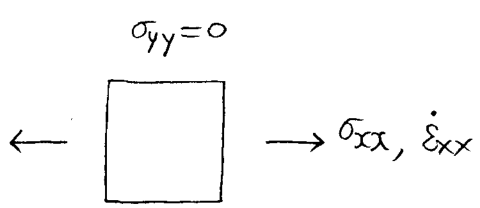
We now work through an example problem that requires a numerical method, to illustrate how numerical methods can be applied to plasticity problems.

§1. Problem Statement



Plane strain condition: $\varepsilon_{zz}=0$

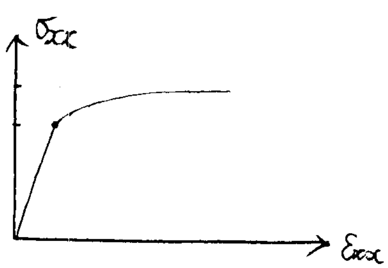
The material is loaded along x

y-surface is traction free: $\sigma_{yy}=0$

Assume von Mises yield criterion, no hardening

$\nu < 0.5$

(If $\nu=0.5$, i.e. incompressible, then the problem can be solved analytically.
The condition $\nu<0.5$ is what makes numerical method necessary)



Find the stress-strain relation

$\sigma_{xx}(\varepsilon_{xx})$

§2. Elastic Regime

Before yield occurs, the stress-strain relation is linear

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \cancel{\frac{\nu}{E}\sigma_{yy}} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \cancel{\frac{1}{E}\sigma_{yy}} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \cancel{\frac{\nu}{E}\sigma_{yy}} - \frac{\nu}{E}\sigma_{zz} = 0$$

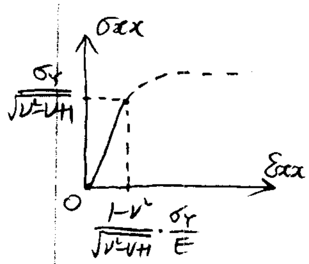
$$\therefore \quad \sigma_{zz} = \nu\,\sigma_{xx}, \qquad \bar{\sigma} = \frac{1+\nu}{3}\sigma_{xx}$$

$$s_{xx} = \sigma_{xx} - \bar{\sigma} = \frac{2-\nu}{3}\sigma_{xx}, \qquad s_{yy} = \sigma_{yy} - \bar{\sigma} = -\frac{1+\nu}{3}\sigma_{xx},$$

$$s_{zz} = \sigma_{zz} - \bar{\sigma} = \frac{2\nu-1}{3}\sigma_{xx}$$

$$J_2 = \frac{1}{2}\left(s_{xx}^2 + s_{yy}^2 + s_{zz}^2\right) = \frac{\sigma_{xx}^2}{3}\left(\nu^2 - \nu + 1\right)$$

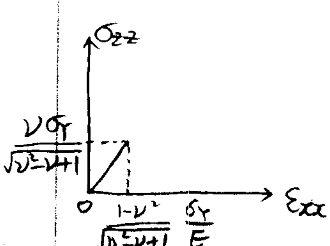
At onset of yield, $J_2 = k^2 = \dfrac{\sigma_Y^2}{3}$

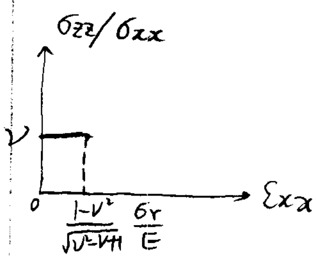


$$\sigma_{xx} = \frac{\sigma_Y}{\sqrt{\nu^2-\nu+1}}$$

(This is higher than σ_Y due to the constraint $\varepsilon_{zz} = 0$.)

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{zz}\right) = \frac{1-\nu^2}{E}\sigma_{xx} = \frac{1-\nu^2}{\sqrt{\nu^2-\nu+1}}\cdot\frac{\sigma_Y}{E}$$





The ratio $\dfrac{\sigma_{zz}}{\sigma_{xx}}$ remains constant ν within the elastic regime.

