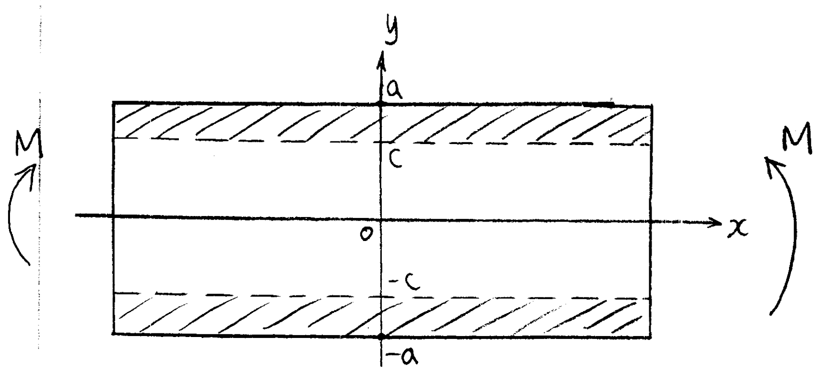
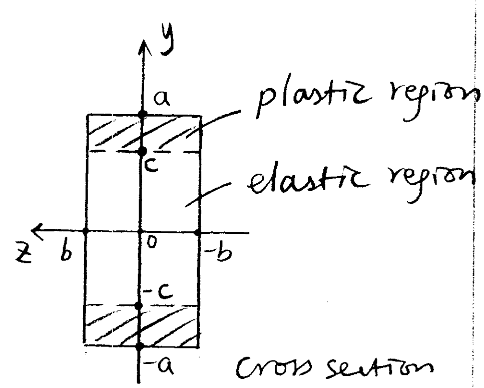
This is our first example in which plastic and elastic regions coexist in the specimen.

This example is taken from Hill's book "The Mathematical Theory of Plasticity", IV.7 p.81

§1. Problem Statement





For a rectangular beam subjected to pure bending moment M, find the thickness of plastic region $(a-c)$ and bending curvature κ as functions of M.

We expect the plastic region to appear if $M > M_Y$, where M_Y is a threshold value. (onset of yield)

We also expect a maximum value M_{max}, at which the plastic region extends to cover the entire cross section.

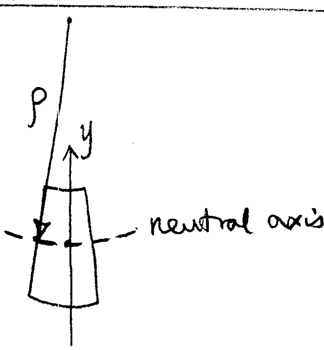
The beam collapses at $M = M_{max}$ and cannot support a greater bending moment.

Find M_{max}/M_Y.

For simplicity, we shall again assume that the material is incompressible ($\nu = 0.5$).

§2. Elastic Bending ($M < M_Y$)

From strength of materials analysis



$$\varepsilon_{xx} = -\frac{y}{\rho} \qquad \left(\rho = \frac{1}{\kappa} : \text{curvature radius of neutral axis}\right)$$

$$\sigma_{xx} = -\frac{Ey}{\rho}, \quad \text{all othere stresses zero}$$

These expressions follow from the assumption that planar cross sections remain planar.

For $\nu = 0.5$, the above expression are exact (see Hill, p. 82)

$$M = \int_{-a}^{a} -\sigma_{xx}\, y\, dy \cdot (2b) = E \cdot \frac{2b}{\rho} \int_{-a}^{a} y^2 dy$$

$$\text{define } I_z = (2b) \cdot \int_{-a}^{a} y^2 dy = (2b)\frac{(2a)^3}{12} = \frac{4a^3 b}{3}$$

$$M = E \cdot \frac{1}{\rho} \cdot I_z$$

$$\kappa = \frac{1}{\rho} = \frac{M}{EI_z}, \qquad \sigma_{xx} = -\frac{My}{I_z}$$

