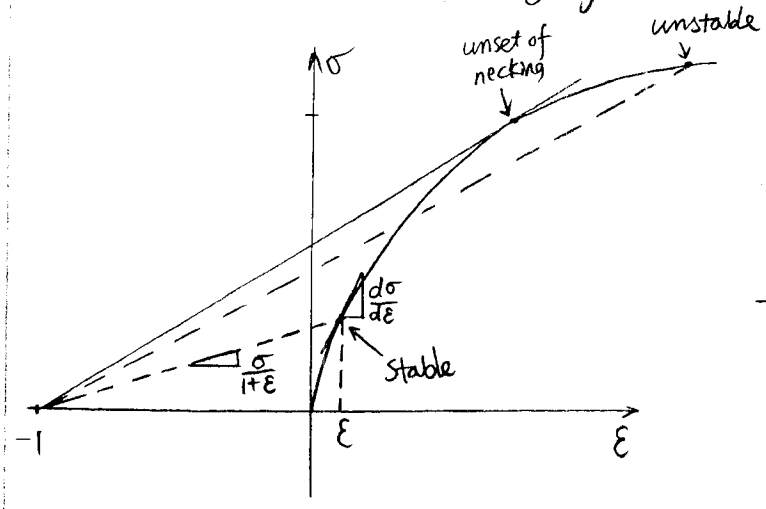
$$dF = A\left(d\sigma - \sigma\frac{d\varepsilon}{1+\varepsilon}\right) = A\,d\varepsilon\left(\frac{d\sigma}{d\varepsilon} - \frac{\sigma}{1+\varepsilon}\right)$$ in terms of engineering strains

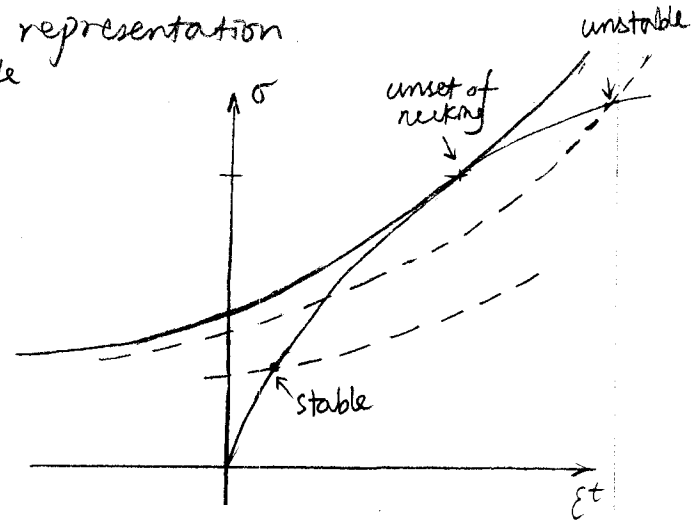
$$dF = A\left(d\sigma - \sigma\,d\varepsilon^t\right) = A\,d\varepsilon^t\left(\frac{d\sigma}{d\varepsilon^t} - \sigma\right)$$ in terms of true strains

Stability criterion $\frac{dF}{dl} > 0$, i.e. $\frac{dF}{d\varepsilon} > 0$, i.e. $\frac{dF}{d\varepsilon^t} > 0$

requires $\boxed{\frac{d\sigma}{d\varepsilon} > \frac{\sigma}{1+\varepsilon}}$ or equivalently $\boxed{\frac{d\sigma}{d\varepsilon^t} > \sigma}$

This condition has a graphical representation





draw a family of curves (dashed line) for $\sigma = \alpha(1+\varepsilon)$

i.e. straight lines passing $(\varepsilon=-1, \sigma=0)$

regardless of α, all these curves satisfy the condition

$$\frac{d\sigma}{d\varepsilon} = \alpha = \frac{\sigma}{1+\varepsilon}$$

At the intersection point between the stress strain curve $\sigma(\varepsilon)$ and any dashed line, if $\frac{d\sigma}{d\varepsilon}$ of the stress strain curve is larger than that of the dashed line, then the point is stable.

$\Theta \equiv \frac{d\sigma}{d\varepsilon}$ is called strain hardening rate

draw a family of curves (dashed line) for $\sigma = \alpha e^{\varepsilon^t}$

(recall $\varepsilon^t = \ln(1+\varepsilon)$)

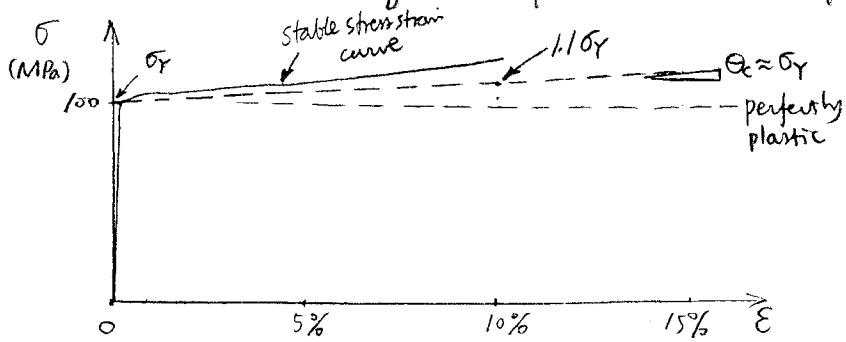
regardless of α, all these curves satisfy the condition

$$\frac{d\sigma}{d\varepsilon^t} = \alpha e^{\varepsilon^t} = \sigma$$

At the intersection point, if $\frac{d\sigma}{d\varepsilon^t}$ of the stress-strain curve is larger than that of the dashed curve, the point is stable.

(note the point $\varepsilon = -1$ is equivalent to $\varepsilon^t = -\infty$)

How much hardening is required for stability in tension?



Suppose

$\sigma_Y = 100\ MPa$

$E = 200\ GPa$

Strain at $\sigma = \sigma_Y$

$\varepsilon_Y = \frac{\sigma_Y}{E} = 0.5 \times 10^{-3} = 0.05\%$

$\varepsilon_Y \approx 0$ at this scale

The required hardening rate is $\theta_c \approx \sigma_Y = 100 MPa$.

This means at $\varepsilon = 10\%$, $\sigma > 1.1\sigma_Y$

i.e. only 10% increase in flow stress for 10% plastic strain

This explains why material may satisfy the tensile stability criterion and yet can still be well approximated by the elastic-perfectly plastic model.

§2 Isotropic Hardening Model

The tensile stress-strain curve is not sufficient to fully describe the plastic behavior. We need to know how the entire yield surface change with plastic deformation.

The isotropic hardening model and the kinematic hardening model are two commonly used approximations to the complex behaviors of real materials.

In isotropic hardening model, we write the yield condition as

$$f(S_{ij}) = \varphi(q) \qquad \text{(Kachanov P.80)}$$

where S_{ij} is the deviatoric part of the stress σ_{ij} $(S_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij})$ (since yield should not depend on average stress $\bar{\sigma}$)

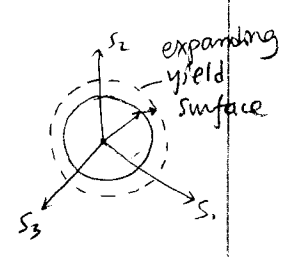
$q > 0$ is some measure of isotropic hardening.

A common measure of q is the work of plastic deformation

$$q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl}$$

Another, less frequent, measure is charateristic of accumulated plastic strain

$$q = \int \sqrt{2\, d\varepsilon_{ij}^{pl}\, d\varepsilon_{ij}^{pl}}$$



We can still use the von Mises criterion for function f

$$f(S_{ij}) = \frac{1}{2} S_{ij} S_{ij} .$$

(The special case of perfect-plasticity (no hardening) corresponds to $\varphi(q) = k^2$ (constant))

For hardening model, we can choose $\varphi(q)$ to reproduce the tensile stress-strain curve.

As an example, let $f(s_{ij}) = \frac{1}{2} s_{ij} s_{ij}$, $q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl}$

and $\varphi(q) = k^2 + \beta q$

and let's find out how the tensile stress-strain curve looks like.

In uniaxial tension $\sigma_{xx} > 0$, $\sigma_{yy} = \sigma_{zz} = 0$, $\bar{\sigma} = \frac{1}{3}\sigma_{xx}$

$$s_{xx} = \sigma_{xx} - \bar{\sigma} = \frac{2}{3}\sigma_{xx}, \quad s_{yy} = -\frac{1}{3}\sigma_{xx}, \quad s_{zz} = -\frac{1}{3}\sigma_{yy}$$

$$f(s_{ij}) = \frac{1}{2} s_{ij} s_{ij} = \frac{1}{2}\left(\frac{4}{9}\sigma_{xx}^2 + \frac{1}{9}\sigma_{xx}^2 + \frac{1}{9}\sigma_{xx}^2\right) = \frac{1}{3}\sigma_{xx}^2$$

plastic flow rule:

$$d\varepsilon_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu} s_{ij}\, dt \qquad \underline{d\varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}}{2\mu}\frac{2}{3}\sigma_{xx}\, dt}$$

$$q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl} = \int s_{ij} \frac{\tilde{\lambda}}{2\mu} s_{ij}\, dt$$

$$dq = \frac{\tilde{\lambda}}{2\mu} s_{ij} s_{ij}\, dt = \frac{\tilde{\lambda}}{2\mu}\frac{1}{3}\sigma_{xx}^2\, dt$$

hardening law:

$$f(s_{ij}) \equiv \frac{1}{2} s_{ij} s_{ij} = \varphi(q) \equiv k^2 + \beta q$$

$$\frac{1}{3}\sigma_{xx}^2 = k^2 + \beta q$$

$$\frac{2}{3}\sigma_{xx}\, d\sigma_{xx} = \beta\, dq = \beta \cdot \frac{\tilde{\lambda}}{2\mu} \cdot \frac{1}{3}\sigma_{xx}^2\, dt$$

$$\underline{d\sigma_{xx} = \frac{\tilde{\lambda}}{2\mu} \cdot \frac{\beta}{2}\sigma_{xx}\, dt}$$