§3. Plastic Regime

Our strategy is to choose the 'principal' unknowns of the problem, express every other unknowns in terms of them, and find sufficient number of equations to solve for the 'principal' unknowns.

We shall choose three 'principal' unknowns: σ_{xx}, σ_{zz}, $\frac{\tilde{\lambda}}{2\mu}$

Every other quantity can be expressed in terms of these three variables and other known quantities.

We then need to identify three equations to solve the problem.

Recall $\sigma_{yy} = 0$

$$\bar{\sigma} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3}(\sigma_{xx} + \sigma_{zz})$$

$$S_{xx} = \sigma_{xx} - \bar{\sigma}, \quad S_{yy} = -\bar{\sigma}, \quad S_{zz} = \sigma_{zz} - \bar{\sigma}$$

$$\bar{\varepsilon}^{el} = \frac{\bar{\sigma}}{3K}, \quad \text{where } K = \frac{E}{3(1-2\nu)} \text{ is bulk modulus}$$

$$\varepsilon^{el}_{xx} = \bar{\varepsilon}^{el} + e^{el}_{xx} = \frac{\bar{\sigma}}{3K} + \frac{S_{xx}}{2\mu}, \quad \varepsilon^{el}_{yy} = \frac{\bar{\sigma}}{3K} + \frac{S_{yy}}{2\mu}, \quad \varepsilon^{el}_{zz} = \frac{\bar{\sigma}}{3K} + \frac{S_{zz}}{2\mu}$$

$$\dot{\varepsilon}^{pl}_{xx} = \frac{\tilde{\lambda}}{2\mu} S_{xx}, \quad \dot{\varepsilon}^{pl}_{yy} = \frac{\tilde{\lambda}}{2\mu} S_{yy}, \quad \dot{\varepsilon}^{pl}_{zz} = \frac{\tilde{\lambda}}{2\mu} S_{zz}$$

Given that plasticity is path dependent, we need to solve the problem in small time increments

Assume we know everything at time t:

$\sigma_{xx}(t)$, $\sigma_{zz}(t)$, etc

and we need to find the solution at $t+\Delta t$:

$\sigma_{xx}(t+\Delta t)$, $\sigma_{zz}(t+\Delta t)$, etc.

We shall also treat $\varepsilon_{xx}(t+\Delta t)$, $\varepsilon_{zz}(t+\Delta t)$ as known quantities because we wish to express everything in terms of ε_{xx}, and the plane strain condition requires $\varepsilon_{zz}=0$.

Known: $\sigma_{xx}(t)$, $\sigma_{zz}(t)$, $\varepsilon_{xx}(t)$, $\varepsilon_{zz}(t)$, $\varepsilon_{xx}(t+\Delta t)$, $\varepsilon_{zz}(t+\Delta t)$,

unknown: $\sigma_{xx}(t+\Delta t)$, $\sigma_{zz}(t+\Delta t)$, $\frac{\tilde{\lambda}}{2\mu}$ (in the period $[t, t+\Delta t]$)

The change of stress from t to $t+\Delta t$ is

$$\Delta\bar{\sigma} = \bar{\sigma}(t+\Delta t) - \bar{\sigma}(t), \quad \text{where } \bar{\sigma}(t+\Delta t) = \frac{1}{3}[\sigma_{xx}(t+\Delta t) + \sigma_{zz}(t+\Delta t)]$$

$$\bar{\sigma}(t) = \frac{1}{3}[\sigma_{xx}(t) + \sigma_{zz}(t)]$$

$$\Delta S_{xx} = S_{xx}(t+\Delta t) - S_{xx}(t)$$

$$\Delta S_{zz} = S_{zz}(t+\Delta t) - S_{zz}(t)$$

This leads to a change of elastic strain

$$\Delta\varepsilon_{xx}^{el} = \frac{\Delta\bar{\sigma}}{3K} + \frac{\Delta S_{xx}}{2\mu}$$

$$\Delta\varepsilon_{zz}^{el} = \frac{\Delta\bar{\sigma}}{3K} + \frac{\Delta S_{zz}}{2\mu}$$

The plastic strain increment from t to $t+\Delta t$ can be approximated by

$$\Delta \varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} \cdot \frac{s_{xx}(t) + s_{xx}(t+\Delta t)}{2}$$

$$\Delta \varepsilon_{zz}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} \cdot \frac{s_{zz}(t) + s_{zz}(t+\Delta t)}{2}$$

Notice that the average deviatoric stress during period $[t, t+\Delta t]$ is approximated by the average value at t and $t+\Delta t$.

