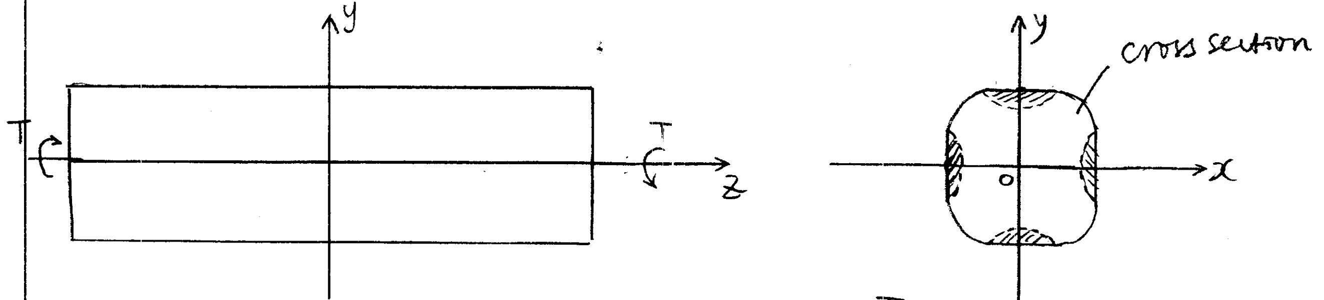
We will study the torsion of prismatic bar beyond yield condition.

Similar to the bending example, we expect the plastic region to initiate at the periphery and grow inward on the cross section.

We shall start with a quick review of the elastic solution using Prandtl's stress function and then extend it ~~to~~ plastic regime.

§1. Problem Statement



A prismatic bar is subjected to torque T along its (z) axis.

Find the plastic region (shaded) and twist per unit length as functions of T.

We expect the plastic region to appear if $T > T_Y$, where T_Y is a threshold value (onset of yield)

We also expect a maximum value T_{max}, at which the plastic region extends to cover the entire cross section

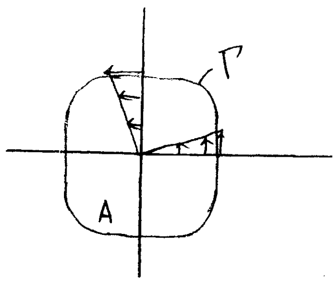
The bar collapses at $T = T_{max}$ and cannot support a greater torque.

Find $\dfrac{T_{max}}{T_Y}$.

§2. Elastic Torsion ($T < T_Y$)

(For more details, see ME340 Winter 2013 Lecture Notes "Torsion".)

Trial solution: $\theta(z) = \beta \cdot z$ β: twist per unit length



displacement field

$$\begin{cases} u_x = -\theta y = -\beta z y \\ u_y = \theta x = \beta z x \\ u_z = \beta \cdot f(x,y) \end{cases}$$

strain field

$$\begin{cases} \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0 \\ \varepsilon_{xy} = 0 \\ \varepsilon_{xz} = \frac{1}{2}(-\beta y + \beta f_{,x}) \\ \varepsilon_{yz} = \frac{1}{2}(\beta x + \beta f_{,y}) \end{cases}$$

stress field

$$\begin{cases} \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \\ \sigma_{xy} = 0 \\ \sigma_{xz} = \mu\beta(-y + f_{,x}) \\ \sigma_{yz} = \mu\beta(x + f_{,y}) \end{cases}$$

Introduce Prandtl's stress function: $\phi(x,y)$ such that

$$\sigma_{zx} = \phi_{,y} \qquad \sigma_{zy} = -\phi_{,x}$$

so that the equilibrium condition $\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0$

is automatically satisfied.

$\phi(x,y)$ satisfies PDE:

$$\boxed{\nabla^2\phi = \phi_{,xx} + \phi_{,yy} = -2\mu\beta} \qquad \text{(Poisson equation)}$$

traction free boundary condition on outer surface of bar

$\rightarrow \phi = \text{const}$ on Γ

without loss of generality, we choose $\boxed{\phi = 0 \text{ on } \Gamma.}$

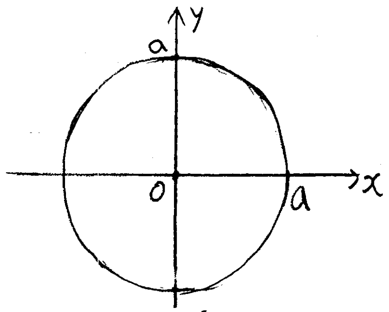
Torque T is related to ϕ through:

$$T = \iint_A (x\sigma_{zy} - y\sigma_{zx})\, dx\,dy = \iint_A (-x\phi_{,x} - y\phi_{,y})\, dx\,dy$$

$$\boxed{T = 2\iint_A \phi \, dx\,dy}$$

$$\tau = \sqrt{(\phi_{,x})^2 + (\phi_{,y})^2} \qquad S_1 = \tau,\ S_2 = -\tau,\ S_3 = 0,\ J_2 = \tau^2$$

