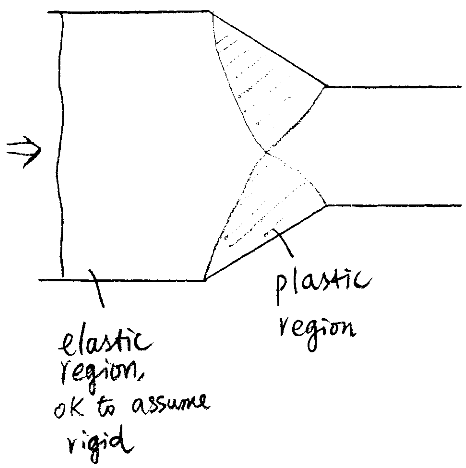
To perform theoretical analysis for general plane strain problems, we shall have to assume the plastic-rigid model for simplicity. This means we shall ignore elastic strain. This is a good approximation when plastic strain is large, e.g. in metal forming.



(extrusion)

We will introduce the slip line method to construct solutions to this class of problems.

Due to time constraints, we limit our scope to the interpretation of existing (classical) slip line solutions.

§1. Equations in Plane Strain Plastic-rigid model

plane strain: $u_z = 0$, $\varepsilon_{zz} = 0$

plastic-rigid: $\varepsilon^{el}_{ij} = 0$, $\xrightarrow{\text{implies}}$ incompressibility: $\nu = \frac{1}{2}$

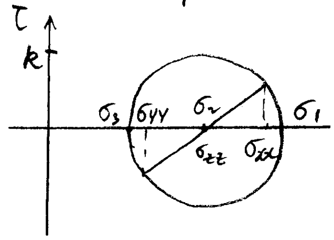
$$\therefore \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$$

non-zeros: $\begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{cases}$ $\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{zz} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \end{cases}$ $\begin{cases} v_x \\ v_y \end{cases}$ (velocity field) $\begin{cases} \dot{\varepsilon}_{xx} = v_{x,x} \\ \dot{\varepsilon}_{yy} = v_{y,y} \\ \dot{\varepsilon}_{xy} = \frac{1}{2}(v_{x,y} + v_{y,x}) \end{cases}$ (plastic) strain rate

In both elastic (rigid) and plastic regions, stress field satisfies <u>equilibrium</u>: (assume zero body force)

$$\begin{cases} \sigma_{xx,x} + \sigma_{yx,y} = 0 & \text{------- (1)} \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 & \text{------- (2)} \end{cases}$$

In the plastic region, the yield condition must be satisfied



note $\sigma_{zz} = \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) = \frac{1}{2}(\sigma_1+\sigma_3) = \sigma_2$

$\sigma_1 > \sigma_2 > \sigma_3$,

$$\bar{\sigma} = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \frac{1}{2}(\sigma_1+\sigma_3) = \sigma_2$$

$$\therefore \quad s_1 = \sigma_1 - \bar{\sigma} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

$$s_2 = \sigma_2 - \bar{\sigma} = 0$$

$$s_3 = \sigma_3 - \bar{\sigma} = -\frac{1}{2}(\sigma_1 - \sigma_3)$$

$$J_2 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) = \left[\frac{1}{2}(\sigma_1 - \sigma_3)\right]^2 = k^2$$

yield condition: $\sigma_1 - \sigma_3 = 2k$

Hence the <u>von Mises</u> yield condition <u>coincides</u> with the <u>Tresca</u> yield condition, due to incompressibility.

In terms of σ_{xx}, σ_{yy}, σ_{xy}, the <u>yield condition</u> is:

$$\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2 = k^2 \quad \text{----- (3)}$$

Because the elastic region is rigid, all strain rates are plastic strain rates, which follow the flow rule:

$$\dot{\varepsilon}_{xx} = \tilde{\lambda}\, s_{xx}$$
$$\dot{\varepsilon}_{yy} = \tilde{\lambda}\, s_{yy}$$
$$\dot{\varepsilon}_{xy} = \tilde{\lambda}\, s_{xy}$$

