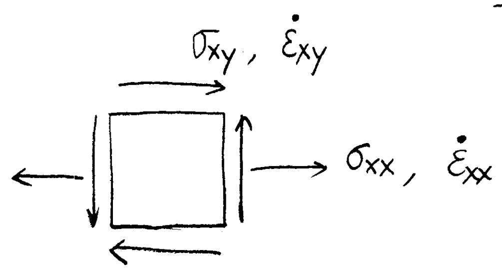
We now work out a simple example problem in which the material is subjected to both tension and shear (which can be achieved in a tube under tension and torsion).

This example illustrates how to solve a plasticity problem, as well as the <u>history-dependent</u> nature of plasticity.

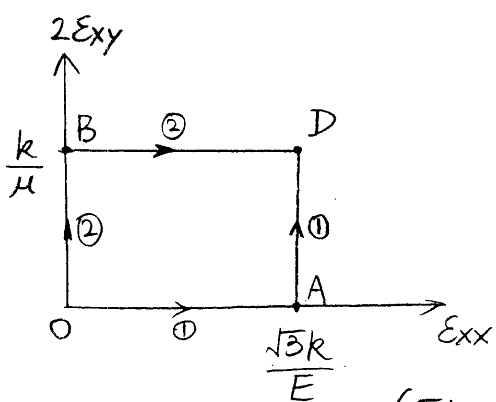
This example is extracted from Dr. Paul Paslay's lecture video "Introduction to Theoretical and Applied Plasticity"
For copies of this DVD, send email to info@blade-energy.com.

§1. Problem Statement



Assume no work hardening, von Mises yield criterion, incompressible material ($\nu=0.5$)

Incompressibility means even the elastic strain ε_{ij}^{el} conserves volume, so that $\varepsilon_{ij}^{el} = e_{ij}^{el}$ (deviatoric)
This simplifies discussions.
Given $\varepsilon_{ij}^{pl} = e_{ij}^{pl}$ (deviatoric)
we have $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl} = e_{ij} = e_{ij}^{el} + e_{ij}^{pl}$



($\sqrt{3}k = \sigma_Y$)

We shall consider <u>two different strain paths</u>:

① OA–AD First load in tension to exactly the yield point then load in shear

② OB–BD First load in shear to exactly the yield point then load in tension

Note: we have assumed incompressibility ($\nu=0.5$)

So $E = 2\mu(1+\nu) = 3\mu$

The goal is to find the stress (as a function of strain) along these two strain paths.

§2. Strain Path ①: Tension – Shear

Obviously, along path OA, all strains are elastic.

$$\sigma_{xx} = E \cdot \varepsilon_{xx} \quad (\text{goes from } 0 \text{ to } \sqrt{3}k = \sigma_Y)$$

$$\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$$

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\,\varepsilon_{xx} = -\frac{1}{2}\varepsilon_{xx}$$

Along path AD, the stress need to stay on yield surface

i.e. $J_2 = k^2$, $\dot{J}_2 = 0$

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2} s_{ij}$$

$\dot{W} = s_{ij}\dot{e}_{ij}$: rate of work done associated with shape change

$$\dot{W}_{tot} \equiv \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_{xy}\cdot 2\dot{\varepsilon}_{xy}$$

(In this example, $\dot{W}$ is also to rate of work done because volume stays constant)

simplification due to incompressibility $\longrightarrow$ $\therefore\ \dot{W} = \sigma_{xy}\cdot 2\dot{\varepsilon}_{xy}$

Since $\sigma_{yy}=\sigma_{zz}=0$, $\bar{\sigma} = \frac{\sigma_{xx}}{3}$, $s_{xx} = \frac{2}{3}\sigma_{xx}$, $s_{yy} = -\frac{1}{3}\sigma_{xx}$, $s_{zz} = -\frac{1}{3}\sigma_{xx}$

$$J_2 = \frac{1}{2}(s_{xx}^2 + s_{yy}^2 + s_{zz}^2) + (s_{yz}^2 + s_{zx}^2 + s_{xy}^2)$$

$$= \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$$

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{xy}^{el} + \dot{\varepsilon}_{xy}^{pl} \underset{(\text{incompressibility})}{=} \dot{e}_{xy}^{el} + \dot{\varepsilon}_{xy}^{pl} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{W}}{2k^2}\sigma_{xy}$$

$$\dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{\varepsilon}_{xy}}{k^2}\sigma_{xy}^2$$

$$\left(1-\frac{\sigma_{xy}^2}{k^2}\right)\dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu}$$

$$2\mu\,\frac{\dot{\varepsilon}_{xy}}{k} = \frac{\dfrac{\dot{\sigma}_{xy}}{k}}{1-\left(\dfrac{\sigma_{xy}}{k}\right)^2}$$

The solution to this equation is

$$2\mu\,\frac{\varepsilon_{xy}(t)}{k} = \operatorname{arctanh}\left(\frac{\sigma_{xy}(t)}{k}\right),$$

$$\left(\frac{d}{dx}\operatorname{arctanh}(x) = \frac{1}{1-x^2}\right)$$

i.e.