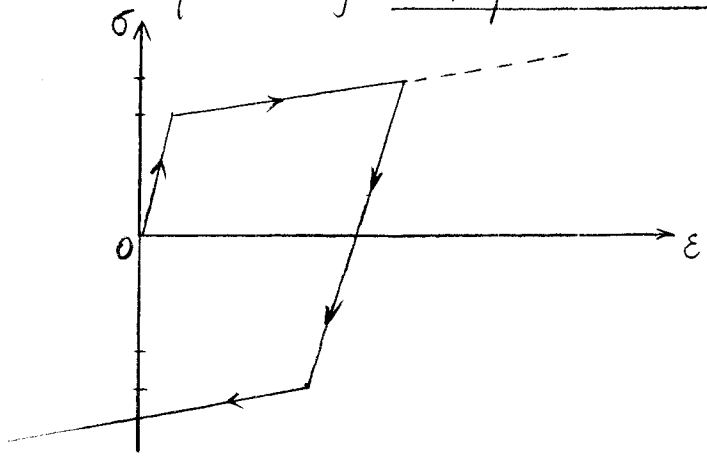
$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}^{pl}} = \frac{3}{4}\beta, \quad \frac{d\varepsilon_{xx}^{tot}}{d\sigma_{xx}} = \frac{d\varepsilon_{xx}^{el}}{d\sigma_{xx}} + \frac{d\varepsilon_{xx}^{pl}}{d\sigma_{xx}} = \frac{1}{E} + \frac{4}{3\beta},$$

Hardening rate: $\theta \equiv \frac{d\sigma_{xx}}{d\varepsilon_{xx}^{tot}} = \left(\frac{1}{E} + \frac{4}{3\beta}\right)^{-1}$

Note that when hardening is present, $\tilde{\lambda}$ can be determined from $d\sigma_{xx}/dt$, which was not possible in perfectly plastic material.

Q: Find the hardening rate in uniaxial tension if $q = \int \sqrt{2 d\varepsilon_{ij}^{pl} d\varepsilon_{ij}^{pl}}$ is used instead.

consequence of <u>isotropic hardening</u> model in reverse loading.

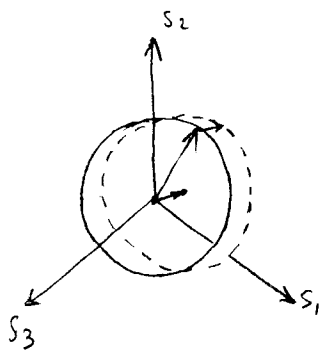


yield stress in reverse loading also increases

no Bauschinger effect.

§3 Kinematic Hardening Model

(Translational hardening in Kachanov)



$$f(s_{ij} - \alpha_{ij}) = k^2$$

$$\alpha_{ij} = c\,\varepsilon_{ij}^{pl}$$

Let's find out its consequence in uniaxial tension

$$s_{xx} = \frac{2}{3}\sigma_{xx}, \quad s_{yy} = -\frac{1}{3}\sigma_{xx}, \quad s_{zz} = -\frac{1}{3}\sigma_{xx}.$$

$$d\varepsilon_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu} s_{ij}\, dt$$

$$\therefore \quad d\varepsilon_{yy}^{pl} = d\varepsilon_{zz}^{pl} = -\frac{1}{2} d\varepsilon_{xx}^{pl}$$

$$\varepsilon_{yy}^{pl} = \varepsilon_{zz}^{pl} = -\frac{1}{2}\varepsilon_{xx}^{pl}$$

$$\alpha_{xx} = c\,\varepsilon_{xx}^{pl}, \quad \alpha_{yy} = -\frac{1}{2} c\,\varepsilon_{xx}^{pl}, \quad \alpha_{zz} = -\frac{1}{2} c\,\varepsilon_{xx}^{pl}$$

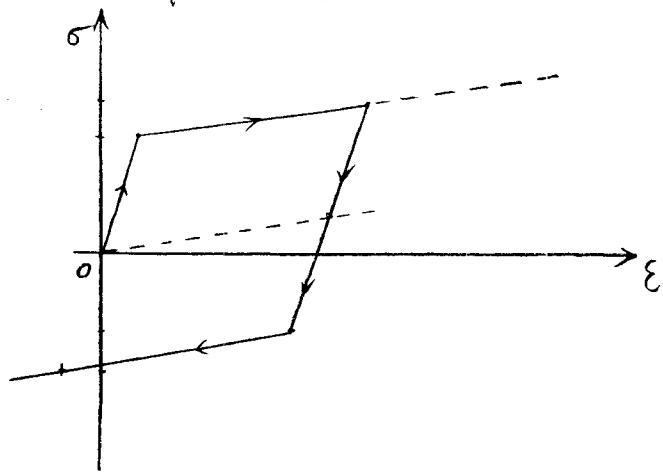
$$f(s_{ij} - \alpha_{ij}) = \frac{1}{2}\left[\left(\frac{2}{3}\sigma_{xx} - c\,\varepsilon_{xx}^{pl}\right)^2 + \left(-\frac{1}{3}\sigma_{xx} + \frac{1}{2}c\,\varepsilon_{xx}^{pl}\right)^2 + \left(-\frac{1}{3}\sigma_{xx} + \frac{1}{2}c\,\varepsilon_{xx}^{pl}\right)^2\right]$$

$$= \frac{1}{3}\left(\sigma_{xx} - \frac{3}{2}c\,\varepsilon_{xx}^{pl}\right)^2 = k^2$$

$$\sigma_{xx} = \sqrt{3}\,k + \frac{2}{3}c\,\varepsilon_{xx}^{pl}$$

$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}^{pl}} = \frac{2}{3}c \qquad \text{Hardening rate: } \Theta \equiv \frac{d\sigma_{xx}}{d\varepsilon_{xx}^{tot}} = \left(\frac{1}{E} + \frac{3}{2c}\right)^{-1}$$

consequence of kinematic hardening model in reverse loading.



Bauschinger effect

of course, a combination of isotropic and kinematic hardening model can be derived, e.g.

$$f(s_{ij} - \alpha_{ij}) = k^2 + \beta q$$

$$\alpha_{ij} = c\,\varepsilon_{ij}^{pl}, \qquad q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl}$$

Q: work out its consequence in uniaxial tension test

§4 Associated Flow Law

The flow rule we have been using: $\dot{\varepsilon}_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu} s_{ij}$

is an <u>associated flow law</u> with the von Mises yield condition:

$$f(s_{ij}) \equiv \frac{1}{2} s_{ij} s_{ij} = \varphi(q)$$

On the other hand, it is not associative with the Tresca yield condition: $|\sigma_1 - \sigma_3| = \varphi(q)$, $\sigma_1 \geq \sigma_2 \geq \sigma_3$

To understand what it means, we shall introduce a function $g(\sigma_{ij})$ (plastic potential) such that the flow rule can be written as

$$\dot{\varepsilon}_{ij}^{pl} = h \frac{\partial g}{\partial \sigma_{ij}}$$ where h is a scalar function.

If $g = f$, then the flow rule is associative (with the yield condition)

for flow — for yield surface

— plastic flow is always normal to yield surface

For example, for the von Mises yield condition, $f = \frac{1}{2} s_{ij} s_{ij}$

the associated flow rule is

$$\dot{\varepsilon}_{ij}^{pl} = h \cdot \frac{\partial f}{\partial \sigma_{ij}} = h \frac{\partial}{\partial s_{ij}} \left(\frac{1}{2} s_{ij} s_{ij}\right) = h \cdot s_{ij}$$

This is the same as $\dot{\varepsilon}_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu} s_{ij}$ if we identify $h = \frac{\tilde{\lambda}}{2\mu}$.

Q: what is the associative flow rule for Tresca's yield condition?

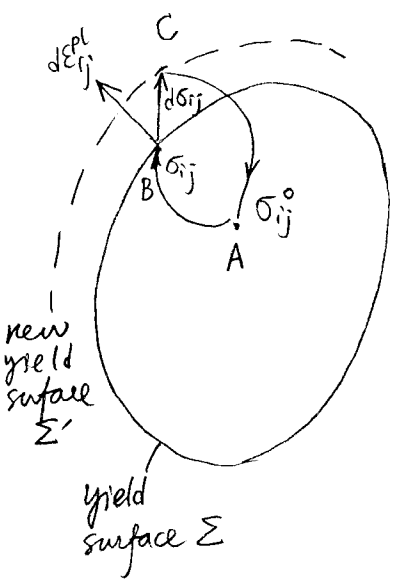
§5. Drucker's Postulate (Kachanov P86)

Consider a hardening medium with original stress σ_{ij}°, on which we apply additional stress $d\sigma_{ij}$ and then remove it.

The loading is slow enough to be considered isothermal.

Drucker's Postulate:

1. In the loading process, additional stress $d\sigma_{ij}$ does positive work
2. For a complete cycle of loading and unloading, additional stress does positive work if plastic deformation takes place.
   (For a hardening material, the work will be zero only for purely elastic deformation.)