This corresponds to the Trapezoid rule of numerical quadrature

Compared with an alternative approximation based on forward Euler

$$\Delta \varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} s_{xx}(t), \quad \Delta \varepsilon_{zz}^{pl} = \frac{\tilde{\lambda}\Delta t}{2\mu} s_{zz}(t)$$

the Trapezoid (or mid-point) rule is not only more accurate, but also (more importantly) more stable (hence tolerating larger Δt).

Now that everything is expressed in terms of $\sigma_{xx}(t+\Delta t)$, $\sigma_{zz}(t+\Delta t)$, $\frac{\tilde{\lambda}}{2\mu}$ we are ready to write down the three equations:

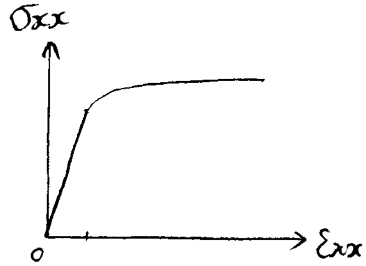
$$\begin{cases} \varepsilon_{xx}(t) + \Delta \varepsilon_{xx}^{el} + \Delta \varepsilon_{xx}^{pl} - \varepsilon_{xx}(t+\Delta t) = 0 & \text{(imposed strain)} \\ \varepsilon_{zz}(t) + \Delta \varepsilon_{zz}^{el} + \Delta \varepsilon_{zz}^{pl} - \varepsilon_{zz}(t+\Delta t) = 0 & \text{(imposed strain)} \\ \frac{1}{2}\left[s_{xx}^2(t+\Delta t) + s_{yy}^2(t+\Delta t) + s_{zz}^2(t+\Delta t) \right] - k^2 = 0 & \text{(yield condition)} \end{cases}$$

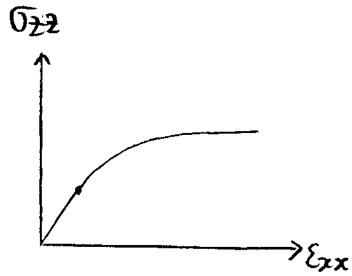
This is a set of 3 non-linear equations for 3 unknowns.

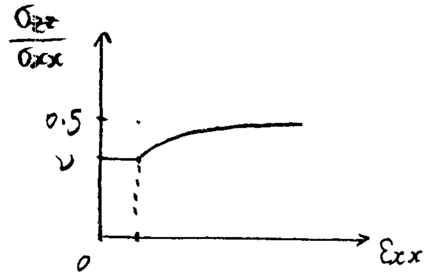
It can be conveniently solved by the 'fsolve' command in Matlab.

The Matlab files are attached below (also on Coursework).

eqns_plane_strain_uniaxial_plast.m implements the 3 equations which are solved by calling 'fsolve' in plane_strain_uniaxial.m







From the numerical solution, we can see that $\frac{\sigma_{zz}}{\sigma_{xx}}$ transitions from ν to 0.5 during the plastic regime.