Maximum stress magnitude occurs at $y = \pm a$

take $y = a$, $\sigma_{xx} = -\dfrac{Ma}{I_z}$, $\sigma_{yy} = \sigma_{zz} = 0$

$$J_2 = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] = \frac{\sigma_{xx}^2}{3} = \frac{1}{3}\left(\frac{Ma}{I_z}\right)^2$$

At onset of yield, $M = M_Y$ $J_2 = k^2$

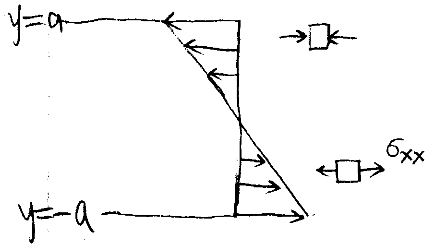
$$\frac{1}{3}\left(\frac{Ma}{I_z}\right)^2 = k^2 \qquad |\sigma_{xx}| = \frac{M_c a}{I_z} = \sqrt{3}\, k = \sigma_Y$$

$$M_Y = \frac{I_z \sigma_Y}{a}$$

The critical curvature at onset of yielding is

$$\kappa_Y = \frac{1}{\rho_Y} = \frac{M_Y}{E I_z} = \frac{\sigma_Y}{E a}$$

$$\rho_Y = \frac{E a}{\sigma_Y}.$$



At $M=M_Y$, the plastic region is still infinitesimal, i.e. $a-c=0$.

Stress field is still linear with y.

§3. Plastic Bending ($M>M_Y$)

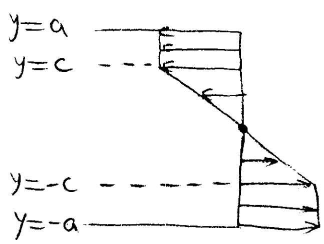
Hill showed that if $\nu=0.5$, then

$\varepsilon_{xx} = -\frac{y}{\rho}$, σ_{xx} is the only non-zero stress

are still true even in the plastic regime ($M>M_c$). (Hill, p.83)

Since σ_{xx} is the only non-zero stress component, every point in the beam is in the state of simple tension (or compression).

Hence
$$\begin{cases} |\sigma_{xx}| = \sigma_Y & \text{in the plastic region: } [-a,-c] \text{ and } [c,a] \\ \sigma_{xx} = -\dfrac{Ey}{\rho} & \text{in the elastic region: } [-c, c] \end{cases}$$



For continuity at $y=c$, $\frac{Ec}{\rho} = \sigma_Y$

$$M = \int_{-a}^{a} -\sigma_{xx}\cdot y\,dy\cdot(2b) = \left(\int_{-c}^{c}\frac{Ey^2}{\rho}dy + 2\int_c^a \sigma_Y\cdot y\,dy\right)\cdot(2b)$$

$$= \left[\frac{E}{\rho}\frac{(2c)^3}{12} + \frac{Ec}{\rho}(a^2-c^2)\right]\cdot(2b)$$

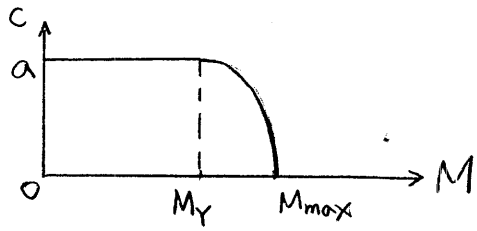
$$= \frac{Ec}{\rho}\left(a^2-\frac{c^2}{3}\right)\cdot 2b = \sigma_Y\left(a^2-\frac{c^2}{3}\right)\cdot 2b$$