Consider loading path $A \to B \to C$
unloading path $C \to A$

Drucker's Postulate (2): $\oint (\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij} > 0$

Given that work done on elastic strains over the complete cycle is zero:

$$\oint (\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij}^{el} = 0$$

We have

$$\oint (\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij}^{pl} = 0$$

But plastic strains only occurs during $B \to C$

$$\therefore \quad \underline{(\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij}^{pl} > 0} \qquad \text{(local maximum principle)}$$

This is true for any σ_{ij}° inside the yield surface Σ.

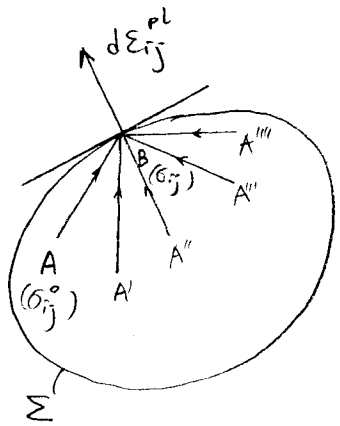
Hence it is a pretty strong requirement.

If we let the point A coincide with point B, we have

during $B \to C$: $d\sigma_{ij}\, d\varepsilon_{ij} > 0$ Drucker's postulate (1)

considering the cycle $B \to C \to B$: $d\sigma_{ij}\, d\varepsilon_{ij}^{pl} > 0$ Drucker's postulate (2)

Associative Flow as a consequence



Assume yield surface is <u>convex</u>

consider point B (σ_{ij}) on the yield surface.

Note the requirement

$$(\sigma_{ij} - \sigma_{ij}^{o})\, d\varepsilon_{ij}^{pl} > 0$$

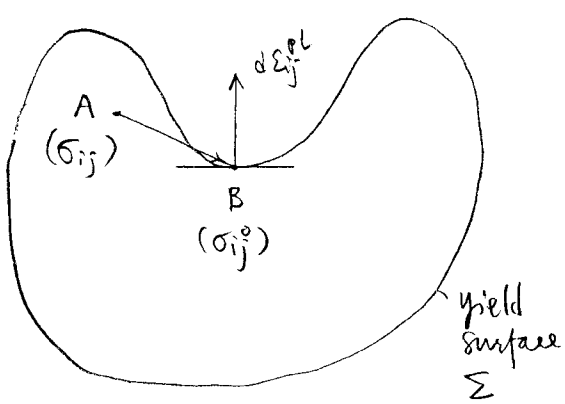
for all σ_{ij}^{o} inside Σ, i.e. for all points A, A', A'', A''', ...

This is only possible if $d\varepsilon_{ij}^{pl}$ is normal to the yield surface Σ at point B.

$\Longrightarrow$ Associative flow !

Hence we see that Drucker's postulate implies associative flow rule ($g = f$).

Convexity of yield Surface



If the yield surface Σ is not convex, then we can always find a point $A(\sigma_{ij})$ such that

$$(\sigma_{ij} - \sigma_{ij}^{o})\, d\varepsilon_{ij}^{pl} < 0$$

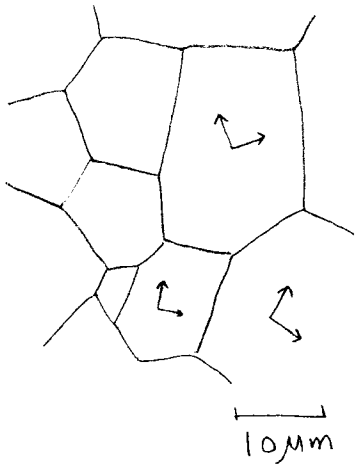
violating Drucker's postulate

(physically, it means that work is extracted in a complete cycle $A \to B \to C \to A$ that produces plastic deformation, which is possible if there is internal stress in the material.)

For perfect plasticity (no hardening), Drucker's postulate is modified to: $d\sigma_{ij}\, d\varepsilon_{ij}^{p} = 0$.

(Valid for every loading process $B \to C$ and for every cycle $B \to C \to B$).

§1. Poly Crystal



A bulk metal is a polycrystal consisting of many single crystal grains (on the order of 1–10μm size) separated by grain boundaries

Each single crystal grain is elastically and plastically <u>anisotropic</u>.

But the aggregate of many grains appears to behave <u>isotropically</u> at larger length scale.

If we want to "derive" (or understand) polycrystal plasticity, a "standard" (or reasonable) approach is to first understand single crystal plasticity and then perform the process of "homogenization".

This is assuming that the grain boundaries (GB) do not play an active role in plastic deformation, other than providing a constraint on plastic strain of neighboring grains. (G.I. Taylor. 1938)

This assumption is confirmed by experiments at ambient conditions.

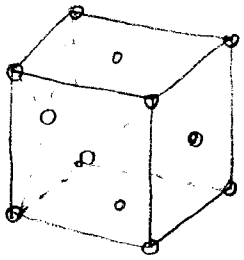
At very high temperatures, GB sliding and GB diffusion contributes significantly to plastic deformation (creep).

§2. Single Crystal Slip Systems

most engineering materials (metals, semiconductors) have cubic crystal structure, with either

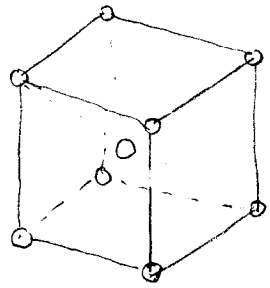
a face-centered cubic (FCC) lattice or

a body-centered cubic (BCC) lattice



FCC lattice

Cu, Al, Au



BCC lattice

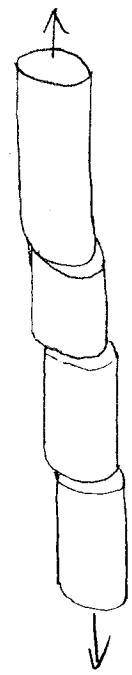
Fe, W, Mo

If we put one (and the same type) atom on each lattice site, we get an FCC or BCC crystal.

But we can also put multiple atoms at each lattice site to get different crystal structures.

For example, the diamond cubic structure (of diamond!) has 2 carbon atoms for each FCC lattice site.

Here we will focus on simple crystal structures with only one (and the same type) atom at each lattice site.

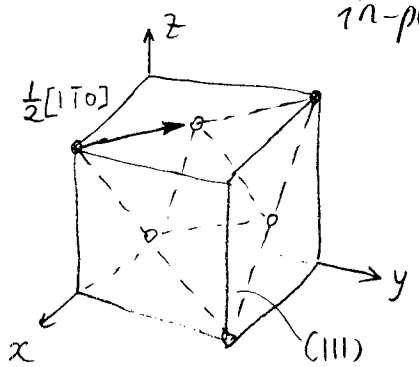


When a single crystal is stretched, it is known to slip on crystallographic planes along specific directions

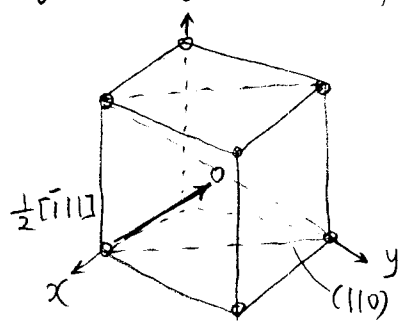
(Micro tensile SEM movie by M. Uchic)

The slip directions are the shortest lattice repeat vectors.

The slip planes are the planes with highest in-plane density (largest inter-plane distance)



FCC



In FCC metal, the dominant slip system is $\{111\}$ planes $\frac{1}{2}\langle 110\rangle$ direction

$\{111\}\langle 110\rangle$ slip for short

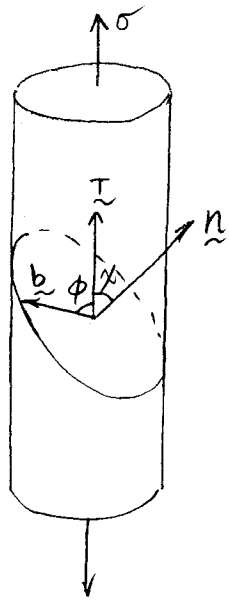
In BCC metal, the dominant slip system is $\{110\}$ planes $\frac{1}{2}\langle 111\rangle$ direction

$\{110\}\langle 111\rangle$ for short

(Slip on $\{112\}$, $\{123\}$ planes have also be- observed in BCC metals)

<u>Q: How many slip systems</u> in FCC and BCC lattices?

§3. Schmid factor



Consider a uniaxial tensile test of a single crystal

$\underset{\sim}{I}$: unit vector along tensile axis

$\underset{\sim}{n}$: unit vector of slip plane normal

$\underset{\sim}{b}$: unit vector along slip direction

σ : tensile stress

χ : angle between $\underset{\sim}{I}$ and $\underset{\sim}{n}$

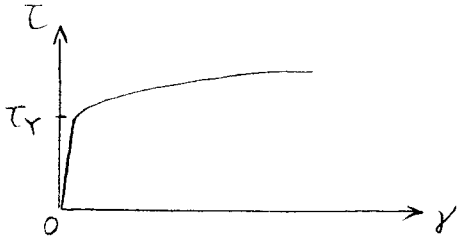
$$\cos\chi = (\underset{\sim}{I} \cdot \underset{\sim}{n})$$

ϕ : angle between $\underset{\sim}{I}$ and $\underset{\sim}{b}$

$$\cos\phi = (\underset{\sim}{I} \cdot \underset{\sim}{b})$$

Let τ be the resolved shear stress on plane $\underset{\sim}{n}$ along $\underset{\sim}{b}$

γ be the shear strain on plane $\underset{\sim}{n}$ along $\underset{\sim}{b}$



$$\tau = S \cdot \sigma$$

$S = \cos\chi \cos\phi$: Schmid factor

$(S < 1)$

$$\sigma_Y = \frac{\tau_Y}{S}$$

S depends on the relative orientation of the loading axis w.r.t. slip system

Hence we expect σ_Y of single crystal to be orientation dependent (but not depend on tension v.s. compression)

This is well obeyed by FCC (&HCP) metals. BCC metals do show tension/compression asymmetry.

