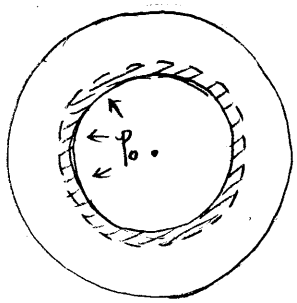
This is our first 2D plane-strain example. The axisymmetry makes the problem simpler than the general 2D plane-strain problem, which will be discussed in the next lecture note

§1. Problem Statement



Consider a thick-walled cylindrical tube (pressure vessel) with internal (gas) pressure p_0

Determine the thickness of plastic region (shaded) with increasing p_0

Determine if plastic flow occurs if p_0 is reduced back to zero.

We expect the plastic region to appear if $p_0 > p_Y$, where p_Y is a threshold value (onset of yield)

There is also a maximum value p_{max}, at which the pressure vessle will burst!

Find $\frac{p_{max}}{p_Y}$.

§2. plane-strain Elasticity

$u_z = 0$, solution independent of z (i.e. $\frac{\partial}{\partial z} = 0$)

Look for: $u_x(x,y)$, $u_y(x,y)$, ε_{xx}, ε_{yy}, ε_{xy}, σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} = 0 \quad \rightarrow \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

equilibrium condition:
$$\sigma_{xx,x} + \sigma_{yx,y} + F_x = 0$$
$$\sigma_{xy,x} + \sigma_{yy,y} + F_y = 0$$

compatibility condition. $\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$

Generalized Hookes Law:
$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}$$
$$\varepsilon_{yy} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}$$
$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy} \qquad E = 2\mu(1+\nu)$$

Introduce Airy stress function: $\phi(x,y)$, such that
$$\sigma_{xx} = \phi_{,yy}, \quad \sigma_{yy} = \phi_{,xx}, \quad \sigma_{xy} = -\phi_{,xy}$$

Then in the absence of body force ($F_x = F_y = 0$),
the equilibrium condition is automatically satisfied.

The compatibility condition becomes (biharmonic equation)
$$\nabla^4\phi \equiv \phi_{,xxxx} + \phi_{,yyyy} - 2\phi_{,xxyy} = 0$$

In polar coordinates, $\phi(r,\theta) = r^m e^{in\theta}$ satisfies biharmonic equation,
as long as $m = n, -n, 2+n, 2-n$.

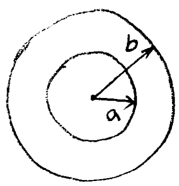
(see ME340 Winter 2013 Lecture Notes, "Polar Coordinates".)

For problems with axial symmetry, $n=0$, (Mitchell solution)
$$\phi = A_1 r^2 + A_2 r^2 \ln r + A_3 \ln r + A_4 \theta$$
$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right), \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$

$$\sigma_{rr} = A_1 \cdot 2 + A_2 \cdot (2\ln r + 1) + A_3 \cdot \frac{1}{r^2}$$

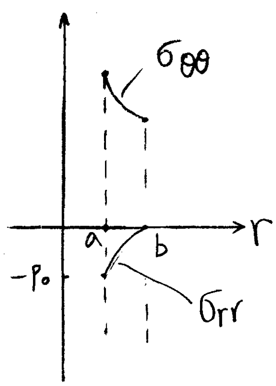
$$\sigma_{r\theta} = \qquad A_4 \cdot \frac{1}{r^2}$$

$$\sigma_{\theta\theta} = A_1 \cdot 2 + A_2 \cdot (2\ln r + 3) + A_3 \cdot \left(-\frac{1}{r^2}\right)$$