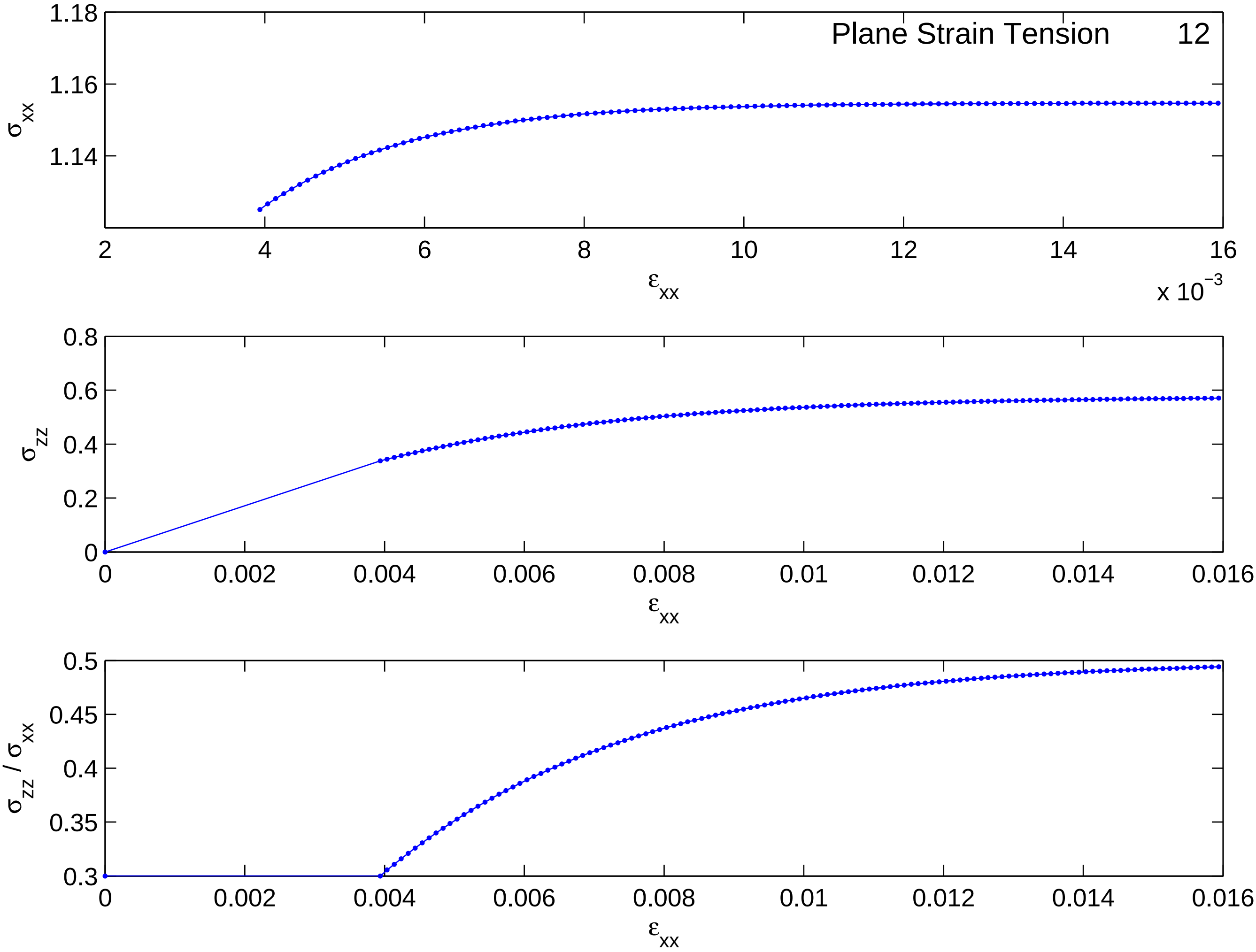
```
% ME342 Theory and Applications of Inelasticity
% Wei Cai, caiwei@stanford.edu
%
% Example: plane strain uniaxial loading

% Material parameters
mu = 100;                       % shear modulus
nu = 0.3;                       % Poisson's ratio
sig_Y = 1;                      % yield stress

K = 2*mu*(1+nu)/3/(1-2*nu); % bulk modulus
E = 2*mu*(1+nu);                % Young's modulus
k = sig_Y/sqrt(3);

% condition at yield
sig_xx = sig_Y/sqrt(nu^2-nu+1); % see HW 1.1(a)
sig_yy = 0;
sig_zz = nu*sig_xx;

% hydrostatic stress
sig_bar   = (sig_xx + sig_yy + sig_zz)/3;

% deviatoric stress
s_xx = sig_xx - sig_bar;
s_yy = sig_yy - sig_bar;
s_zz = sig_zz - sig_bar;
J2 = (s_xx^2 + s_yy^2 + s_zz^2)/2;  % J2 should equal k^2 at yield

% strain (elastic at onset of yield)
eps_xx = sig_bar/(3*K) + s_xx/(2*mu);
eps_yy = sig_bar/(3*K) + s_yy/(2*mu);
eps_zz = sig_bar/(3*K) + s_zz/(2*mu);

% strain array (going beyond yield point)
eps_xx_data = eps_xx + [0:120]*1e-4;

% initialize arrays to store results
sig_xx_data = zeros(size(eps_xx_data));
sig_zz_data = zeros(size(eps_xx_data));
lambda_dt_over_2mu_data = zeros(size(eps_xx_data));
s_xx_data = zeros(size(eps_xx_data));
s_yy_data = zeros(size(eps_xx_data));
s_zz_data = zeros(size(eps_xx_data));
J2_data = zeros(size(eps_xx_data));

% the first data point is the yield point considered above
sig_xx_data(1) = sig_xx;
sig_zz_data(1) = sig_zz;
s_xx_data(1)   = s_xx;
s_yy_data(1)   = s_yy;
s_zz_data(1)   = s_zz;
```

```
J2_data(1)     = J2;