§4 Schmid Law

There are multiple slip systems in a single crystal.

It is reasonable to expect that

For single crystals under uniaxial stress, the slip system with the highest Schmid factor is activated first and remain the most active — Schmid Law

However, for certain (high symmetry) orientations, two or more slip systems have identical Schmid factor, so they will be simultaneously activated.

Hence the tensile axis can be categorized as <u>single-slip</u> or <u>multi-slip</u>.

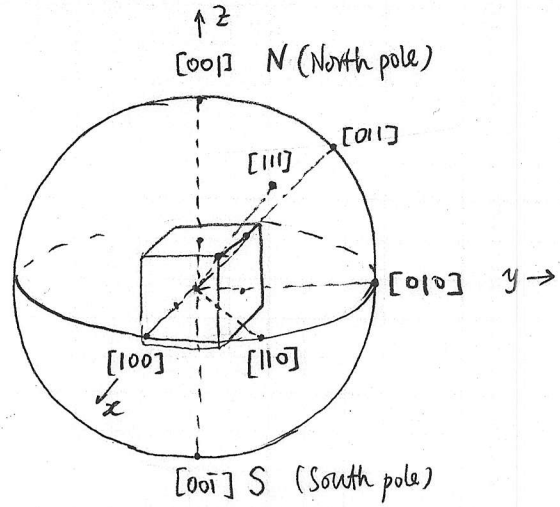
In a polycrystal, G.I. Taylor show that in general a crystal grain must activate 5 slip systems to be able to acommodate the strain constraints imposed by its neighbors.

<u>Q. Why five?</u>

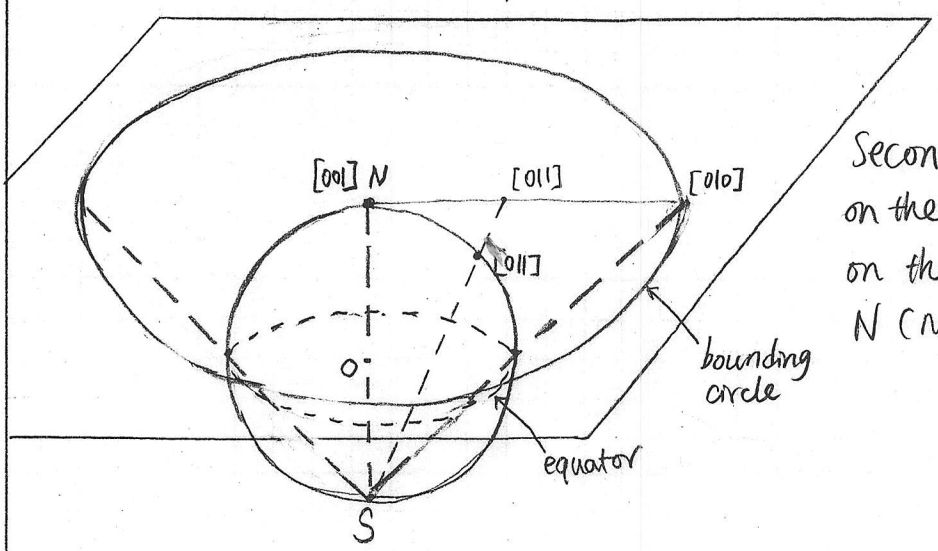
G.I. Taylor, J. Inst. Metals 62, 307 (1938)
May Lecture, 1938
"Plastic Strain in Metals"
on Courseawk/Materials/Extra Reading.

§5. Stereographic Projection

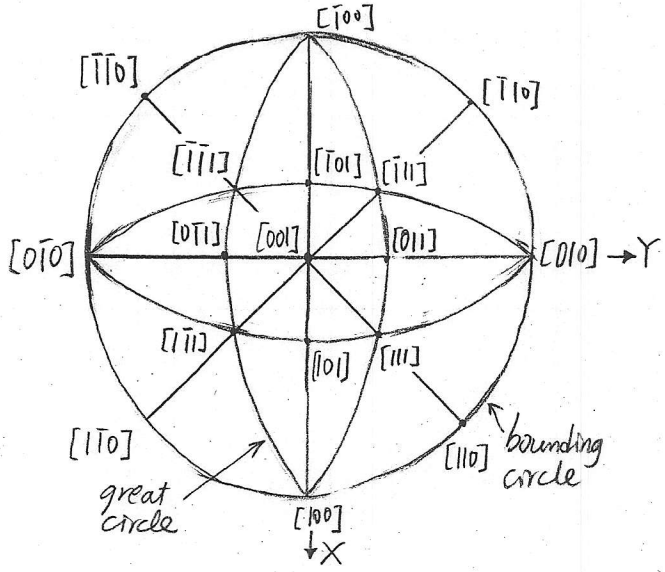
Stereographic Projection is a tool to map all directions (on a unit sphere) on to a flat plane to facilitate discussions. The mapping preserves angles but do not preserve area or length.



First, project all directions of the crystal onto the unit sphere.



Second, project every point on the top hemi-sphere on the plane tangent at N (north-pole)



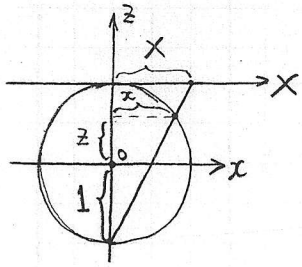
Third, look down at the projection plane.

(x, y, z) On the sphere →	(X, Y) on the plane
circles	circles
equator	bounding circle
Northern hemisphere	points inside bounding circle
great circles	circles intersecting bounding circle at opposite end of diameter

Points on unit sphere (x, y, z), $x^2+y^2+z^2=1$,

Northern hemisphere $z \geq 0$, $z=\sqrt{1-x^2-y^2}$

Points on projection plane $(X, Y) = \left(\frac{2x}{1+z}, \frac{2y}{1+z}\right)$



$$X^2+Y^2 = \frac{4(x^2+y^2)}{(1+z)^2} = \frac{4(1-z^2)}{(1+z)^2} = \frac{4(1-z)}{1+z}$$

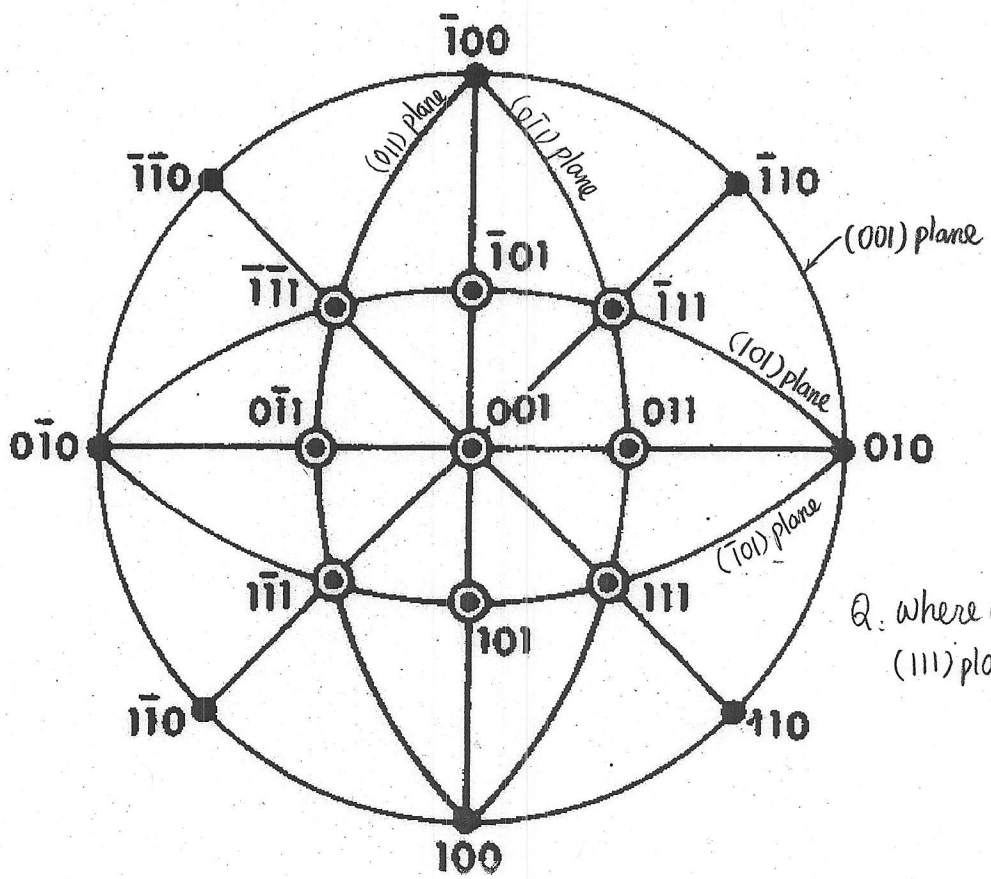
$$1+\frac{X^2}{4}+\frac{Y^2}{4} = \frac{2}{1+z}$$

Reverse mapping:

$$x = \frac{X}{1+\frac{X^2}{4}+\frac{Y^2}{4}}, \quad y = \frac{Y}{1+\frac{X^2}{4}+\frac{Y^2}{4}}, \quad z = \frac{2}{1+\frac{X^2}{4}+\frac{Y^2}{4}} - 1$$

- Bounding circle is the smallest great circle on the projection plane.