(notice we have changed $\frac{\tilde{\lambda}}{2\mu}$ to $\tilde{\lambda}$ since $\mu \to \infty$ here)

$$\bar{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_3) = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$$

$$S_{xx} = \sigma_{xx} - \bar{\sigma} = \frac{1}{2}(\sigma_{xx} - \sigma_{yy})$$

$$S_{yy} = \sigma_{yy} - \bar{\sigma} = \frac{1}{2}(\sigma_{yy} - \sigma_{xx})$$

$$S_{xy} = \sigma_{xy}$$

$\therefore$ <u>flow rule</u> can be written as

$$\dot{\varepsilon}_{xx} = \hat{\lambda}\ \frac{\sigma_{xx} - \sigma_{yy}}{2}$$

$$\dot{\varepsilon}_{yy} = \hat{\lambda}\ \frac{\sigma_{yy} - \sigma_{xx}}{2}$$

$$\dot{\varepsilon}_{xy} = \hat{\lambda}\ \sigma_{xy}$$

Combined with the relation between $\dot{\varepsilon}_{ij}$ and v_i, we have

$$v_{x,x} = \hat{\lambda}\ \frac{\sigma_{xx} - \sigma_{yy}}{2}$$

$$v_{y,y} = \hat{\lambda}\ \frac{\sigma_{yy} - \sigma_{xx}}{2}$$

$$\frac{v_{x,y} + v_{y,x}}{2} = \hat{\lambda}\ \sigma_{xy}$$

Adding the first two equations above, we get

$$v_{x,x} + v_{y,y} = 0 \quad \text{-----} \quad (4) \qquad \text{incompressibility condition}$$

Subtracting the first two equations, we get

$$v_{x,x} - v_{y,y} = \hat{\lambda}\ (\sigma_{xx} - \sigma_{yy})$$

Dividing by the third equation, we eliminate $\hat{\lambda}$

$$\frac{v_{x,x} - v_{y,y}}{v_{x,y} + v_{y,x}} = \frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}} \quad \text{-----} (5)$$

Consider 5 unknowns: σ_{xx}, σ_{yy}, σ_{xy}, v_x, v_y

there are 5 equations:

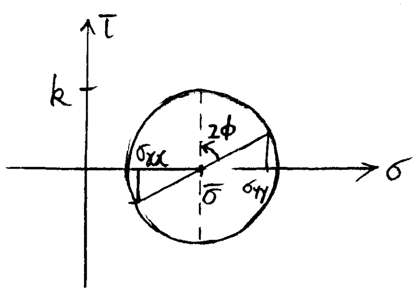
(1), (2)	equilibrium
(3)	yield condition
(4)	incompressibility
(5)	flow rule

§2. Slip line fields

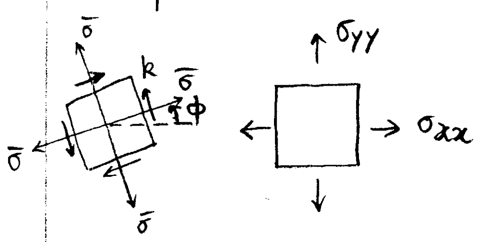
For the 5 unknowns, if the boundary conditions are sufficient to determine the stress field, the problem is called <u>statically determinate</u> (and easier to solve), and the velocity fields can be obtained after the stress fields have been obtained.

Otherwise, the problem is called statically indeterminate and stress fields and velocity fields must be solved together.

We will focus on statically determinate problems here.



In the plastic region, the stress state must lie on the Mohr's circle with center $\bar{\sigma}$ and radius k (maximum shear)



Let ϕ be the angle of rotation to the orientation of maximum shear, where all normal stresses equal to $\bar{\sigma}$

$$\begin{cases} \sigma_{xx} = \bar{\sigma} - k \sin 2\phi \\ \sigma_{yy} = \bar{\sigma} + k \sin 2\phi \\ \sigma_{xy} = k \cos 2\phi \end{cases}$$

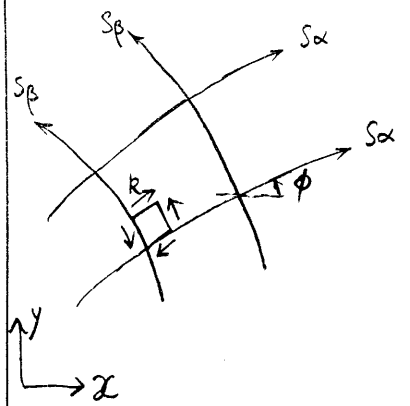
$\bar{\sigma}$ and ϕ are functions of (x, y)

The yield condition, Eq. (3), is automatically satisfied.