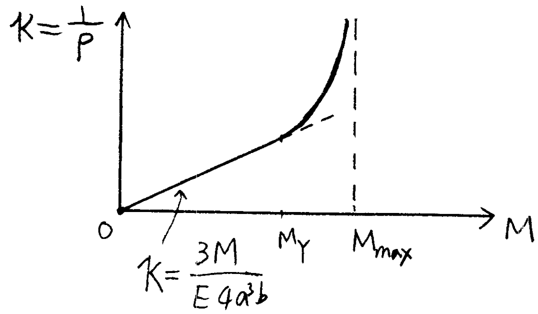
$$C = \sqrt{3} \cdot \sqrt{a^2 - \frac{M}{\sigma_Y \cdot 2b}} \qquad \text{for } M > M_c = \frac{I_z \sigma_Y}{a}$$

$$K = \frac{1}{\rho} = \frac{\sigma_Y}{EC} = \frac{\sigma_Y}{E \cdot \sqrt{3} \cdot \sqrt{a^2 - M/(\sigma_Y \cdot 2b)}}$$

$M = M_{max}$ when $C = 0$, $a^2 = \frac{M_{max}}{\sigma_Y \cdot 2b}$, $\quad M_{max} = \sigma_Y \cdot 2a^2 b$

$$\frac{M_{max}}{M_c} = \frac{\sigma_Y \cdot 2a^2 b}{\sigma_Y \cdot \frac{I_z}{a}} = \frac{2a^2 b}{\frac{4}{3} a^2 b} = \frac{3}{2}$$

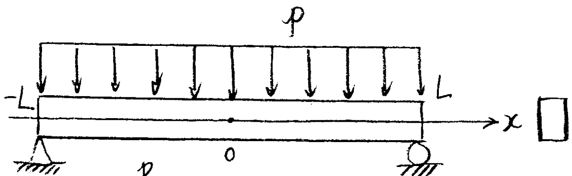


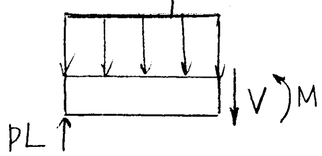


§4. Simply supported beam under uniform load

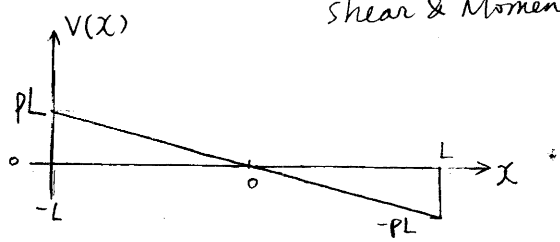
The result obtained above can be used to study a beam in which the internal moment is not uniform.

(Prager & Hodge, §7. P.44)





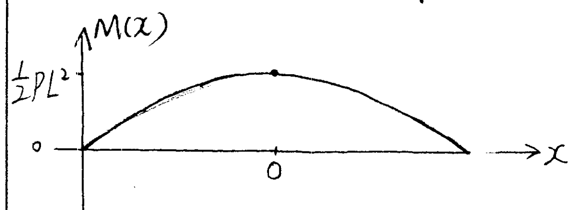
Shear & Moment diagram



$V(-L) = pL, \quad V(L) = -pL$

$V'(x) = -p$

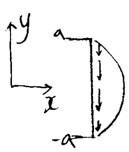
$V(x) = -px$



$M(-L) = 0, \quad M(L) = 0$

$M'(x) = V(x)$

$M(x) = \frac{1}{2}p(L^2 - x^2)$



shear force gives rise to σ_{xy}

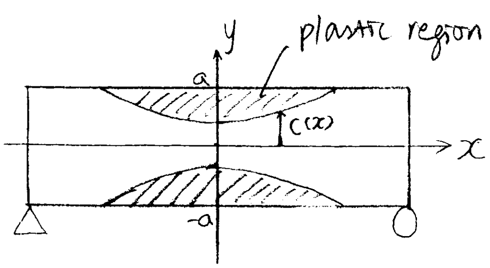
if entire beam is in elastic regime,

$$\sigma_{xy} = -V(x) \cdot \frac{3}{2}(a^2 - y^2)$$

bending moment gives rise to σ_{xx}.

if entire beam is in elastic regime,

$$\sigma_{xx} = -\frac{M(x)\,y}{I_z}$$



yield condition: $J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$

in practice: $|\sigma_{xy}| \ll |\sigma_{xx}|$

$\therefore J_2 \approx \frac{1}{3}\sigma_{xx}^2$

Hence yield condition is the same as in §2.