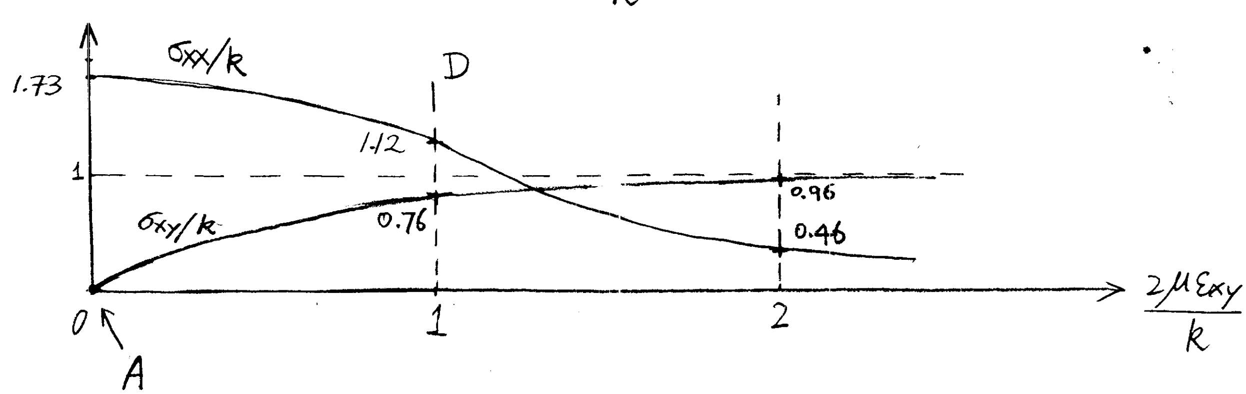
$$\frac{\sigma_{xy}(t)}{k} = \tanh\left(2\mu\,\frac{\varepsilon_{xy}(t)}{k}\right)$$

Given that $\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$

$$\frac{\sigma_{xx}}{k} = \sqrt{3}\cdot\sqrt{1-\left(\frac{\sigma_{xy}}{k}\right)^2}$$

$$\left(1-[\tanh(x)]^2 = \frac{(\cosh x)^2-(\sinh x)^2}{(\cosh x)^2} = \frac{1}{(\cosh x)^2}\right)$$

$$\frac{\sigma_{xx}(t)}{k} = \frac{\sqrt{3}}{\cosh\left(\frac{2\mu\varepsilon_{xy}(t)}{k}\right)}$$



At point A, $\sigma_{xx} = \sqrt{3}k$ and $\sigma_{xy} = 0$

With increasing shear strain, the normal stress σ_{xx} decreases while the shear stress σ_{xy} increases.

At point D, $\boxed{\sigma_{xx} = 1.12k \text{ and } \sigma_{xy} = 0.76k}$

If shear strain keeps on increasing beyond D, eventually

$\sigma_{xx} \to 0$, $\sigma_{xy} \to k$ (yield condition in pure shear)

§3. Strain Path ②: Shear - Tension

Obviously, along path OB, all strains are elastic

$$\sigma_{xy} = 2\mu \cdot \varepsilon_{xy} \quad \text{(goes from } 0 \text{ to } k\text{)}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

Along path BD, the stress need to stay on yield surface

$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$$

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2} s_{ij}, \quad \dot{\varepsilon}_{xx}^{pl} = \frac{\dot{W}}{2k^2} s_{xx} = \frac{\dot{W}}{2k^2}\cdot\frac{2}{3}\sigma_{xx} = \frac{\dot{W}}{3k^2}\sigma_{xx}$$

simplification due to incompressibility $\rightarrow$ $\dot{W} = \dot{W}_{tot} = \sigma_{xx}\dot{\varepsilon}_{xx}$

$$\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{xx}^{el} + \dot{\varepsilon}_{xx}^{pl} = \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{W}}{3k^2}\sigma_{xx}$$

$$\dot{\varepsilon}_{xx} = \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{\varepsilon}_{xx}}{3k^2}\sigma_{xx}^2$$

$$\left(1 - \frac{\sigma_{xx}^2}{3k^2}\right)\dot{\varepsilon}_{xx} = \frac{\dot{\sigma}_{xx}}{E}, \quad E\frac{\dot{\varepsilon}_{xx}}{\sqrt{3}k} = \frac{\frac{\dot{\sigma}_{xx}}{\sqrt{3}k}}{1 - \left(\frac{\sigma_{xx}}{\sqrt{3}k}\right)^2}$$

The solution to this equation is

$$E\frac{\varepsilon_{xx}(t)}{\sqrt{3}\,k} = \operatorname{arctanh}\left(\frac{\sigma_{xx}(t)}{\sqrt{3}\,k}\right), \text{ i.e. } \frac{\sigma_{xx}(t)}{\sqrt{3}\,k} = \tanh\left(E\frac{\varepsilon_{xx}(t)}{\sqrt{3}\,k}\right)$$