for i = 2:length(eps_xx_data),
    param = [mu, nu, k, sig_xx_data(i-1), sig_zz_data(i-1), ...
             eps_xx_data(i-1), eps_xx_data(i)];
    %options = optimset('Display','iter','TolFun',1e-10);
    options = optimset('Display','off','TolFun',1e-10);
    trial = [sig_xx_data(i-1), sig_zz_data(i-1), 0];
    sol = fsolve('eqns_plane_strain_uniaxial_plast', trial, options, param);
    sig_xx = sol(1); sig_zz = sol(2); lambda_dt_over_2mu = sol(3);

    % compute hydrostatic stress and deviatoric stress
    sig_bar = (sig_xx + sig_zz)/3;
    s_xx = sig_xx - sig_bar;
    s_yy = sig_yy - sig_bar;
    s_zz = sig_zz - sig_bar;
    J2 = (s_xx^2 + s_yy^2 + s_zz^2)/2;

    % save results to array
    sig_xx_data(i) = sig_xx; sig_zz_data(i) = sig_zz;
    lambda_dt_over_2mu_data(i) = lambda_dt_over_2mu;
    s_xx_data(i) = s_xx; s_yy_data(i) = s_yy; s_zz_data(i) = s_zz;
    J2_data(i) = J2;
end

% plot solution
% plot stress
figure(1);
subplot(3,1,1);
plot(eps_xx_data, sig_xx_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('\sigma_{xx}');
subplot(3,1,2);
plot([0 eps_xx_data], [0 sig_zz_data], '.-');
xlabel('\epsilon_{xx}');
ylabel('\sigma_{zz}');
subplot(3,1,3);
plot([0 eps_xx_data], [nu sig_zz_data./sig_xx_data], '.-');
xlabel('\epsilon_{xx}');
ylabel('\sigma_{zz} / \sigma_{xx}');

% plot deviatoric stress
figure(2);
subplot(3,1,1);
plot(eps_xx_data, s_xx_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('s_{xx}');
subplot(3,1,2);
plot(eps_xx_data, s_yy_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('s_{yy}');
```