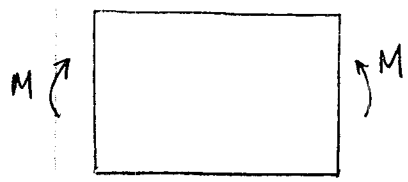
Assuming the maximum bending moment ($\frac{1}{2}pL^2$ at $x=0$) already exceeds M_Y, then in the plastic section (see §3, p.3)

$$M(x) = \sigma_Y\left(a^2 - \frac{c^2}{3}\right)2b = \frac{1}{2}p(L^2 - x^2)$$

$$c(x) = \sqrt{3} \cdot \sqrt{a^2 - \frac{p}{4b\sigma_Y}(L^2 - x^2)}$$

(* exercise: find deflection of the beam's neutral axis)

§5. Unloading



Consider again a beam in pure bending

Suppose M has increased from 0 to $M_1 > M_Y$

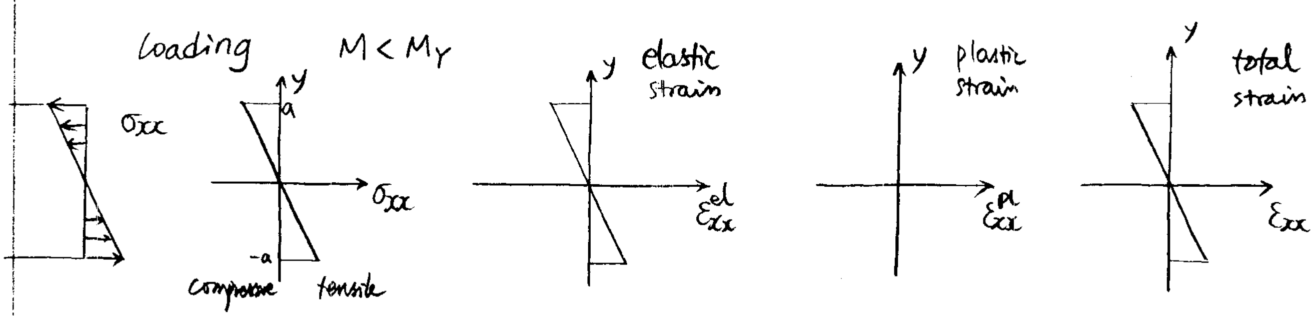
We now let M decrease from M_1 back to 0.

Q: What is the residual stress in the beam?

What is the curvature of the beam when it is fully unloaded ($M=0$)?

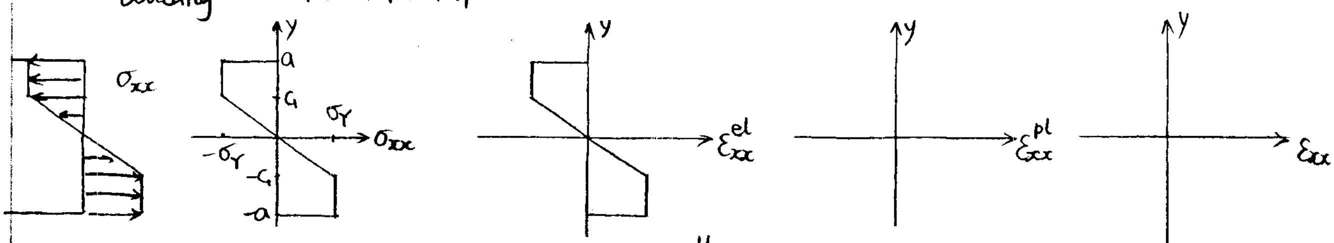
Is the residual stress strong enough to cause plastic flow (in the reverse direction)?