Note: $\sqrt{3}\,k = \sigma_Y$, $E = 3\mu$ (incompressibility) $= \tanh\left(\sqrt{3}\mu\frac{\varepsilon_{xx}(t)}{k}\right)$

Given that $\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$

$$\frac{\sigma_{xy}}{k} = \sqrt{1 - \left(\frac{\sigma_{xx}}{\sqrt{3}k}\right)^2}$$

$$\frac{\sigma_{xy}(t)}{k} = \frac{1}{\cosh\left(E\frac{\varepsilon_{xx}(t)}{\sqrt{3}\,k}\right)} = \frac{1}{\cosh\left(\sqrt{3}\mu\frac{\varepsilon_{xx}(t)}{k}\right)}$$

So along path BD,

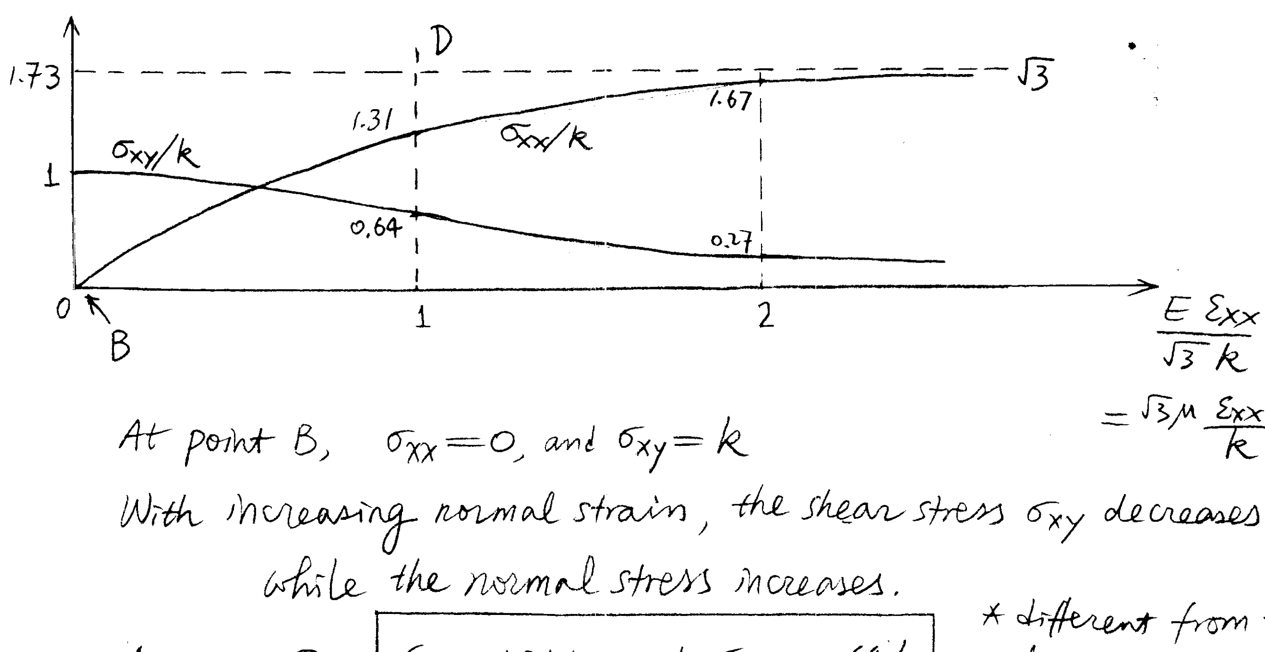
$$\frac{\sigma_{xx}}{k} = \sqrt{3} \tanh\left(\sqrt{3}\mu \frac{\varepsilon_{xx}}{k}\right)$$

$$\frac{\sigma_{xy}}{k} = \frac{1}{\cosh\left(\sqrt{3}\mu \frac{\varepsilon_{xx}}{k}\right)}$$

compared with path AD, the roles of σ_{xx} and σ_{xy} are reversed.

(* exercise: plot plastic strain ε_{xx}^{pl}, ε_{xy}^{pl} as functions of ε_{xx}.)

(* exercise: consider a strain path in which ε_{xx} goes from 0 to $\frac{\sigma_Y}{E}$ and then ε_{yy} goes from 0 to $\frac{\sigma_Y}{E}$, find σ_{xx} and σ_{yy})



At point B, $\sigma_{xx} = 0$, and $\sigma_{xy} = k$

With increasing normal strain, the shear stress σ_{xy} decreases while the normal stress increases.

At point D, $\boxed{\sigma_{xx} = 1.31\,k \text{ and } \sigma_{xy} = 0.64\,k}$

* different from the values on p.3 — history dependence.

If normal strain keeps on increasing beyond D, eventually $\sigma_{xx} \to \sqrt{3}k$ (yield stress in tension), $\sigma_{xy} \to 0$.