The <u>equilibrium condition</u> can be written in terms of $\bar{\sigma}$ and ϕ

$$\begin{cases} \dfrac{\partial \bar{\sigma}}{\partial x} + k\left(-\dfrac{\partial}{\partial x}\sin 2\phi + \dfrac{\partial}{\partial y}\cos 2\phi\right) = 0 \\ \dfrac{\partial \bar{\sigma}}{\partial y} + k\left(\dfrac{\partial}{\partial x}\cos 2\phi + \dfrac{\partial}{\partial y}\sin 2\phi\right) = 0 \end{cases}$$

The equations are hyperbolic with characteristics coinciding with slip lines (introduced below).



Consider a curvilinear, locally orthogonal family of curves S_α, S_β covering the plastic region

(S_α, S_β) can be considered as a curvilinear coordinate system.

The tangent of α-curves always have the orientation $\phi(x,y)$

Hence both α-curves and β-curves are along the direction of maximum shear.

In terms of the curvilinear coordinates, the equilibrium conditions look particularly simple:

$$\begin{cases} \dfrac{\partial}{\partial S_\alpha}(\bar\sigma - 2k\phi) = 0 \\[2ex] \dfrac{\partial}{\partial S_\beta}(\bar\sigma + 2k\phi) = 0 \end{cases}$$

In other words,

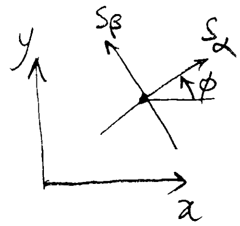
$$\begin{cases} \bar\sigma - 2k\phi = \text{constant} & \text{along } \alpha\text{-line} \\ \bar\sigma + 2k\phi = \text{constant} & \text{along } \beta\text{-line} \end{cases}$$

(These are typical properties of characteristic curves of PDEs.)

If we know the shapes of the α- and β-curves, and the $(\bar\sigma, \phi)$ value at some point, it is easy to find $(\bar\sigma, \phi)$ values at the entire region covered by these curves.

The challenge is to construct these curves in the plastic region.

§3. Geometrical Properties of Slip Lines



If we know the stress state $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ at a point (x, y), we know the local orientation of the slip lines, because the α-line must make angle ϕ with the x-axis.

This means that the α-line satisfy the condition

$$\frac{dy}{dx} = \tan\phi \,, \qquad \frac{\bar{\sigma}}{2k} - \phi = \xi \quad (\text{constant})$$

and the β-line satisfy the condition

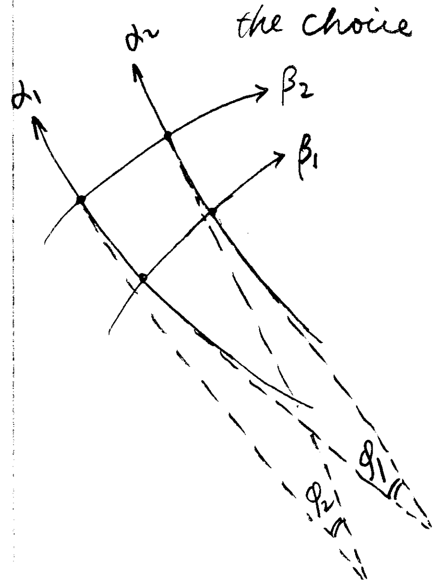
$$\frac{dy}{dx} = -\cot\phi \,, \qquad \frac{\bar{\sigma}}{2k} + \phi = \eta \quad (\text{constant})$$