Loading $M < M_Y$



$\sigma_{xx} = -\frac{My}{I_z}$ $\qquad \varepsilon^{el}_{xx} = -\frac{My}{EI_z}$ $\qquad \varepsilon^{pl}_{xx} =$ $\qquad \varepsilon_{xx} =$

Loading $M = M_1 > M_Y$

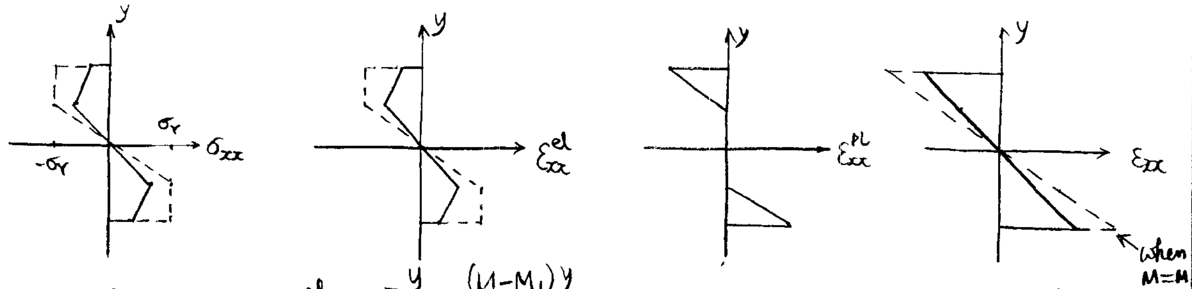


$$\sigma_{xx} = \begin{cases} -\frac{Ey}{\rho_1} & -c_1 \le y \le c_1 \\ \pm\sigma_Y & |y| > c_1 \end{cases} \qquad \varepsilon^{el}_{xx} = \begin{cases} -\frac{y}{\rho_1} \\ \pm\frac{\sigma_Y}{E} \end{cases} \qquad \varepsilon^{pl}_{xx} = \Big\{ \qquad \varepsilon_{xx} = -\frac{y}{\rho_1}$$

$$c_1 = \sqrt{3}\cdot\sqrt{a^2 - \frac{M_1}{\sigma_Y \cdot 2b}}$$

$$\frac{1}{\rho_1} = \frac{\sigma_Y}{E c_1} = \frac{\sigma_Y}{E\sqrt{3}\cdot\sqrt{a^2 - M_1/(\sigma_Y\cdot 2b)}}$$

Unloading $0 < M < M_1$



$$\sigma_{xx} = E \cdot \varepsilon_{xx}^{el}$$

$$\varepsilon_{xx}^{el} = \begin{cases} -\dfrac{y}{\rho_1} - \dfrac{(M-M_1)\,y}{E I_z} \\ \pm \dfrac{\sigma_Y}{E} - \dfrac{(M-M_1)\,y}{E I_z} \end{cases}$$

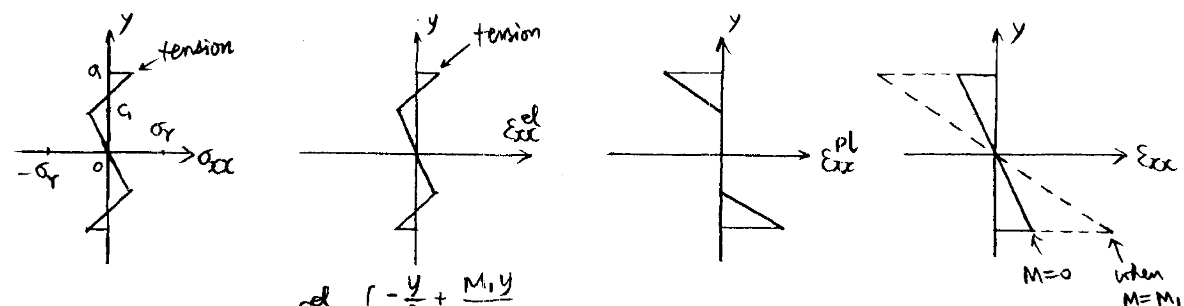
$$\varepsilon_{xx} = -\frac{y}{\rho_1} - \frac{(M-M_1)\,y}{E I_z}$$