Boundary condition for pressurized tube

$$\begin{cases} \sigma_{rr}\big|_{r=a} = -p_0 \\ \sigma_{r\theta}\big|_{r=a} = 0 \end{cases} \qquad \begin{cases} \sigma_{rr}\big|_{r=b} = 0 \\ \sigma_{r\theta}\big|_{r=b} = 0 \end{cases}$$



$$A_2 = A_3 = 0, \quad A_1 = \frac{p_0}{2} \cdot \frac{a^2}{b^2 - a^2}, \quad A_3 = -\frac{p_0 a^2 b^2}{b^2 - a^2}$$

$$\begin{cases} \sigma_{rr} = \dfrac{p_0 a^2}{b^2 - a^2}\left(1 - \dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = \dfrac{p_0 a^2}{b^2 - a^2}\left(1 + \dfrac{b^2}{r^2}\right) \end{cases}$$

$$\sigma_1 = \sigma_{\theta\theta}, \quad \sigma_2 = \sigma_{rr}, \quad \sigma_3 = \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{p_0 a^2}{b^2 - a^2} \cdot 2\nu$$

$$\bar{\sigma} = \frac{1}{3}(\sigma_{\theta\theta} + \sigma_{rr} + \sigma_{zz}) = \frac{1+\nu}{3}(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{2(1+\nu)}{3} \cdot \frac{p_0 a^2}{b^2 - a^2}$$

$$S_1 = \sigma_1 - \bar{\sigma} = \frac{p_0 a^2}{b^2 - a^2}\left(\frac{1-2\nu}{3} + \frac{b^2}{r^2}\right)$$

$$S_2 = \sigma_2 - \bar{\sigma} = \frac{p_0 a^2}{b^2 - a^2}\left(\frac{1-2\nu}{3} - \frac{b^2}{r^2}\right)$$

$$S_3 = \sigma_3 - \bar{\sigma} = -\frac{p_0 a^2}{b^2 - a^2} \cdot \frac{2(1-2\nu)}{3}$$

$$J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2) = \left(\frac{p_0 a^2}{b^2 - a^2}\right)\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{r^4}\right]$$

note $K = \dfrac{2\mu(1+\nu)}{3(1-2\nu)}$ bulk modulus

define $p' \equiv \dfrac{p_0 a^2}{b^2 - a^2}$

$$3\left(\frac{\mu}{\mu + 3K}\right)^2 = \frac{(1-2\nu)^2}{3}$$

$$J_2 = p'\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{r^4}\right]$$

$$\begin{cases} \sigma_{rr} = p' \cdot \left(1 - \dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = p' \cdot \left(1 + \dfrac{b^2}{r^2}\right) \end{cases}$$

maximum J_2 occurs at $r=a$.

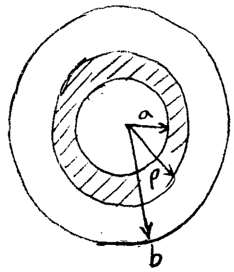
At onset of yield $J_2\Big|_{r=a} = (p')^2\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{a^4}\right] = k^2$

$$p_0^Y = k \cdot \frac{b^2-a^2}{a^2} \cdot \left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{a^4}\right]^{-1/2}$$

§3. Elastic region after yield

For $p_0 > p_0^Y$, the tube contains a plastic region.

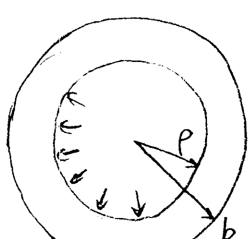
Suppose p_0 increases montonically from 0 to a value greater than p_0^Y. Then the plastic region starts from the inner wall ($r=a$) and grow thicker.



Let $r=\rho$ be the elastic-plastic boundary.

The elastic region $\rho \le r \le b$ has similar boundary conditions as before.

$$\begin{cases} \sigma_{rr}\Big|_{r=\rho} = -p_\rho \\ \sigma_{r\theta}\Big|_{r=\rho} = 0 \end{cases} \qquad \begin{cases} \sigma_{rr}\Big|_{r=b} = 0 \\ \sigma_{r\theta}\Big|_{r=b} = 0 \end{cases}$$



elastic region

So the stress field in the elastic region must be:

$$\begin{cases} \sigma_{rr} = \dfrac{p_\rho \rho^2}{b^2-\rho^2}\left(1-\dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = \dfrac{p_\rho \rho^2}{b^2-\rho^2}\left(1+\dfrac{b^2}{r^2}\right) \end{cases}$$

$$J_2\Big|_{r=\rho} = \frac{p_\rho \rho^2}{b^2-\rho^2}\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{\rho^4}\right] = k^2$$

$$\therefore \quad p_\rho = k\,\frac{b^2-\rho^2}{\rho^2}\left[\frac{(1-2\nu)^2}{3} + \frac{b^4}{\rho^4}\right]^{-1/2}$$