§2.1 Circular cross section

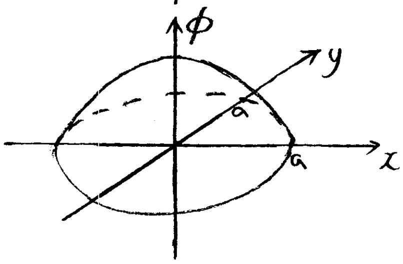


$$\phi(x,y) = -\frac{\mu\beta}{2}(x^2+y^2-a^2) \qquad r \equiv \sqrt{x^2+y^2}$$

$$\phi(r) = -\frac{\mu\beta}{2}(r^2-a^2)$$

$T = \mu\beta K_t$ $\qquad K_t$: torsional rigidity

$K_t = \frac{\pi}{2}a^4 = J$ for circular cross section

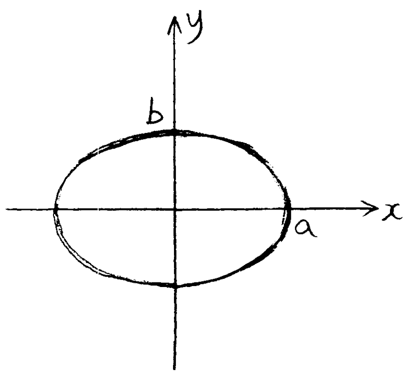


$$\tau \equiv \sqrt{\sigma_{xz}^2+\sigma_{yz}^2} = \left|\frac{\partial\phi}{\partial r}\right| = \mu\beta r$$

$$\tau_{max} = \mu\beta a = \frac{T}{K_t}a = \frac{2T}{\pi a^3} \quad \text{at } r=a$$

When $T=T_Y$, $\tau_{max} = k = \frac{\sigma_Y}{\sqrt{3}}$, $\therefore \underline{T_Y = \frac{\pi}{2}ka^3}$

§2.2 Elliptical cross section

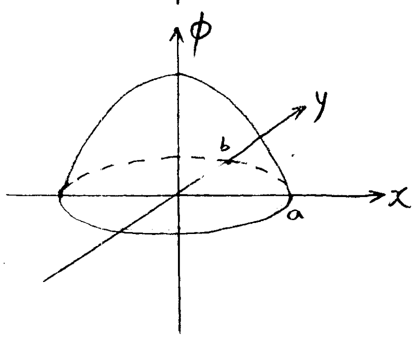


$$\phi(x,y) = -\frac{\mu\beta a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

$T = \mu\beta K_t$ $\qquad$ (equality reached for $a=b$)

$$K_t = \frac{\pi a^3b^3}{a^2+b^2} \leqslant J = \frac{\pi ab}{4}(a^2+b^2)$$

$J \equiv \int r^2 dA$ polar moment of inertia

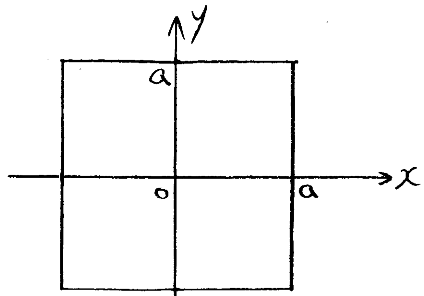


$$\tau_{max} = \left|\sigma_{xz}\right|\Big|_{\substack{x=0\\y=b}} = \left|\frac{\partial\phi}{\partial y}\right|\Bigg|_{\substack{x=0\\y=b}} = \frac{\mu\beta a^2b^2}{a^2+b^2}\cdot\frac{2b}{b^2}$$

$$\tau_{max} = \frac{2\mu\beta a^2 b}{a^2+b^2} = \frac{T}{K_t}\frac{2a^2b}{a^2+b^2} = \frac{2T}{\pi ab^2}$$