- Each great circle represent a plane (passing through O) the normal of this plane can be obtained by taking cross product of any two vectors in this plane



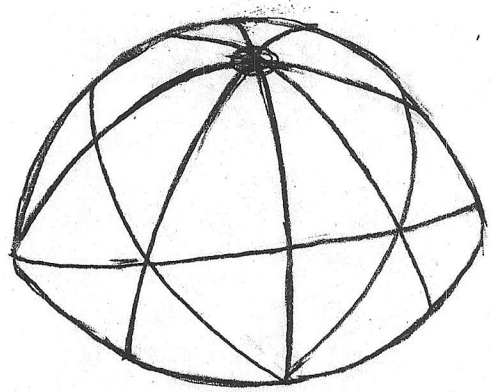
Q. where is (111) plane?

§6. Stereographic Triangle

Notice that the area inside the bounding circle is divided into 24 triangles (with curved sides)

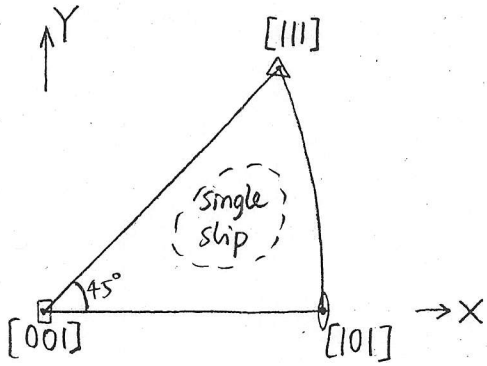
every triangle has a ⟨001⟩ node, a ⟨011⟩ node, and a ⟨111⟩ node.

every triangle covers the same area on the unit sphere (they don't have the same area on the projection plane)



Every triangle is equivalent to each other by symmetry (cubic symmetry of the lattice)

Hence, to examine the orientation dependence of tensile tests, we only need to examine tensile axis within a triangle.



The standard Triangle

Q: find it in the bounding circle on the previous page

Note: [001] has 4-fold symmetry □
[101] has 2-fold symmetry ⬯
[111] has 3-fold symmetry △

In general, if the tensile axis lie close to the center of the standard triangle, then single slip is prefered.

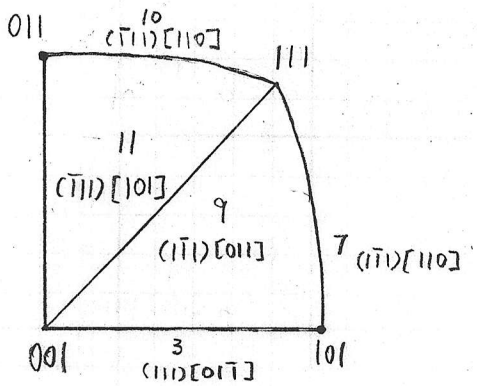
Multi-slip occurs if the tensile axis moves close to the edges of the triangle.

§7 Active slip System in FCC Crystal

Q: which slip system (out of 12) is activated when the tensile axis is inside the standard triangle?

Q: How close is the second highest Schmid factor to the highest?

Using Matlab code to analyze all 12 slip systems for each tensile axis in the standard triangle. (fcc_slip_system_map.m)



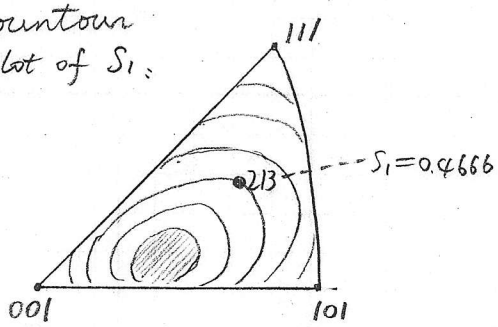
ID		slip system
1	(111)	$[1\ \bar{1}\ 0]$
2		$[1\ 0\ \bar{1}]$
3		$[0\ 1\ \bar{1}]$
4	$(11\bar{1})$	$[1\ \bar{1}\ 0]$
5		$[1\ 0\ 1]$
6		$[0\ 1\ 1]$
7	$(1\bar{1}1)$	$[1\ 1\ 0]$
8		$[1\ 0\ \bar{1}]$
9		$[0\ 1\ 1]$
10	$(\bar{1}11)$	$[1\ 1\ 0]$
11		$[1\ 0\ 1]$
12		$[0\ 1\ \bar{1}]$

Define:

S_1: highest Schmid factor

S_2: second highest Schmid factor

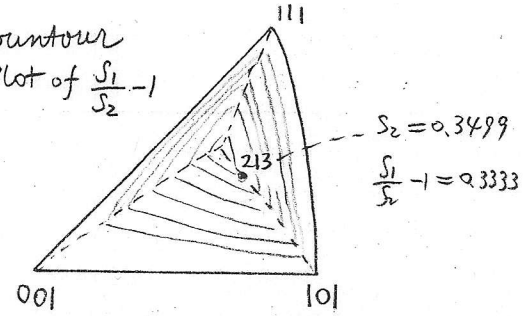
Countour Plot of S_1:



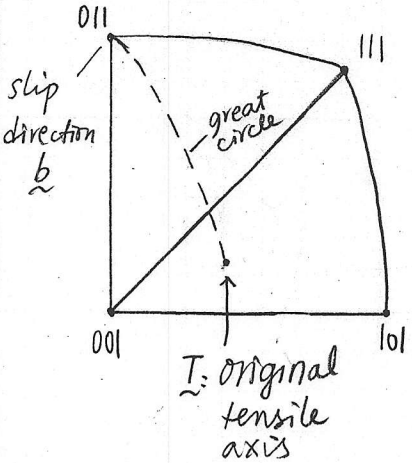
Q: Is the slip system of the second highest Schmid factor always the same in the Standard triangle?

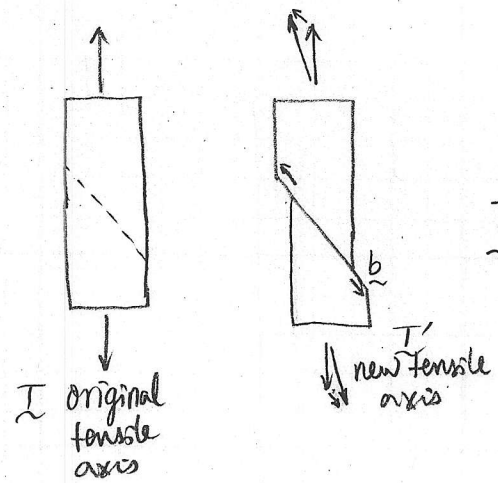
A: _____

Countour Plot of $\frac{S_1}{S_2} - 1$



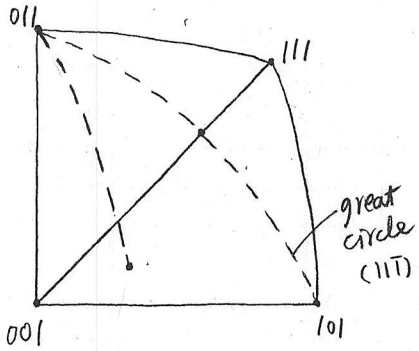
§8. Rotation of Crystal Axes for Single Crystal





$$\underset{\sim}{T}' = \underset{\sim}{T} + \alpha \underset{\sim}{b}$$

Tensile axis rotate along a great circle toward the direction of slip $\underset{\sim}{b}$

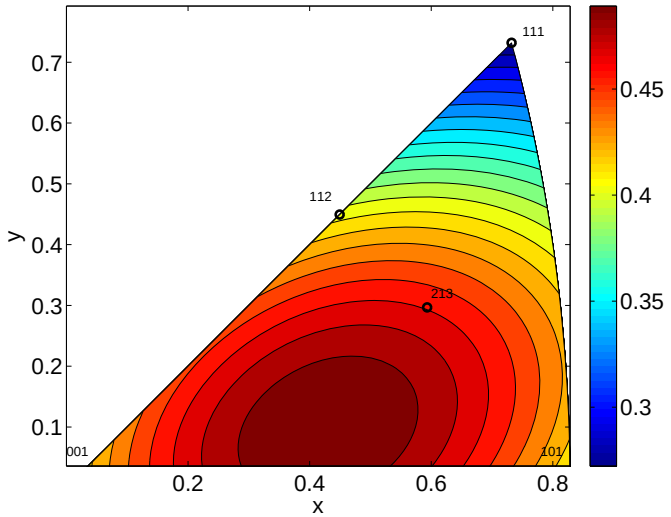


Q: Will the tensile axis leave the standard triangle?