This is the magnitude of the normal stress $|\sigma_{rr}|$ at the elastic-plastic boundary.

define $p'' \equiv \dfrac{p_\rho \rho^2}{b^2 - \rho^2} = k \left[\dfrac{(1-2\nu)^2}{3} + \dfrac{b^4}{\rho^4} \right]^{-1/2}$

Then in the elastic region $\rho \le r \le b$.

$$\begin{cases} \sigma_{rr} = p'' \cdot \left(1 - \dfrac{b^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = p'' \cdot \left(1 + \dfrac{b^2}{r^2}\right) \end{cases}$$

(very similar to the expression in p.3)

With increasing p_0, ρ increases from a to b.

§4. Numerical Solution in Elastic Regime

Before discussing the plastic regime, let us see how the same solution in the elastic regime as derived above can be obtained numerically. This prepares us for the numerical solution in the plastic regime.

Equilibrium condition in polar coordinates:

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\theta}}{\partial \theta} + \dfrac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0 \\ \dfrac{1}{r}\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial \sigma_{r\theta}}{\partial r} + \dfrac{2\sigma_{r\theta}}{r} + F_\theta = 0 \end{cases}$$

In our example, $\sigma_{r\theta} = 0$, $F_r = F_\theta = 0$, $\frac{\partial}{\partial \theta} = 0$

$$\therefore \quad \boxed{\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0} \qquad \text{(equilibrium condition)}$$

The compatibility condition can be obtained from the definition of strain (in polar coordinates)

$$\begin{cases} \varepsilon_{rr} = \dfrac{\partial}{\partial r} u_r \\ \varepsilon_{\theta\theta} = \dfrac{1}{r}\left(u_r + \dfrac{\partial u_\theta}{\partial \theta}\right) \\ \varepsilon_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) \end{cases}$$

By symmetry, in our example, we have only one non-zero displacement field: $u_r(r)$

$$\begin{cases} \varepsilon_{rr} = \dfrac{\partial}{\partial r} u_r \\ \varepsilon_{\theta\theta} = \dfrac{1}{r} u_r \\ \varepsilon_{r\theta} = 0 \end{cases}$$

hydrostatic strain:

$$\bar{\varepsilon} \equiv \frac{1}{3}(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) = \frac{1}{3}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right)$$

$$\therefore \quad \varepsilon_{rr} = \frac{\partial}{\partial r}(r\,\varepsilon_{\theta\theta}) = \varepsilon_{\theta\theta} + r\frac{\partial \varepsilon_{\theta\theta}}{\partial r}$$

$$\boxed{\frac{\partial \varepsilon_{\theta\theta}}{\partial r} = \frac{1}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta})} \qquad \text{(compatibility condition)}$$

Generalized Hooke's Law:

$$\sigma_{rr} = (\lambda+2\mu)\,\varepsilon_{rr} + \lambda\,\varepsilon_{\theta\theta}$$
$$\sigma_{\theta\theta} = \lambda\,\varepsilon_{rr} + (\lambda+2\mu)\,\varepsilon_{\theta\theta}$$
$$\sigma_{zz} = \lambda\,\varepsilon_{rr} + \lambda\,\varepsilon_{\theta\theta}$$

$$\sigma_{rr} - \sigma_{\theta\theta} = 2\mu\,(\varepsilon_{rr} - \varepsilon_{\theta\theta})$$

Plug into equilibrium condition

$$\boxed{(\lambda+2\mu)\frac{\partial \varepsilon_{rr}}{\partial r} + \lambda\frac{\partial \varepsilon_{\theta\theta}}{\partial r} + \frac{2\mu}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta}) = 0}$$

