We now work out a simple example problem in which the material is subjected to both tension and shear (which can be achieved in a tube under tension and torsion).

This example illustrates how to solve a plasticity problem, as well as the history-dependent nature of plasticity.

This example is extracted from Dr. Paul Paslay's lecture video "Introduction to Theoretical and Applied Plasticity."

For copies of this DVD, send email to info@blade-energy.com.

§1. Problem Statement

Assume no work hardening,

von Mises yield criterion,
incompressible material \( (\nu=0.5) \)

Incompressibility means even the elastic strain \( \varepsilon_{ij}^{el} \) conserves volume, so that \( \varepsilon_{ij}^{el} = \varepsilon_{ij}^{pl} \) (deviatoric).

This simplifies discussions.

Given \( \varepsilon_{ij}^{pl} = \varepsilon_{ij}^{pl} \) (deviatoric),

we have

\[
\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}
\]

We shall consider two different strain paths:

1. \( OA-AD \) First load in tension to exactly the yield point then load in shear

2. \( OB-BD \) First load in shear to exactly the yield point then load in tension
ME342  Tension & Shear  Cai

Note: we have assumed incompressibility (ν=0.5)

So  \( E = 2\mu(1+\nu) = 3\mu \)

The goal is to find the stress (as a function of strain) along these two strain paths.

\[ \sigma_{xx} = E \cdot \varepsilon_{xx} \quad \text{ (goes from 0 to } \frac{1}{3} k = \sigma_y \) \]

\[ \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \]

\[ \varepsilon_{yy} = \varepsilon_{zz} = -\nu \varepsilon_{xx} = -\frac{1}{2} \varepsilon_{xx} \]

\[ \text{Along path AD, the stress need to stay on yield surface} \]

\[ \text{ i.e. } J_2 = k^2, \quad J_3 = 0 \]

\[ \dot{\varepsilon}_{ij} = \frac{W}{2k^2} S_{ij} \]

\[ W = S_{ij} \dot{\varepsilon}_{ij} \text{ : rate of work done associated with shape change} \]

\[ W = G_{ij} \dot{\varepsilon}_{ij} = \sigma_{ij} \cdot 2 \dot{\varepsilon}_{xy} \]

(In this example, \( W \) is also to rate of work done because volume stays constant)

\[ \text{simplification due to incompressibility} \]

\[ \sigma_{yy} = \sigma_{zz} = 0, \quad \overline{\sigma} = \frac{\sigma_{xx}}{3}, \quad s_{xx} = \frac{2}{3} \sigma_{xx}, \quad s_{yy} = -\frac{1}{3} \sigma_{xx}, \quad s_{zz} = -\frac{1}{3} \sigma_{xx} \]

\[ J_2 = \frac{1}{2} (s_{xx}^2 + s_{yy}^2 + s_{zz}^2) + (s_{yz}^2 + s_{zx}^2 + s_{xy}^2) \]

\[ = \frac{1}{3} \sigma_{xx}^2 + \sigma_{xy}^2 = k^2 \]

\[ \dot{\varepsilon}_{xy} = \dot{\varepsilon}_{xy} + \dot{\varepsilon}_{xy} = \dot{\varepsilon}_{xy} + \dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{W}{2k^2} \sigma_{xy} \]

\( \text{(incompressibility)} \)

\[ \dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{\varepsilon}_{xy}}{k^2} \]
\[
\left(1 - \frac{\sigma_{xy}^2}{k^2}\right) \dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu},
\]

\[
2\mu \frac{\dot{\varepsilon}_{xy}}{k} = \frac{\dot{\sigma}_{xy}}{k} \left(1 - \left(\frac{\sigma_{xy}}{k}\right)^2\right)
\]

The solution to this equation is

\[
2\mu \frac{\varepsilon_{xy}(t)}{k} = \operatorname{arctanh} \left(\frac{\sigma_{xy}(t)}{k}\right),
\]

i.e.