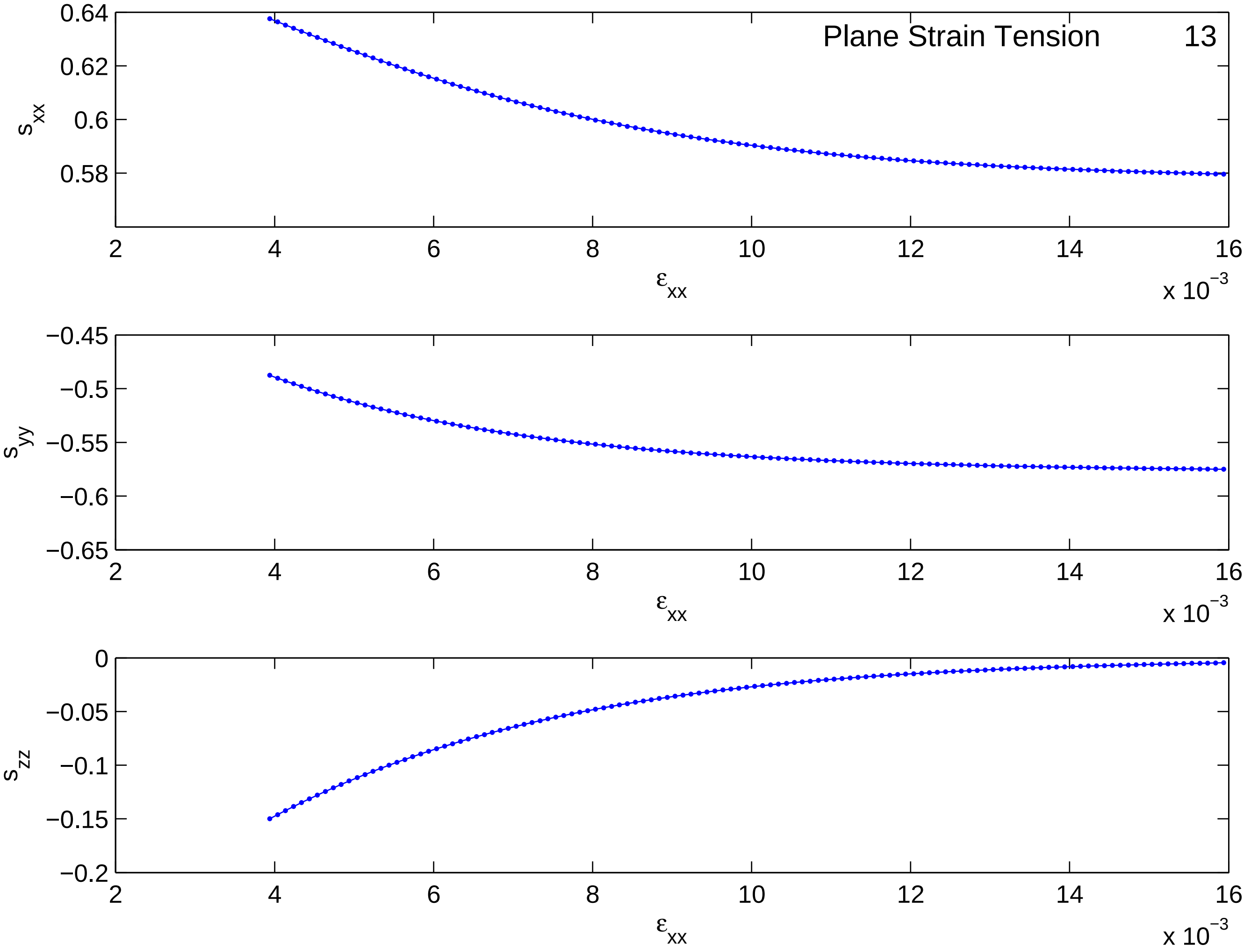
```
subplot(3,1,3);
plot(eps_xx_data, s_zz_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('s_{zz}');

% plot lambda_dt_over_2mu and J2 (should equal k^2)
figure(3);
subplot(2,1,1);
plot(eps_xx_data, [NaN lambda_dt_over_2mu_data(2:end)], '.-');
xlabel('\epsilon_{xx}');
ylabel('\lambda \Delta t / (2\mu)');
subplot(2,1,2);
plot(eps_xx_data, J2_data, '.-');
xlabel('\epsilon_{xx}');
ylabel('J_2');
ylim([0.9 1.1]*k^2);
```