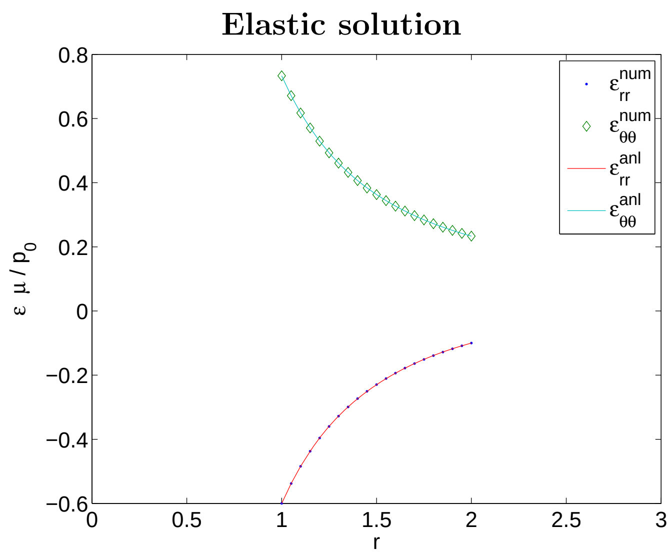
So we have two equations for the two unknowns: ε_{rr}, $\varepsilon_{\theta\theta}$.

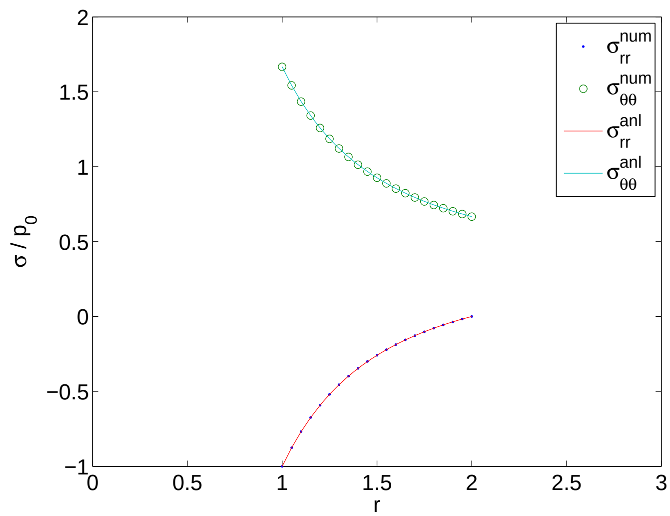
Boundary conditions:

$$\begin{cases} \sigma_{rr}\big|_{r=a} = (\lambda+2\mu)\,\varepsilon_{rr}\big|_{r=a} + \lambda\,\varepsilon_{\theta\theta}\big|_{r=a} = -p_0 \\ \sigma_{rr}\big|_{r=b} = (\lambda+2\mu)\,\varepsilon_{rr}\big|_{r=b} + \lambda\,\varepsilon_{\theta\theta}\big|_{r=b} = 0 \end{cases}$$

A Matlab code that finds $\varepsilon_{rr}(r)$ and $\varepsilon_{\theta\theta}(r)$ that satisfy the compatibility, equilibrium, and boundary condition is written — cylind_tube_elast.m , eqns_cylind_tube elast.m

It makes uses of Matlab's fsolve function.


Elastic solution
ε μ / p0
r
ε_rr^num
ε_θθ^num
ε_rr^anl
ε_θθ^anl


σ / p0
r
σ_rr^num
σ_θθ^num
σ_rr^anl
σ_θθ^anl

§5. Plastic Region

Our next task is to find the relation between p_0 and ρ

(ρ is the radius of elastic-plastic boundary, when $p_0 > p_0^Y$)

and to find the stress and strain field inside the plastic region $a \le r \le \rho$.

We shall choose our unknowns as $\underbrace{\varepsilon_{rr},\ \varepsilon_{\theta\theta}}_{\text{total strain}}$, $\underbrace{S_{rr}}_{\text{deviatoric stress}}$

and establish 3 PDE's.

All other quantities of interest can be expressed in terms of these three.