So that

$$\bar{\sigma} = k(\xi + \eta)$$

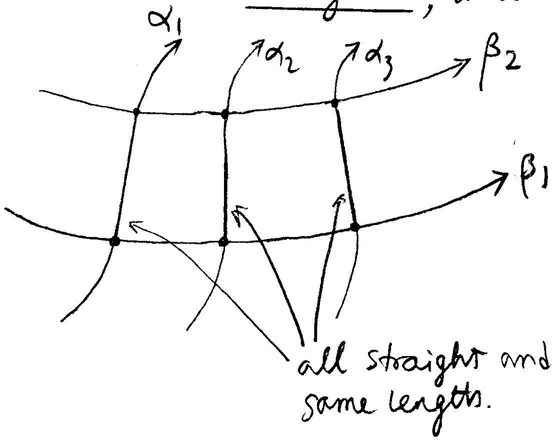
$$\theta = \frac{1}{2}(\eta - \xi)$$

<u>Property 1</u>: The change of orientation of two α-lines at intersections with another β line is independent of the choice of the β-line

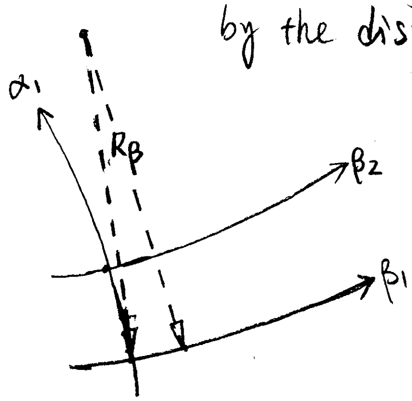


$$\underline{\varphi_1 = \varphi_2}$$

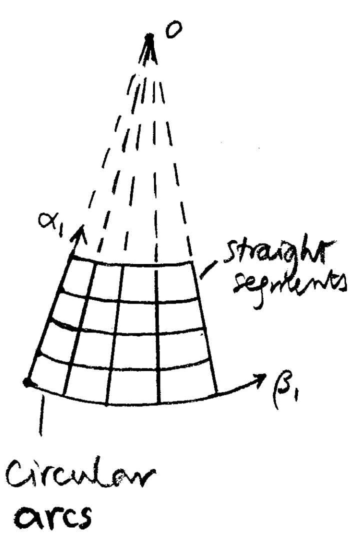
<u>Property 2</u>: If a segment of slip line (e.g. α-line) is <u>straight</u>, then $\bar{\sigma}$, θ, ξ, η, σ_{xx}, σ_{yy}, σ_{xy} are <u>constant</u> along it. and <u>all α lines</u> cut off by the same β lines are <u>straight</u>, and all have the <u>same lengths</u>.



<u>Property 3</u>: Travelling along an α-line, the radius of curvature of the β-line at the intersection changes by the distance traveled



$$\frac{\partial R_\beta}{\partial S_\alpha} = -1, \qquad \frac{\partial R_\alpha}{\partial S_\beta} = -1$$



An example that illustrates both Property 2 and Property 3.

This region corresponds to a "simple stress state" (see below).

§4. Velocity field

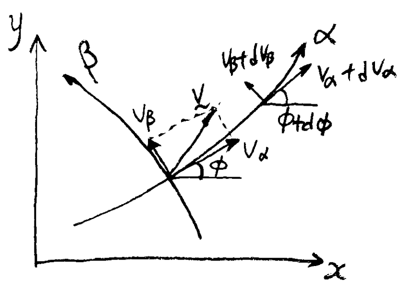
For statically determinate problems, the velocity fields $V_x(x,y)$, $V_y(x,y)$ are found after the slip line fields and stress fields have been obtained.

$$\frac{V_{x,x} - V_{y,y}}{V_{x,y} + V_{y,x}} = \frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}} = \frac{-2k\sin 2\phi}{2k\cos 2\phi} = -\tan 2\phi \quad \text{---- from (4)}$$

Hence (already solved)

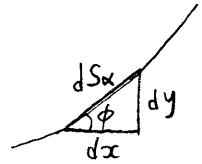
$$\begin{cases} (V_{x,y} + V_{y,x}) \tan[2\phi(x,y)] + (V_{x,x} - V_{y,y}) = 0 \\ V_{x,x} + V_{y,y} = 0 \qquad \text{------ (5)} \quad \text{incompressibility} \end{cases}$$

This system of equations is (again) of hyperbolic type, and its characteristics coincide with the slip lines.