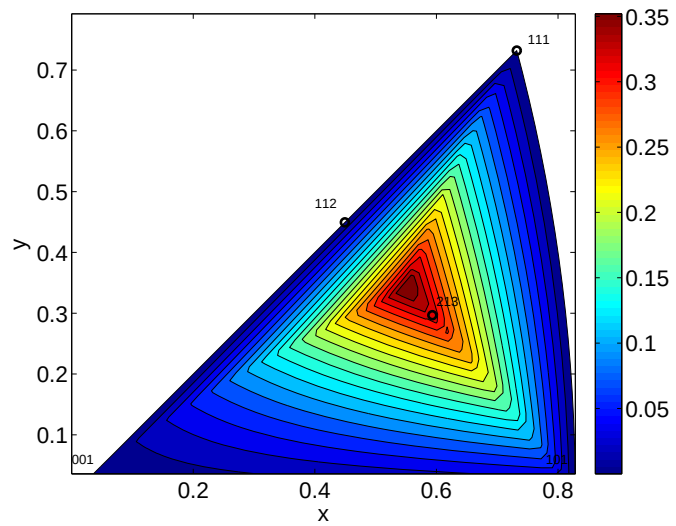
A: _____

Schmid Factor in Single Crystal

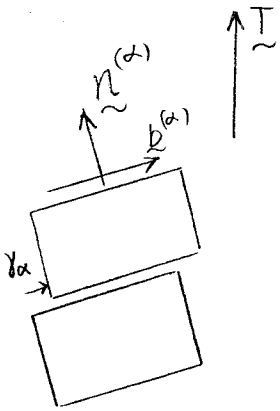
highest Schmid factor S_1



$S_1 / S_2 - 1$



§9. Taylor Factor for Polycrystal



Consider slip system α with

slip plane normal vector $\underset{\sim}{n}^{(\alpha)}$

slip direction $\underset{\sim}{b}^{(\alpha)}$

both $\underset{\sim}{n}^{(\alpha)}$ and $\underset{\sim}{b}^{(\alpha)}$ are unit vectors

Suppose the slip system experience shear γ_α

$$\gamma_\alpha = \frac{\text{slip distance} * \text{slipped area}}{\text{volume of crystal}}$$

Then the (shear) strain produced by slip system α is

$$\underset{\approx}{\varepsilon}^{(\alpha)} = \frac{\gamma_\alpha}{2}\left(\underset{\sim}{n}^{(\alpha)} \otimes \underset{\sim}{b}^{(\alpha)} + \underset{\sim}{b}^{(\alpha)} \otimes \underset{\sim}{n}^{(\alpha)}\right)$$

dyadic product

in component form, $\varepsilon_{ij}^{(\alpha)} = \frac{\gamma_\alpha}{2}\left(n_i^{(\alpha)} b_j^{(\alpha)} + n_j^{(\alpha)} b_i^{(\alpha)}\right)$

The slip also produces a rotation of the tensile axis $\underset{\sim}{T}$

$$\underset{\sim}{\omega}^{(\alpha)} = \frac{\gamma_\alpha}{2}\left(\underset{\sim}{n}^{(\alpha)} \times \underset{\sim}{b}^{(\alpha)}\right)$$

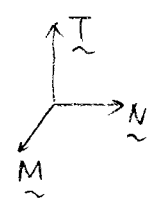
cross product

$\underset{\sim}{\omega}^{(\alpha)}$ points in the axis of rotation

$|\underset{\sim}{\omega}^{(\alpha)}|$ is the rotation angle (rad)

Taylor assumes that every grain experience tensile strain ε^t along $\underset{\sim}{T}$ and compressive strain $0.5\,\varepsilon^t$ in perpendicular directions.

$$\underset{\approx}{\varepsilon} = (\underset{\sim}{T} \otimes \underset{\sim}{T})\,\varepsilon^t - \frac{1}{2}(\underset{\sim}{M} \otimes \underset{\sim}{M} + \underset{\sim}{N} \otimes \underset{\sim}{N})\,\varepsilon^t$$



note $\underset{\sim}{T} \otimes \underset{\sim}{T} + \underset{\sim}{M} \otimes \underset{\sim}{M} + \underset{\sim}{N} \otimes \underset{\sim}{N} = \underset{\approx}{I}$ identity

$$\therefore \quad \underset{\approx}{\varepsilon} = (\underset{\sim}{T} \otimes \underset{\sim}{T})\,\varepsilon^t - \left(\underset{\approx}{I} - \underset{\sim}{T} \otimes \underset{\sim}{T}\right)\frac{\varepsilon^t}{2}$$

$$= \frac{3}{2}(\underset{\sim}{T} \otimes \underset{\sim}{T})\,\varepsilon^t - \frac{1}{2}\underset{\approx}{I}\,\varepsilon^t$$

in component form $\varepsilon_{ij} = \frac{\varepsilon^t}{2}(3 T_i T_j - \delta_{ij})$

For a randomly chosen grain, $\underset{\sim}{T}$ is arbitrarily oriented relative to the crystallographic axis. Hence the imposed strain $\underset{\approx}{\varepsilon}$ is also quite arbitrary.

$\underset{\approx}{\varepsilon}$ must be accommodated by a number of slip systems.

For FCC crystal, there are 12 slip systems.

$$\varepsilon_{ij} = \sum_{\alpha=1}^{12} \varepsilon_{ij}^{(\alpha)} = \sum_{\alpha=1}^{12} \gamma_\alpha \cdot \frac{1}{2}\left(n_i^{(\alpha)} b_j^{(\alpha)} + n_j^{(\alpha)} b_i^{(\alpha)}\right)$$

Taylor showed that in general 5 non-zero γ_α's are necessary (and sufficient) to satisfy the imposed ε_{ij}

But the solution to the above equation is not unique.

Taylor argued we should choose γ_α's such that $\sum_\alpha |\gamma_\alpha|$ is minimized.

$$M \equiv \min_{\{\gamma_\alpha\}} \sum_\alpha \frac{|\gamma_\alpha|}{\varepsilon^t}$$

The averaged M over all $\underset{\sim}{T}$ is now called the Taylor factor.

Taylor factor relates the critical resolved shear stress τ_c on each slip system with the flow stress σ_f of polycrystals (with random grain orientation)

$$\sigma_f = M \cdot \tau_c$$

For FCC crystal with $\{111\}\langle 110\rangle$ slip systems, $M = 3.067$ (see below)

For BCC crystal with $\{110\}\langle 111\rangle$ slip systems, the result for Taylor factor is identical. why?

§10. Solving Taylor's Problem by Linear Programming

Chin and Mammel, Trans. Metall. Soc. AIME 239, 1400-1405 (1967)

Taylor's problem is a standard problem in Linear Programming (LP)

The Matlab function linprog is able to solve the following LP problem

$$\min_{x} \quad f^T \cdot x \quad \text{such that} \quad \begin{cases} A \cdot x \leq b \\ A_{eq} \cdot x = b_{eq} \\ lb \leq x \leq ub \end{cases}$$

(lower bound) (upper bound)

$\left(\begin{array}{l} x, f, b, lb, ub \text{ vectors} \\ A, A_{eq} \text{ matrices} \end{array} \right)$

The absolute value ($|\gamma_\alpha|$) in Taylor's problem can be easily accounted for in LP by introducing more variables.

Here, we will not use inequality constraints ($A x \leq b$)
so we will set $A = [\,]$; $b = [\,]$; (empty array)

Define $x = (x_1 \cdots x_{12}, x_{13}, \cdots x_{24})^T$ such that

$$x_\alpha = \max\left\{ \frac{\gamma_\alpha}{\varepsilon^t}, 0 \right\}$$

$$\alpha = 1, 2, \cdots 12$$

$$x_{\alpha+12} = \max\left\{ -\frac{\gamma_\alpha}{\varepsilon^t}, 0 \right\}$$

So that $x_\alpha \geq 0$ for all $\alpha = 1, 2, \cdots 24$

Hence $lb = \{\underbrace{0, 0, \cdots}_{24 \text{ elements}}\}$ $\quad ub = [\,]$;

Define $\hat{\varepsilon}_{ij}^{(\alpha)} = \frac{1}{2}\left(n_i^{(\alpha)} b_j^{(\alpha)} + n_j^{(\alpha)} b_j^{(\alpha)}\right)$

$$\hat{\varepsilon}_{ij} = \frac{1}{2}\left(3\, T_i T_j - \delta_{ij}\right)$$

$\therefore$ γ_α satisfy the constraint

$$\hat{\varepsilon}_{ij} = \sum_{\alpha=1}^{12} \gamma_\alpha \hat{\varepsilon}_{ij}^{(\alpha)}$$

Use Voigt notation to convert strain tensors into vectors.

$$\hat{\varepsilon}_I = (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3, \hat{\varepsilon}_4, \hat{\varepsilon}_5, \hat{\varepsilon}_6)^T, \quad \hat{\varepsilon}_1 = \hat{\varepsilon}_{11},\ \hat{\varepsilon}_2 = \hat{\varepsilon}_{22},\ \hat{\varepsilon}_3 = \hat{\varepsilon}_{33},\ \hat{\varepsilon}_4 = \hat{\varepsilon}_{23},\ \hat{\varepsilon}_5 = \hat{\varepsilon}_{31},\ \hat{\varepsilon}_6 = \hat{\varepsilon}_{12}$$

$$\hat{\varepsilon}_I = \sum_{\alpha=1}^{12} \gamma_\alpha \hat{\varepsilon}_I^{(\alpha)} = \sum_{\alpha=1}^{12} x_\alpha \hat{\varepsilon}_I^{(\alpha)} - x_{\alpha+12} \hat{\varepsilon}_I^{(\alpha)}$$