Hydrostatic strain: $\bar{\varepsilon} = \frac{1}{3}(\varepsilon_{rr} + \varepsilon_{\theta\theta})$ note $\varepsilon_{zz} = 0$

deviatoric strain: $e_{rr} = \varepsilon_{rr} - \bar{\varepsilon} = \frac{2}{3}\varepsilon_{rr} - \frac{1}{3}\varepsilon_{\theta\theta}$

$$e_{\theta\theta} = \varepsilon_{\theta\theta} - \bar{\varepsilon} = -\frac{1}{3}\varepsilon_{rr} + \frac{2}{3}\varepsilon_{\theta\theta}$$

$$e_{zz} = \varepsilon_{zz} - \bar{\varepsilon} = -\frac{1}{3}\varepsilon_{rr} - \frac{1}{3}\varepsilon_{\theta\theta}$$

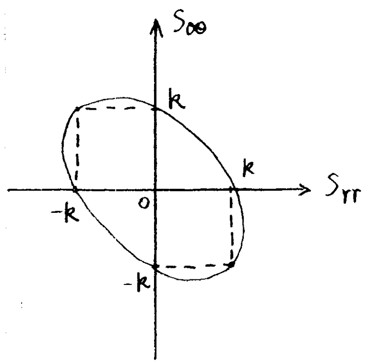
Hydrostatic stress: $\bar{\sigma} = 3K\bar{\varepsilon} = K(\varepsilon_{rr} + \varepsilon_{\theta\theta})$

deviatoric stress: $S_{rr},\ S_{\theta\theta},\ S_{zz}$

$$S_{rr} + S_{\theta\theta} + S_{zz} = 0, \quad S_{zz} = -(S_{rr} + S_{\theta\theta})$$

in the plastic region, stress must satisfy yield condition

$$J_2 = \frac{1}{2}(S_{rr}^2 + S_{\theta\theta}^2 + S_{zz}^2) = S_{rr}^2 + S_{\theta\theta}^2 + S_{rr}S_{\theta\theta} = k^2$$



$$S_{\theta\theta} = \frac{1}{2}\left(-S_{rr} \pm \sqrt{4k^2 - 3S_{rr}^2}\right)$$

Choose + sign, why?

Stress: $\sigma_{rr} = S_{rr} + \bar{\sigma} = S_{rr} + K(\varepsilon_{rr} + \varepsilon_{\theta\theta})$

$$\sigma_{\theta\theta} = S_{\theta\theta} + \bar{\sigma}$$

$$\sigma_{rr} - \sigma_{\theta\theta} = S_{rr} - S_{\theta\theta}$$

We now set up the 3 PDE's for ε_{rr}, $\varepsilon_{\theta\theta}$, S_{rr}.

equilibrium condition: $$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

now becomes

$$\frac{\partial S_{rr}}{\partial r} + K\left(\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right) + \frac{S_{rr} - S_{\theta\theta}}{r} = 0 \quad \cdots\cdots ①$$

Compatibility condition:

$$\frac{\partial \varepsilon_{\theta\theta}}{\partial r} = \frac{1}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta}) \quad \cdots\cdots ②$$

Plastic flow rule will lead to the following equation:

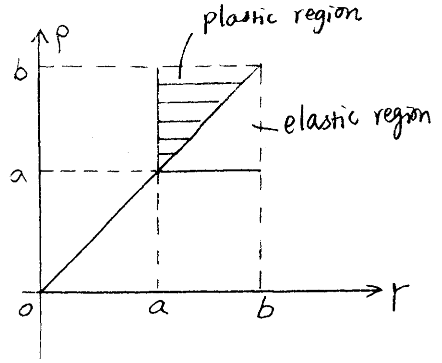
$$2\mu\left[\left(\frac{2}{3}\frac{\partial \varepsilon_{rr}}{\partial \rho} - \frac{1}{3}\frac{\partial \varepsilon_{\theta\theta}}{\partial \rho}\right) S_{\theta\theta} - \left(\frac{2}{3}\frac{\partial \varepsilon_{\theta\theta}}{\partial \rho} - \frac{1}{3}\frac{\partial \varepsilon_{rr}}{\partial \rho}\right) S_{rr}\right] = \frac{\partial S_{rr}}{\partial \rho} S_{\theta\theta} - \frac{\partial S_{\theta\theta}}{\partial \rho} S_{rr} \quad \cdots ③$$

note this is the only PDE with $\frac{\partial}{\partial \rho}$.