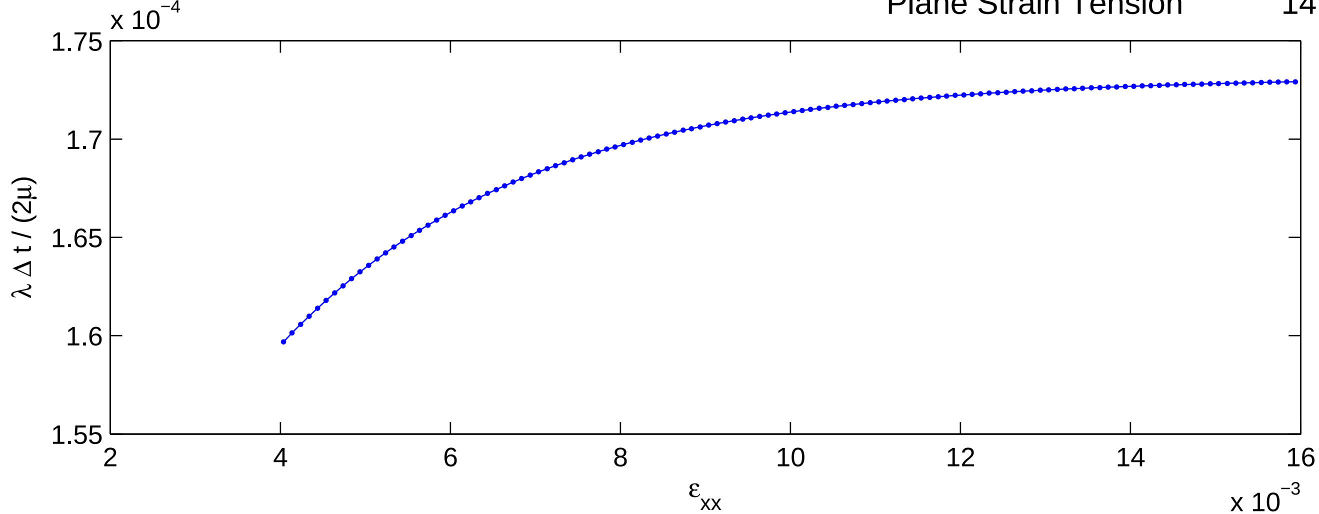
```
% ME342 Theory and Applications of Inelasticity
% Wei Cai, caiwei@stanford.edu
%
% equations to be solved by plane_strain_uniaxial.m

function F = eqns_plane_strain_uniaxial_plast(vars, param)
% F(1): eps_xx_tot (specified)
% F(2): eps_zz_tot = 0
% F(3): yield condition

sig_xx = vars(1);
sig_zz = vars(2);
lambda_dt_over_2mu = vars(3);
sig_yy = 0;

mu = param(1);
nu = param(2);
k  = param(3);
sig_xx_0 = param(4);
sig_zz_0 = param(5);
sig_yy_0 = 0;
eps_xx_0 = param(6); % total strain at previous time step
eps_xx   = param(7); % total strain at current  time step
eps_zz_0 = 0;
eps_zz   = 0;

K = 2*mu*(1+nu)/3/(1-2*nu); % bulk modulus

% change of stress
dsig_xx = sig_xx - sig_xx_0;
dsig_yy = sig_yy - sig_yy_0;
dsig_zz = sig_zz - sig_zz_0;

dsig_bar = (dsig_xx + dsig_yy + dsig_zz)/3;
ds_xx = dsig_xx - dsig_bar;
ds_yy = dsig_yy - dsig_bar;
ds_zz = dsig_zz - dsig_bar;

% increment of elastic strain
deps_elast_xx = dsig_bar/(3*K) + ds_xx/(2*mu);
deps_elast_yy = dsig_bar/(3*K) + ds_yy/(2*mu);
deps_elast_zz = dsig_bar/(3*K) + ds_zz/(2*mu);

% hydrostaic stress
sig_bar   = (sig_xx + sig_yy + sig_zz)/3;
sig_bar_0 = (sig_xx_0 + sig_yy_0 + sig_zz_0)/3;

% deviatoric stress
s_xx = sig_xx - sig_bar;
s_yy = sig_yy - sig_bar;
s_zz = sig_zz - sig_bar;
```