When $T=T_Y$, $\tau_{max} = k = \frac{\sigma_Y}{\sqrt{3}}$, $\therefore \underline{T_Y = \frac{\pi}{2}kab^2}$

§2.3 Square cross section

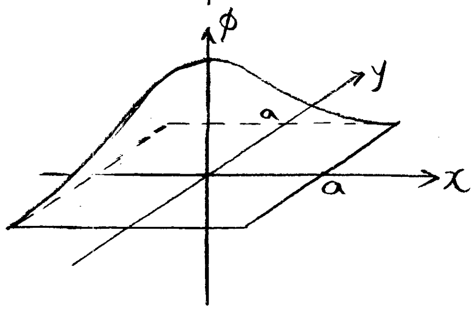


$\phi(x,y)$ must be solved numerically.

$$T = \mu\beta K_t$$

$$K_t \approx 2.24\, a^4 \leqslant J = \frac{8}{3}a^4$$

$$\frac{K_t}{J} \approx 0.84$$



$$\tau_{max} = \left|\sigma_{yz}\right|\Big|_{\substack{x=\pm a\\ y=0}} = \left|\sigma_{xz}\right|\Big|_{\substack{x=0\\ y=\pm a}}$$

$$\approx 0.60\frac{T}{a^3}$$

when $T = T_Y$, $\tau_{max} = k = \frac{\sigma_Y}{\sqrt{3}}$ $\therefore T_Y \approx 1.7\, k a^3$

* exercise: plot contour of $\phi(x,y)$ for circular, elliptic and square cross sections.

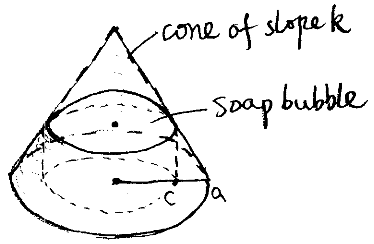
§3. Plastic Torsion $(T > T_Y)$

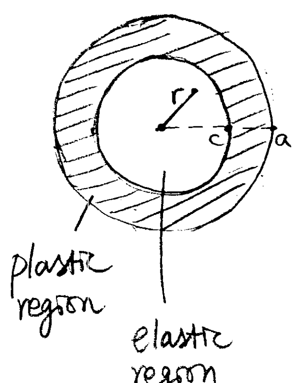
For perfectly plastic material (no hardening), the shear stress τ inside the torsion bar cannot exceed $k = \frac{\sigma_Y}{\sqrt{3}}$ (Von Mises).

In the plastic region $\tau = k$, i.e. $|\nabla\phi(x,y)| = k$.

In other words, the slope of function $\phi(x,y)$ cannot exceed k.

§3.1 Circular cross section





$$r = \sqrt{x^2+y^2}$$

$$\phi(x,y) = k(a-r) \qquad c \le r \le a$$

$$\phi(x,y) = -\frac{\mu\beta}{2}(r^2 - c^2) + k(a-r) \qquad 0 \le r \le c$$

continuity of $d\phi/dr$ at $r = c$

$$\underset{\text{known}}{k} = \mu\beta c \quad (\mu \text{ known};\ \beta, c \text{ unknown}) \qquad \beta = \frac{k}{\mu c}$$

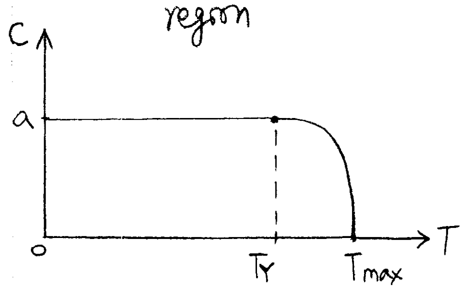
$$T = \int_0^a \left|\frac{\partial\phi}{\partial r}\right| \cdot r \cdot 2\pi r\, dr$$

$$= \int_0^c \mu\beta r^3 \cdot 2\pi\, dr + \int_c^a k \cdot r^2 \cdot 2\pi\, dr$$

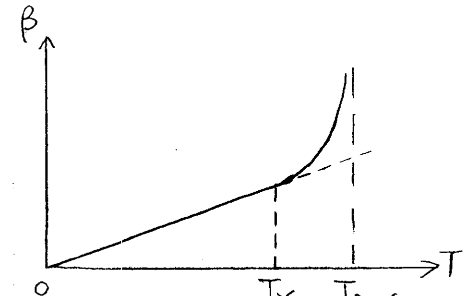
$$= \frac{\pi}{2}\mu\beta c^4 + \frac{2\pi}{3}k(a^3 - c^3)$$

$$= \frac{\pi}{2}kc^3 + \frac{2\pi}{3}k(a^3 - c^3)$$

$$= \pi k\left(\frac{2}{3}a^3 - \frac{1}{6}c^3\right) = \frac{\pi k}{6}(4a^3 - c^3)$$