PDE ③ is derived in P.10

Compare these equations with those in the elastic regime (§4).
The compatibility condition is the same.
But the equilibrium condition here cannot be written in terms of ε_{rr}, $\varepsilon_{\theta\theta}$ alone (because of the plastic strain)

Instead, S_{rr} is introduced here as the 3rd unknown and the 3rd PDE is supplied by the plastic flow rule.



The equations now need to be solved in the 2D space of r and ρ.

The solution in the elastic region is already known.

The plastic strain <u>rate</u> is in the same direction as the deviatoric stress

$$\begin{cases} \dot{\varepsilon}^{pl}_{rr} = \frac{\tilde{\lambda}}{2\mu} S_{rr} \\ \dot{\varepsilon}^{pl}_{\theta\theta} = \frac{\tilde{\lambda}}{2\mu} S_{\theta\theta} \\ \dot{\varepsilon}^{pl}_{zz} = \frac{\tilde{\lambda}}{2\mu} S_{zz} \\ \dot{\varepsilon}^{pl}_{r\theta} = \frac{\tilde{\lambda}}{2\mu} S_{r\theta} = 0 \end{cases}$$

(plastic flow rule)

The deviatoric elastic strain is in the same direction as the deviatoric stress

$$\begin{cases} e^{el}_{rr} = \frac{1}{2\mu} S_{rr} \\ e^{el}_{\theta\theta} = \frac{1}{2\mu} S_{\theta\theta} \\ e^{el}_{zz} = \frac{1}{2\mu} S_{zz} \\ e^{el}_{r\theta} = \frac{1}{2\mu} S_{r\theta} = 0 \end{cases} \rightarrow \begin{cases} \dot{e}^{el}_{rr} = \frac{1}{2\mu} \dot{S}_{rr} \\ \dot{e}^{el}_{\theta\theta} = \frac{1}{2\mu} \dot{S}_{\theta\theta} \end{cases}$$

Total strain rate (deviatoric part)

$$\begin{cases} \dot{e}_{rr} = \dot{e}^{el}_{rr} + \dot{\varepsilon}^{pl}_{rr} = \frac{1}{2\mu} (\dot{S}_{rr} + \lambda S_{rr}) \\ \dot{e}_{\theta\theta} = \dot{e}^{el}_{\theta\theta} + \dot{\varepsilon}^{pl}_{\theta\theta} = \frac{1}{2\mu} (\dot{S}_{\theta\theta} + \lambda S_{\theta\theta}) \\ \dot{e}_{zz} = \dot{e}^{el}_{zz} + \dot{\varepsilon}^{pl}_{zz} = \frac{1}{2\mu} (\dot{S}_{zz} + \lambda S_{zz}) \\ \dot{e}_{r\theta} = \dot{e}^{el}_{r\theta} + \dot{\varepsilon}^{pl}_{r\theta} = 0 \end{cases} \xrightarrow{\text{eliminate } \tilde{\lambda}}$$

$$\left(2\mu \dot{e}_{rr} - \dot{S}_{rr}\right) S_{\theta\theta} = \tilde{\lambda} S_{rr} S_{\theta\theta} = (2\mu \dot{e}_{\theta\theta} - \dot{S}_{\theta\theta}) S_{rr}$$

$$2\mu\left(\dot{e}_{rr} S_{\theta\theta} - \dot{e}_{\theta\theta} S_{rr}\right) = \dot{S}_{rr} S_{\theta\theta} - \dot{S}_{\theta\theta} S_{rr}$$

Imagine that the elastic-plastic boundary is moving at rate $\dot{\rho}$

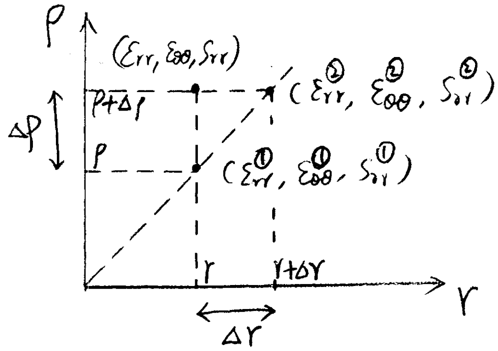
Divide the equation by $\dot{\rho}$, note $\frac{\dot{e}_{rr}}{\dot{\rho}} = \frac{de_{rr}}{dt} \Big/ \frac{d\rho}{dt} = \frac{\partial e_{rr}}{\partial \rho}$