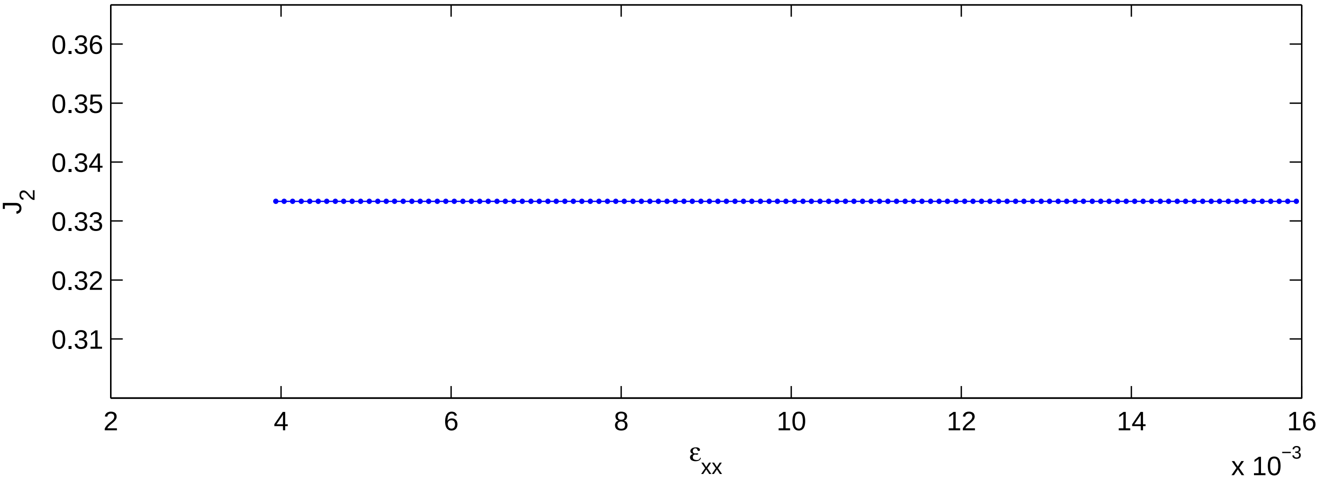
```
s_xx_0 = sig_xx_0 - sig_bar_0;
s_yy_0 = sig_yy_0 - sig_bar_0;
s_zz_0 = sig_zz_0 - sig_bar_0;

% average deviatoric stress
ave_s_xx = (s_xx + s_xx_0)/2;
ave_s_yy = (s_yy + s_yy_0)/2;
ave_s_zz = (s_zz + s_zz_0)/2;

% increment of plastic strain
deps_plast_xx = lambda_dt_over_2mu * ave_s_xx;
deps_plast_yy = lambda_dt_over_2mu * ave_s_yy;
deps_plast_zz = lambda_dt_over_2mu * ave_s_zz;

% construct equation
F = [0 0 0]';

% total strain in xx
F(1) = (eps_xx_0 + deps_elast_xx + deps_plast_xx) - eps_xx;

% total strain in zz
F(2) = (eps_zz_0 + deps_elast_zz + deps_plast_zz) - eps_zz;

% yield condition
J2 = (s_xx^2 + s_yy^2 + s_zz^2)/2;
F(3) = J2 - k^2;
```



Plane Strain Tension
12
σxx
1.18
1.16
1.14
2
4
6
8
10
12
14
16
εxx
x 10−3
σzz
0.8
0.6
0.4
0.2
0
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
εxx
σzz / σxx
0.5
0.45
0.4
0.35
0.3
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
εxx


Plane Strain Tension
13
0.64
0.62
0.6
0.58
s_xx
2
4
6
8
10
12
14
16
ε_xx
x 10^−3
−0.45
−0.5
−0.55
−0.6
−0.65
s_yy
ε_xx
x 10^−3
0
−0.05
−0.1
−0.15
−0.2
s_zz
ε_xx
x 10^−3


x 10^−4
1.75
1.7
1.65
1.6
1.55
λ Δ t / (2μ)
2
4
6
8
10
12
14
16
εxx
x 10^−3


0.36
0.35
0.34
0.33
0.32
0.31
J2
2
4
6
8
10
12
14
16
εxx
x 10^−3