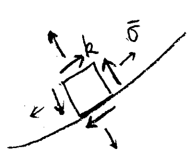


Let V_α, V_β be the component of $\underset{\sim}{v}$ along α-line and β-line respectively.

$$\begin{cases} V_x = V_\alpha \cos\phi - V_\beta \sin\phi \\ V_y = V_\alpha \sin\phi + V_\beta \cos\phi \end{cases}$$



note the stress state along slip lines is pure shear (k) plus hydrostatic stress ($\bar{\sigma}$)



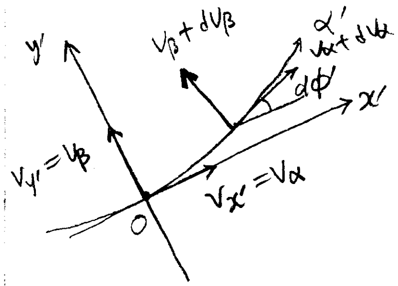
So the normal strain rate along slip line is zero (ignore elastic strain)

The rate of <u>relative elongation along slip lines are zero</u>.

The condition of zero elongation along slip lines can be expressed as

$$\boxed{\begin{array}{ll} dV_\alpha - V_\beta\, d\phi = 0 & \text{along } \alpha\text{-line} \\ dV_\beta + V_\alpha\, d\phi = 0 & \text{along } \beta\text{-line} \end{array}}$$

These are compatibility conditions for the velocities, Geiringer (1930)



To prove Geiringer's relations, we choose a local coordinate system $x'-y'$ so that the x'-axis is tangent to the α-line at point O

In the neighborhood of point O,

$$\begin{cases} V_{x'} = V_\alpha \cos\phi' - V_\beta \sin\phi' \\ V_{y'} = V_\alpha \sin\phi' + V_\beta \cos\phi' \end{cases}$$

Right at point O, $\phi' = 0$, so that $V_{x'} = V_\alpha$, $V_{y'} = V_\beta$

The condition of zero elongation along α-line can be written as

$$\dot{\varepsilon}_{x'x'} = \frac{\partial V_{x'}}{\partial x'} = 0$$

$$\frac{\partial V_\alpha}{\partial x'} \cos\phi' + V_\alpha (-\sin\phi') \frac{\partial \phi'}{\partial x'} - \frac{\partial V_\beta}{\partial x'} \sin\phi' - V_\beta \cos\phi' \frac{\partial \phi'}{\partial x'} = 0$$

At point O, $\phi' = 0$, $\cos\phi' = 1$, $\sin\phi' = 0$

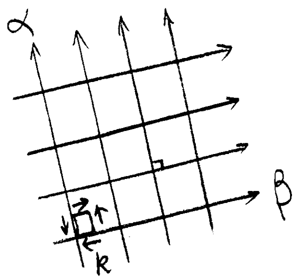
$$\frac{\partial V_\alpha}{\partial x'} - V_\beta \frac{\partial \phi'}{\partial x'} = 0$$

$\therefore \quad dV_\alpha - V_\beta\, d\phi' = 0 \quad$ along x'

Rotate to x-y coordinate system $\quad \phi = \phi' + \phi|_0$

$$dV_\alpha - V_\beta\, d\phi = 0 \quad \text{along } \alpha\text{-line}$$

§5. Simple stress states



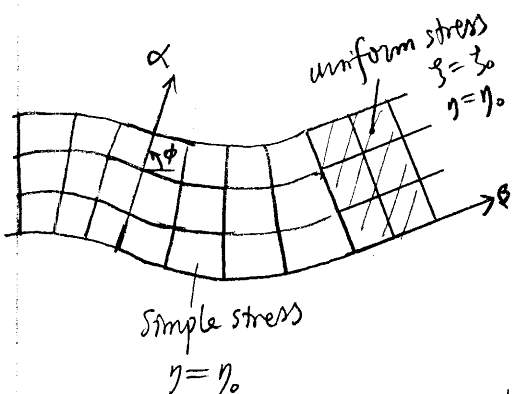
uniform stress state

both α-lines and β-lines are straight lines.

both ξ and η are constants. (ξ_0, η_0)

All stress components are uniform in this region.

$$\bar{\sigma} = k(\xi_0 + \eta_0), \quad \phi = \frac{1}{2}(\eta_0 - \xi_0)$$



Simple stress state

One family (α-line here) is straight.
the other family (β-line here) are generated by orthogonal curves.