During unloading, as long as stress stay below σ_Y,

ε_{xx}^{pl} remains unchanged.

all strain changes are accommodated by ε_{xx}^{el}

Unloading $M = 0$



$$\sigma_{xx} = E \cdot \varepsilon_{xx}^{el}$$

$$\varepsilon_{xx}^{el} = \begin{cases} -\dfrac{y}{\rho_1} + \dfrac{M_1 y}{E I_z} \\ \pm \dfrac{\sigma_Y}{E} + \dfrac{M_1 y}{E I_z} \end{cases}$$

$$\varepsilon_{xx} = -\frac{y}{\rho_1} + \frac{M_1 y}{E I_z} = -\frac{y}{\rho_0}$$

residual curvature
$$\frac{1}{\rho_0} = \frac{1}{\rho_1} - \frac{M_1}{E I_z} = \frac{\sigma_Y}{E \cdot \sqrt{3} \cdot \sqrt{a^2 - M_1/(\sigma_Y \cdot 2b)}} - \frac{M_1}{E \cdot I_z}$$

residual stress
$$\sigma_{xx}\Big|_{y=c_1} = -\sigma_Y + \frac{M_1 c_1}{I_z}$$

$$\sigma_{xx}\Big|_{y=a} = -\sigma_Y + \frac{M_1 a}{I_z}$$

Recall $M_Y \leq M_1 \leq M_{max}$ $\quad (M_{max} = \frac{3}{2} M_Y)$ $\quad (M_Y = \frac{I_z \sigma_Y}{a})$

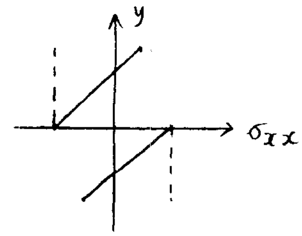
When $M_1 = M_Y$ (lower limit) $c_1 = a$.

$$\sigma_{xx}\big|_{y=c_1} = \sigma_{xx}\big|_{y=a} = -\sigma_Y + \frac{M_Y a}{I_z} = 0$$ → no residual stress

When $M_1 = M_{max}$ (upper limit) $c_1 = 0$.

$$\sigma_{xx}\big|_{y=0} = -\sigma_Y$$

$$\sigma_{xx}\big|_{y=a} = -\sigma_Y + \frac{\frac{3}{2} I_z \sigma_Y}{a} \cdot \frac{a}{I_z} = \frac{1}{2}\sigma_Y$$



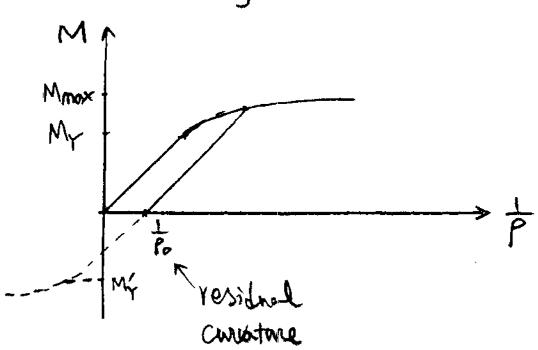
In general, for $M_Y < M < M_{max}$

$$-\sigma_Y < \sigma_{xx}\big|_{y=c_1} < 0$$

$$0 < \sigma_{xx}\big|_{y=a} < \frac{\sigma_Y}{2}$$

Hence plastic flow $\left\{ \begin{array}{l} \text{should} \\ \text{should not} \end{array} \right\}$ (choose one) occur during unloading

Q: After unloading, suppose we apply a bending moment in the reverse direction, at which bending moment M_Y' will plastic flow occur again? Where will yield occur first?



(Bauschinger effect if $|M_Y'| < M_Y$)