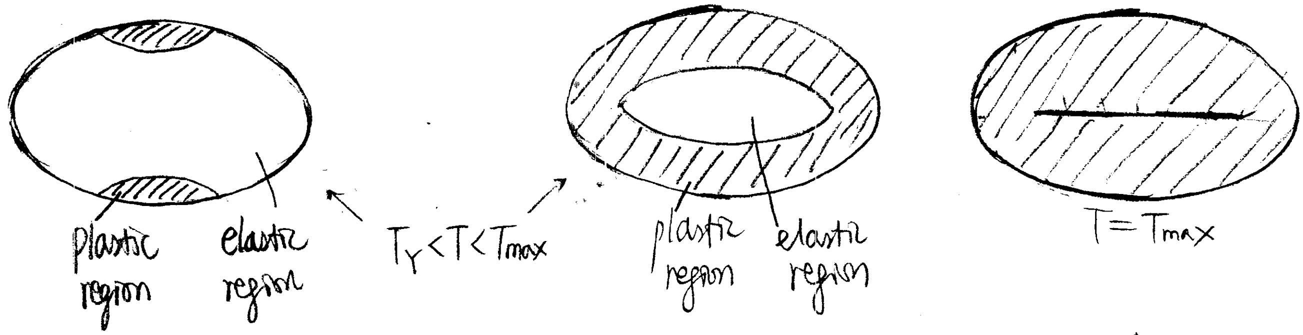
$$c = \left(4a^3 - \frac{6T}{\pi k}\right)^{1/3}, \qquad c = 0 \text{ at } T = T_{max} = \frac{2\pi}{3}ka^3$$

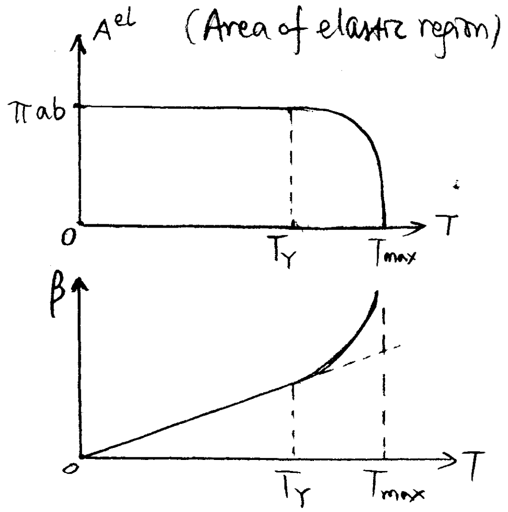


$$\beta = \frac{k}{\mu}\left(4a^3 - \frac{6T}{\pi k}\right)^{-1/3} \qquad \boxed{\frac{T_{max}}{T_Y} = \frac{4}{3}}$$

*compare with bending

§3.2 Elliptic cross section



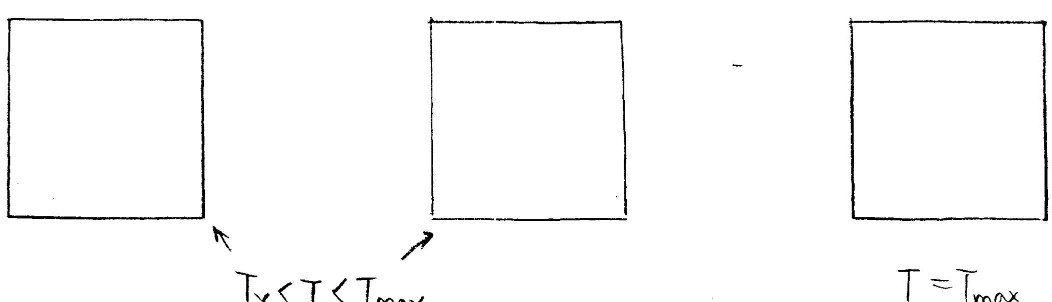


(Numerical solution required)

§3.3 Square cross section

(Left as exercise problem.)

sketch elastic & plastic regions



(Numerical solution required)

$$\frac{T_{max}}{T_Y} = \underline{\hspace{4cm}}$$