Here α-lines are straight, so
$\eta = \eta_0$ is a constant.

(How to prove?
First, η = const along each α-line if it is straight
Second, η = const along β-line by definition
Therefore, η = const in the entire region of simple stress)

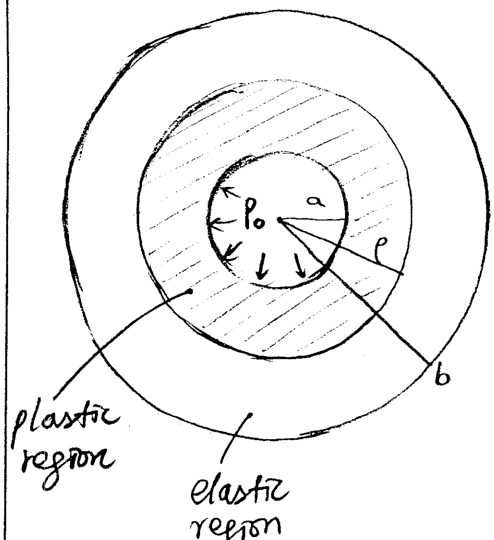
A region adjoining a region of uniform stress is always in a state of simple stress.

simple stress state:

$$\frac{\bar{\sigma}}{2k} + \phi = \eta_0$$

$$\bar{\sigma} = 2k(\eta_0 - \phi)$$

§5. Axisymmetric field

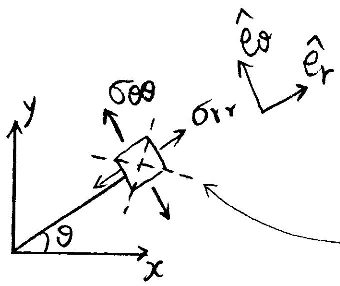


We return to the pressurized cylindrical tube problem ($p_0 > p_0^Y$), and try to construct a slip line field solution in the plastic region ($a \le r \le \rho$).

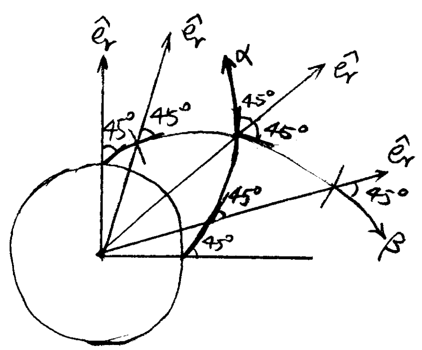
By axi-symmetry, $\sigma_{r\theta} = 0$

Hence σ_{rr} and $\sigma_{\theta\theta}$ are principal stresses, i.e.

$\hat{e}_r$ and $\hat{e}_\theta$ are principal stress directions.



The maximum shear stress direction must be at 45° with $\hat{e}_r$ and $\hat{e}_\theta$.



This means that both α-lines and β-lines must always be at 45° with $\hat{e}_r$ and $\hat{e}_\theta$

A curve that always makes the <u>same angle</u>(γ) with the <u>radial direction</u> is the <u>Logarithmic-spiral</u>.

$$r = a\, e^{b\theta}$$

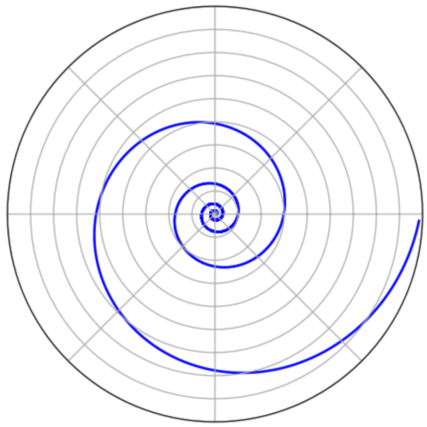
or $$\theta = \frac{1}{b} \ln \frac{r}{a}$$