$$2\mu \left(\frac{\partial e_{rr}}{\partial \rho} S_{\theta\theta} - \frac{\partial e_{\theta\theta}}{\partial \rho} S_{rr} \right) = \frac{\partial S_{rr}}{\partial \rho} S_{\theta\theta} - \frac{\partial S_{\theta\theta}}{\partial \rho} S_{rr}$$

plug in $e_{rr} = \frac{2}{3}\varepsilon_{rr} - \frac{1}{3}\varepsilon_{\theta\theta}$, $e_{\theta\theta} = \frac{2}{3}\varepsilon_{\theta\theta} - \frac{1}{3}\varepsilon_{rr}$

we arrive at PDE ③ on P.9.

§6. Numerical Solution in Plastic Region



Suppose the solution is already known at (r, ρ): $(\varepsilon_{rr}^{①}, \varepsilon_{\theta\theta}^{①}, S_{rr}^{①})$

and $(r+\Delta r, \rho+\Delta\rho)$: $(\varepsilon_{rr}^{②}, \varepsilon_{\theta\theta}^{②}, S_{rr}^{②})$

Here we describe an algorithm to find the solution at

$(r, \rho+\Delta\rho)$: $(\varepsilon_{rr}, \varepsilon_{\theta\theta}, S_{rr})$

The discretized PDE becomes:

equilibrium condition:

$$\frac{S_{rr}^{②} - S_{rr}}{\Delta r} + K\left[\frac{\varepsilon_{rr}^{②} - \varepsilon_{rr}}{\Delta r} + \frac{\varepsilon_{\theta\theta}^{②} - \varepsilon_{\theta\theta}}{\Delta r}\right] + \frac{1}{r+\Delta r/2}\left[\left(\frac{S_{rr}^{②} + S_{rr}}{2}\right) - \left(\frac{S_{\theta\theta}^{②} + S_{\theta\theta}}{2}\right)\right] = 0$$

compatibility condition:

$$\frac{\varepsilon_{\theta\theta}^{②} - \varepsilon_{\theta\theta}}{\Delta r} - \frac{1}{r+\Delta r/2}\left[\left(\frac{\varepsilon_{rr}^{②} + \varepsilon_{rr}}{2}\right) - \left(\frac{\varepsilon_{\theta\theta}^{②} - \varepsilon_{\theta\theta}}{2}\right)\right] = 0$$

Plastic flow rule:

$$2\mu\left[\left(\frac{2}{3}\frac{\varepsilon_{rr} - \varepsilon_{rr}^{①}}{\Delta\rho} - \frac{1}{3}\frac{\varepsilon_{\theta\theta} - \varepsilon_{\theta\theta}^{①}}{\Delta\rho}\right)\left(\frac{S_{\theta\theta} + S_{\theta\theta}^{①}}{2}\right) - \left(\frac{2}{3}\frac{\varepsilon_{\theta\theta} - \varepsilon_{\theta\theta}^{①}}{\Delta\rho} - \frac{1}{3}\frac{\varepsilon_{rr} - \varepsilon_{rr}^{①}}{\Delta\rho}\right)\left(\frac{S_{rr} + S_{rr}^{①}}{2}\right)\right]$$

$$- \frac{S_{rr} - S_{rr}^{①}}{\Delta\rho} \cdot \frac{S_{\theta\theta} + S_{\theta\theta}^{①}}{2} + \frac{S_{\theta\theta} - S_{\theta\theta}^{①}}{\Delta\rho} \cdot \frac{S_{rr} + S_{rr}^{①}}{2} = 0$$

$$S_{\theta\theta} = \frac{1}{2}\left(-S_{rr} + \sqrt{4R^2 - 3S_{rr}^2}\right)$$

Numerical solution obtained by

cylind_tube_plast.m, eqns_cylind_tube_plast.m.

Plastic solution