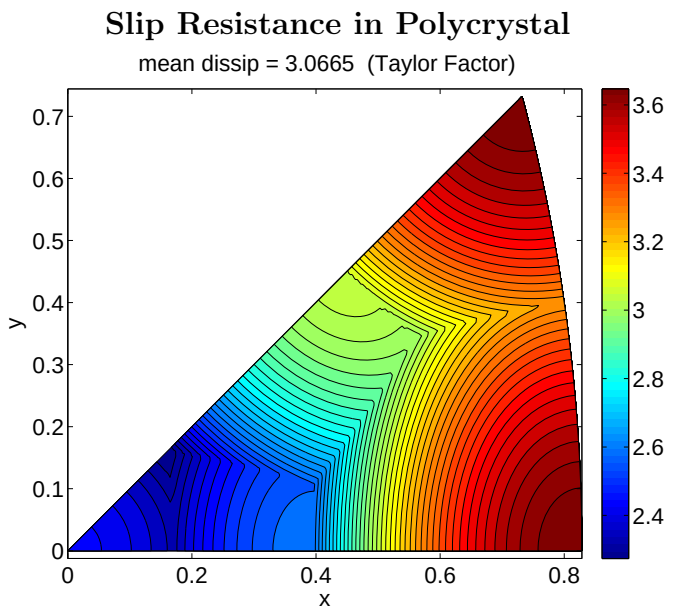
This can be written in the form of $A_{eq} \cdot x = b_{eq}$ where

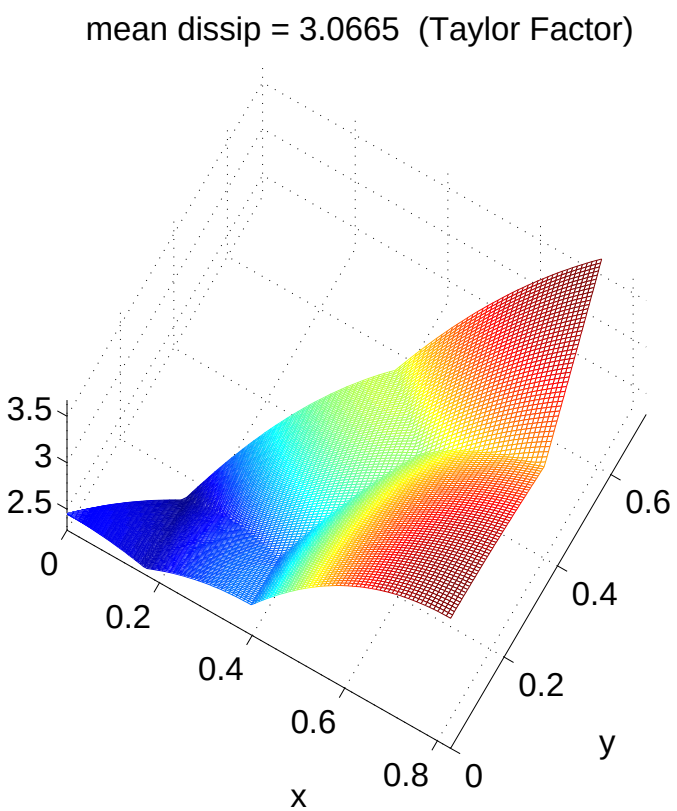
$$A_{eq} = \left(\begin{array}{cccc|cccc} \varepsilon_1^{(1)} & \varepsilon_1^{(2)} & \cdots & \varepsilon_1^{(12)} & -\varepsilon_1^{(1)} & -\varepsilon_1^{(2)} & \cdots & -\varepsilon_1^{(12)} \\ \varepsilon_2^{(1)} & \varepsilon_2^{(2)} & \cdots & \varepsilon_2^{(12)} & -\varepsilon_2^{(1)} & -\varepsilon_2^{(2)} & \cdots & -\varepsilon_2^{(12)} \\ & & & & & & & \\ \varepsilon_6^{(1)} & \varepsilon_6^{(2)} & \cdots & \varepsilon_6^{(12)} & -\varepsilon_6^{(1)} & -\varepsilon_6^{(2)} & \cdots & -\varepsilon_6^{(12)} \end{array}\right) \qquad b_{eq} = \begin{pmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \vdots \\ \hat{\varepsilon}_6 \end{pmatrix}$$

12 slip systems | negative of the 12 slip systems

See fcc-slip-system_polycrystal.m

M is computed for each tensile axis ($\underset{\sim}{I}$) in the standard triangle. (see contour plot on next page) M is largest (~3.6) near [101] and [111] (hard grains) and smallest (~2.5) near [001] (soft grains). The average M is 3.067 (Taylor factor).


Slip Resistance in Polycrystal
mean dissip = 3.0665 (Taylor Factor)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
y
0
0.2
0.4
0.6
0.8
x
3.6
3.4
3.2
3
2.8
2.6
2.4


mean dissip = 3.0665 (Taylor Factor)
3.5
3
2.5
0
0.2
0.4
0.6
0.8
x
0
0.2
0.4
0.6
y

§10. Grain Rotation in Polycrystal.

After x_α is solved by LP (linprog in Matlab)

we find $\gamma_\alpha = (x_\alpha - x_{\alpha+12})\,\varepsilon^t \quad \alpha = 1, 2, \cdots 12$

Total rotation of Tensile axis is

$$\underset{\sim}{\omega} = \sum_{\alpha=1}^{12} \frac{\gamma_\alpha}{2} \left(\underset{\sim}{n}^{(\alpha)} \times \underset{\sim}{b}^{(\alpha)} \right)$$

which is different for each $\underset{\sim}{T}$ in the standard triangle.

The direction of $\underset{\sim}{T}$ rotation can be visualized as arrows in the standard triangle.

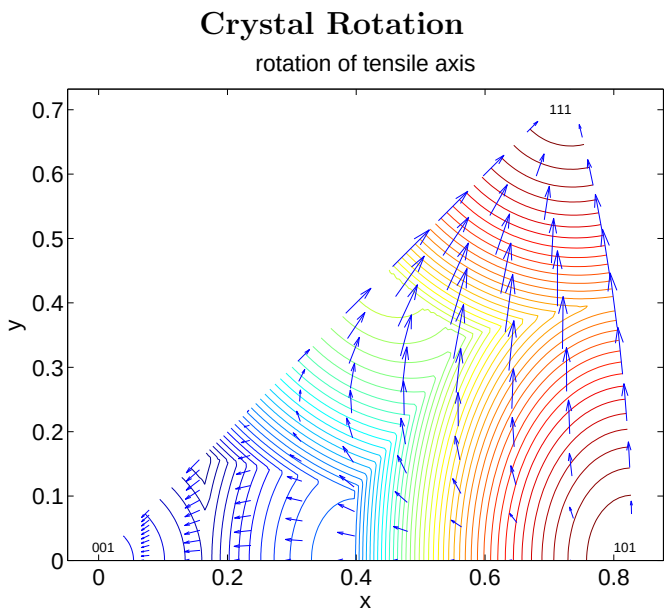
In tension, grains rotate so that their {111} or {100} orientations become aligned with the tensile axis.

Their {110} orientations move away from the tensile axis.

The opposite is true for compression

Hence, polycrystal with initial random grain orientations will develop preferred grain orientation (i.e. texture formation) after severe plastic deformation (extrusion or rolling)

Results discussed here are consistent with Taylor, G. I. J. Inst Metals 62, 307, 1938 "Plastic strain in metals".