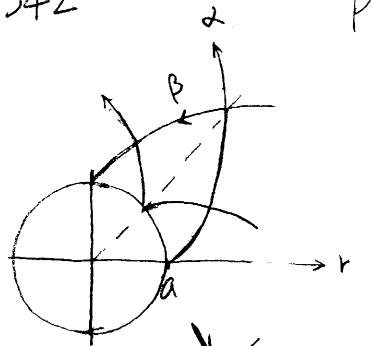
$$\arctan \frac{1}{b} = \gamma = 45^\circ . \quad \frac{1}{b} = \tan 45^\circ = 1 \rightarrow b = 1$$

Logarithmic Spiral

http://en.wikipedia.org/wiki/Logarithmic_spiral

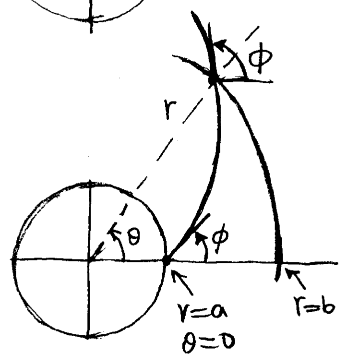
$$r = ae^{b\theta}$$
$$\theta = \frac{1}{b}\ln(r/a),$$





α-lines: $r = a\,e^{\theta-\theta_0}$, $\theta = \theta_0 + \ln\frac{r}{a}$

β-lines: $r = a\,e^{-(\theta-\theta_0)}$, $\theta = \theta_0 - \ln\frac{r}{a}$



consider an α-line, $\theta = \ln\frac{r}{a}$

note $\phi = \frac{\pi}{4} + \theta$

$\therefore\ \Delta\phi = \Delta\theta$ as we travel along the α-line

Recall $\frac{\bar{\sigma}}{2k} - \phi = \text{const}$ along α-line

$\therefore\ \Delta\bar{\sigma} = 2k\,\Delta\theta$ along the α-line

$$\bar{\sigma} = \bar{\sigma}\big|_{r=a} + 2k\ln\frac{r}{a}$$

Since $\sigma_{\theta\theta} - \sigma_{rr} = 2k$ (yield condition)

$$\sigma_{rr}\big|_{r=a} = -p_0$$

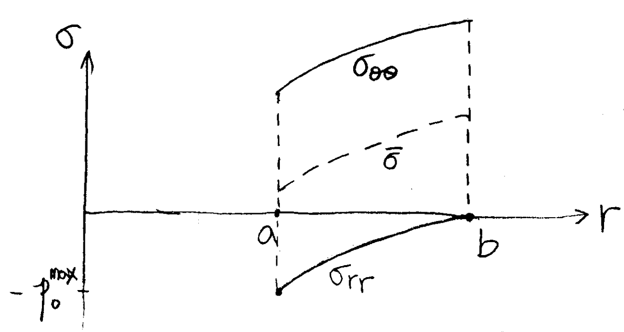
$$\sigma_{\theta\theta}\big|_{r=a} = -p_0 + 2k$$

$$\bar{\sigma}\big|_{r=a} = \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta})_{r=a} = -p_0 + k$$

$$\therefore\ \bar{\sigma} = -p_0 + k + 2k\ln\frac{r}{a}$$

$$\sigma_{rr} = \bar{\sigma} - k = -p_0 + 2k\ln\frac{r}{a}$$

$$\sigma_{\theta\theta} = \bar{\sigma} + k = -p_0 + 2k + 2k\ln\frac{r}{a}$$



Applying to full plastic state ($p_0 = p_0^{max}$)

$$0 = \sigma_{rr}\big|_{r=b} = -p_0^{max} + 2k\ln\frac{b}{a}$$

$$p_0^{max} = 2k\ln\frac{b}{a}$$

(compare with numerical solution)

This is the solution in the rigid-plastic limit $\nu \to 1/2$, $\mu \to \infty$