Crystal Rotation
rotation of tensile axis
111
001
101
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
y
0
0.2
0.4
0.6
0.8
x

```
% FCC slip systems for single crystal under uniaxial tension
%                 plane     slip vector
slip_system = [  1  1  1    1 -1  0
                 1  1  1    1  0 -1
                 1  1  1    0  1 -1
                 1  1 -1    1 -1  0
                 1  1 -1    1  0  1
                 1  1 -1    0  1  1
                 1 -1  1    1  1  0
                 1 -1  1    1  0 -1
                 1 -1  1    0 -1 -1
                -1  1  1   -1 -1  0
                -1  1  1   -1  0 -1
                -1  1  1    0  1 -1 ];
N_slip_system = length(slip_system(:,1));

%domain_type = 1; % a simple parameterization covering several triangles
domain_type = 2; % parameterization of the standard triangle
                 %  G. Y. Chin and W. L. Mammel,
                 %   Computer Solutions of the Taylor Analysis for Axisymmetric Flow
                 %   Trans. Metall. Soc. AIME 239, 1400-1405 (1967).


if domain_type == 1
    theta = linspace(1e-4,1-1e-4,50)*acos(1/sqrt(3));
    phi   = linspace(-0.2+1e-4,1-1e-4,50)*(pi/2);
elseif domain_type == 2
    theta = linspace(1e-4,1-1e-4,50)*(pi/4);
    phi   = linspace(1e-4,1-1e-4,50)*(pi/4);
else
    disp('unknown domain_type');
end

X = zeros(length(theta), length(phi));
Y = X; x = X; y = X; z = X;
maxS = X;  maxSind = X;
secondS = X;  secondSind = X;

T_select = [ 0 0 1
             1 0 1
             1 1 1
             0 1 1
             1 1 2
             2 1 3 ];

for ai = 1:length(theta),
    for bi = 1:length(phi),
        if domain_type == 1
            x(ai,bi) = sin(theta(ai))*cos(phi(bi));
            y(ai,bi) = sin(theta(ai))*sin(phi(bi));
            z(ai,bi) = cos(theta(ai));
```

```
        elseif domain_type == 2
            factor = atan(sin(theta(ai)))/(pi/4);
            x(ai,bi) = sin(theta(ai))*cos(phi(bi)*factor);
            y(ai,bi) =                sin(phi(bi)*factor);
            z(ai,bi) = cos(theta(ai))*cos(phi(bi)*factor);
        end
        X(ai,bi) = 2*x(ai,bi)/(1+z(ai,bi));
        Y(ai,bi) = 2*y(ai,bi)/(1+z(ai,bi));
        T = [x(ai,bi) y(ai,bi) z(ai,bi)];

        S = zeros(N_slip_system,1);
        for i = 1:N_slip_system
            n = slip_system(i,1:3);
            b = slip_system(i,4:6);
            S(i) = abs(dot(T,n)/norm(T)/norm(n) * dot(T,b)/norm(T)/norm(b));
        end
        [sortedS ind] = sort(S,1,'descend');
        maxS(ai,bi) = sortedS(1);    maxSind(ai,bi) = ind(1);
        secondS(ai,bi) = sortedS(2); secondSind(ai,bi) = ind(2);
    end
end

X_select = zeros(length(T_select(:,1)),1);
Y_select = X_select;
maxS_select = X_select;
maxSind_select = X_select;
secondS_select = X_select;
secondSind_select = X_select;

for si=1:length(X_select),
    T = T_select(si,:); T = T / norm(T);
    X_select(si) = 2*T(1)/(1+T(3));
    Y_select(si) = 2*T(2)/(1+T(3));
    S = zeros(N_slip_system,1);
    for i = 1:N_slip_system,
        n = slip_system(i,1:3);
        b = slip_system(i,4:6);
        S(i) = abs(dot(T,n)/norm(T)/norm(n) * dot(T,b)/norm(T)/norm(b));
    end
    [sortedS ind] = sort(S,1,'descend');
    maxS_select(si) = sortedS(1);    maxSind_select(si) = ind(1);
    secondS_select(si) = sortedS(2); secondSind_select(si) = ind(2);
end

%plot results
figure(1);
contourf(X,Y,maxS,20);
hold on
plot3(X_select,Y_select,ones(size(X_select)),'ko','LineWidth',2);
hold off
xlabel('x');
```

```
ylabel('y');
axis equal
title('highest Schmid factor');

figure(2);
mesh(X,Y,maxSind);
hold on
plot3(X_select,Y_select,13*ones(size(X_select)),'ko','LineWidth',2);
hold off
view([0 90]);
xlabel('x');
ylabel('y');
axis equal
title('slip system ID with highest S');

figure(3);
mesh(X,Y,secondSind);
hold on
plot3(X_select,Y_select,13*ones(size(X_select)),'ko','LineWidth',2);
hold off
view([0 90]);
xlabel('x');
ylabel('y');
axis equal
title('slip system ID with second highest S');

figure(4);
contourf(X,Y,maxS./secondS-1,20);
colorbar
hold on
plot3(X_select,Y_select,ones(size(X_select)),'ko','LineWidth',2);
hold off
xlabel('x');
ylabel('y');
axis equal
title('S_2/S_1 - 1');
```

```
% FCC slip systems for poly-crystal under uniaxial tension
% Find the 5 slip systems activated for each tensile axis
% following Taylor 1938
%  G. Y. Chin and W. L. Mammel,
%   Computer Solutions of the Taylor Analysis for Axisymmetric Flow
%   Trans. Metall. Soc. AIME 239, 1400-1405 (1967).

%                plane     slip vector
slip_system = [  1  1  1    1 -1  0
                 1  1  1    1  0 -1
                 1  1  1    0  1 -1
                 1  1 -1    1 -1  0
                 1  1 -1    1  0  1
                 1  1 -1    0  1  1
                 1 -1  1    1  1  0
                 1 -1  1    1  0 -1
                 1 -1  1    0 -1 -1
                -1  1  1   -1 -1  0
                -1  1  1   -1  0 -1
                -1  1  1    0  1 -1 ];
N_slip_system = length(slip_system(:,1));

theta = [0:0.01:1]*(pi/4);
phi   = [0:0.01:1]*(pi/4);
dtheta = theta(2)-theta(1);
dphi   = phi(2)-phi(1);
Ntrial = 3;
dTfactor = 1e-4;
X = zeros(length(theta), length(phi));
Y = X; x = X; y = X; z = X;
activeS = zeros(length(theta), length(phi), N_slip_system);
dissip   = zeros(length(theta), length(phi));  weight = dissip;
Taxis   = zeros(length(theta), length(phi), 3); netrot = Taxis; dTaxis = Taxis;
Taxisnew = Taxis; Xnew = X; Ynew = Y; dX = X; dY = Y;

strain = zeros(N_slip_system,6);
rotation = zeros(N_slip_system,3);
for i = 1:length(slip_system(:,1)),
    n = slip_system(i,1:3); n = n/norm(n);
    b = slip_system(i,4:6); b = b/norm(b);
    tmp = (n'*b + b'*n)/2;
    strain(i,:) = [tmp(1,1) tmp(2,2) tmp(3,3) tmp(2,3) tmp(3,1) tmp(1,2)];
    rotation(i,:) = cross(n,b)/2;
end

% compute participation ratio of each slip system
for ai = 1:length(theta),
    if mod(ai,10)==0
        disp(sprintf('ai = %d / %d ', ai, length(theta)));
    end
    for bi = 1:length(phi),
```

```
        factor = atan(sin(theta(ai)))/(pi/4);
        x(ai,bi) = sin(theta(ai))*cos(phi(bi)*factor);
        y(ai,bi) =                sin(phi(bi)*factor);
        z(ai,bi) = cos(theta(ai))*cos(phi(bi)*factor);
        X(ai,bi) = 2*x(ai,bi)/(1+z(ai,bi));
        Y(ai,bi) = 2*y(ai,bi)/(1+z(ai,bi));
        T = [x(ai,bi) y(ai,bi) z(ai,bi)];
        Taxis(ai,bi,:) = T;
        tmp = (T'*T - (T*T'/3)*eye(3))*(3/2);
        tensile_strain = [tmp(1,1) tmp(2,2) tmp(3,3) tmp(2,3) tmp(3,1) tmp(1,2)];
        %options = optimset('Display','iter','TolFun',1e-8);
        options = optimset('Display','off','TolFun',1e-8);
        % double # of slip systems so that coeff is non-negative
        Aeq = [strain', -strain'];
        beq = [tensile_strain'];
        lb = zeros(length(slip_system(:,1))*2,1);
        f= ones(length(slip_system(:,1))*2,1);
        coeff = linprog(f, [], [], Aeq, beq, lb, [], [], options);
        % half # of coeff to store as positive and negative values
        compact_coeff = coeff(1:end/2)-coeff(end/2+1:end);
        activeS(ai,bi,:) = compact_coeff;
        dissip(ai,bi) = sum(abs(compact_coeff));
        netrot(ai,bi,:) = rotation'*compact_coeff;
    end
end

% compute change of Taxis
for ai = 1:length(theta),
    for bi = 1:length(phi),
        dTaxis(ai,bi,:) = cross(netrot(ai,bi,:),Taxis(ai,bi,:));
        Taxisnew(ai,bi,:) = Taxis(ai,bi,:) + dTaxis(ai,bi,:)*dTfactor;
        Xnew(ai,bi) = 2*Taxisnew(ai,bi,1)/(1+Taxisnew(ai,bi,3));
        Ynew(ai,bi) = 2*Taxisnew(ai,bi,2)/(1+Taxisnew(ai,bi,3));
    end
end
dX = Xnew - X; dY = Ynew - Y;

% compute area of mesh
area = zeros(size(weight));
for ai = 1:length(theta)-1,
    for bi = 1:length(phi)-1,
        dr1 = [x(ai+1,bi) y(ai+1,bi) z(ai+1,bi)] - [x(ai,bi) y(ai,bi) z(ai,bi)];
        dr2 = [x(ai,bi+1) y(ai,bi+1) z(ai,bi+1)] - [x(ai,bi) y(ai,bi) z(ai,bi)];
        dr3 = [x(ai+1,bi) y(ai+1,bi) z(ai+1,bi)] ...
            - [x(ai+1,bi+1) y(ai+1,bi+1) z(ai+1,bi+1)];
        dr4 = [x(ai,bi+1) y(ai,bi+1) z(ai,bi+1)] ...
            - [x(ai+1,bi+1) y(ai+1,bi+1) z(ai+1,bi+1)];
        area(ai,bi) = (norm(cross(dr1,dr2))+norm(cross(dr3,dr4)))/2;
    end
end
```

```
% compute integration (quadrature) weight
for ai = 1:length(theta),
    for bi = 1:length(phi),
        weight(ai,bi) = area(ai,bi)/4;
        if ai > 1
            weight(ai,bi) = weight(ai,bi) + area(ai-1,bi)/4;
        end
        if bi > 1
            weight(ai,bi) = weight(ai,bi) + area(ai,bi-1)/4;
        end
        if ai > 1 && bi > 1
            weight(ai,bi) = weight(ai,bi) + area(ai-1,bi-1)/4;
        end
    end
end
mean_dissip = sum(sum(dissip.*weight))/(4*pi/48);

% plot results
fs = 17;
figure(1);
contourf(X,Y,dissip,50);
colorbar
set(gca,'FontSize',fs);
xlabel('x'); ylabel('y');
axis equal
title(sprintf('mean dissip = %.4f  (Taylor Factor)',mean_dissip));

figure(2);
mesh(X,Y,dissip);
set(gca,'FontSize',fs);
xlabel('x'); ylabel('y');
axis equal
view([30 80]);
title(sprintf('mean dissip = %.4f  (Taylor Factor)',mean_dissip));

figure(3);
skip = 10;
contour(X,Y,dissip,50); hold on
quiver(X(1:skip:end,1:skip:end), Y(1:skip:end,1:skip:end), ...
      dX(1:skip:end,1:skip:end),dY(1:skip:end,1:skip:end));
hold off
set(gca,'FontSize',fs);
xlabel('x'); ylabel('y');
axis equal
title('rotation of tensile axis');
t001 = text(-0.01,0.02,'001');
t101 = text( 0.80,0.02,'101');
t111 = text( 0.70,0.70,'111');
```