\[
\frac{\sigma_{xy}(t)}{k} = \tanh \left(2\mu \frac{\varepsilon_{xy}(t)}{k}\right)
\]

given that \(\frac{1}{3} \sigma_{xx} + \sigma_{xy} = k^2\)

\[
\frac{\sigma_{xx}}{k} = \sqrt{3} \cdot \sqrt{1 - \left(\frac{\sigma_{xy}}{k}\right)^2}
\]

\[
\frac{\sigma_{xx}(t)}{k} = \frac{\sqrt{3}}{\cosh \left(2\mu \frac{\varepsilon_{xy}(t)}{k}\right)}
\]

At point A, \(\sigma_{xx} = \sqrt{3} k\) and \(\sigma_{xy} = 0\)

With increasing shear strain, the normal stress \(\sigma_{xx}\) decreases while the shear stress \(\sigma_{xy}\) increases.

At point D, \(\sigma_{xx} = 1/2 k\) and \(\sigma_{xy} = 0.76 k\)

If shear strain keeps on increasing beyond D, eventually

\(\sigma_{xx} \to 0, \quad \sigma_{xy} \to k\) (yield condition in pure shear)
§3. Strain Path 2: Shear-Tension

Obviously, along path OB, all strains are elastic

\[ \sigma_{xy} = 2 \mu \cdot \varepsilon_{xy} \quad \text{(goes from 0 to k)} \]

\[ \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{zt} = \sigma_{tx} = 0 \]

\[ \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zt} = \varepsilon_{tx} = 0 \]

Along path BD, the stress need to stay on yield surface

\[ J_2 = \frac{1}{3} \sigma_{xx}^2 + \sigma_{xy}^2 = k^2 \]

\[ \dot{\varepsilon}_{ij}^p = \frac{W}{2k^2} S_{ij}, \quad \dot{\varepsilon}_{xx}^p = \frac{W}{2k^2} S_{xx} = \frac{W}{2k^2} \cdot \frac{2}{3} \sigma_{xx} = \frac{W}{3k^2} \sigma_{xx} \]

Simplification due to incompressibility \[ \rightarrow \dot{W} = W_{tot} = \sigma_{xx} \dot{\varepsilon}_{xx} \]

\[ \dot{\varepsilon}_{xx} = \dot{\varepsilon}_{xx}^e + \dot{\varepsilon}_{xx}^p = \frac{\dot{\sigma}_{xx}}{E} + \frac{W}{3k^2} \sigma_{xx} \]

\[ \dot{\varepsilon}_{xx} = \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{\varepsilon}_{xx}}{3k^2} \sigma_{xx}^2 \]

\[ \left( 1 - \frac{\sigma_{xx}^2}{3k^2} \right) \dot{\varepsilon}_{xx} = \frac{\dot{\sigma}_{xx}}{E} \]

\[ E \frac{\dot{\varepsilon}_{xx}(t)}{\sqrt{3} k} = \tanh \left( \frac{\dot{\sigma}_{xx}(t)}{\sqrt{3} k} \right), \quad \text{i.e.} \quad \dot{\sigma}_{xx}(t) = \tanh \left( E \frac{\dot{\varepsilon}_{xx}(t)}{\sqrt{3} k} \right) \]

Note: \[ \sqrt{3} k = \sigma_Y, \quad E = 3 \mu \quad \text{(incompressibility)} \]

The solution to this equation is

Given that \[ \frac{1}{3} \sigma_{xx}^2 + \sigma_{xy}^2 = k^2 \]

\[ \frac{\dot{\sigma}_{xy}}{k} = \sqrt{1 - \left( \frac{\dot{\sigma}_{xx}}{3k} \right)^2} \]

\[ \frac{\dot{\sigma}_{xy}(t)}{k} = \frac{1}{\cosh \left( E \frac{\dot{\varepsilon}_{xx}(t)}{\sqrt{3} k} \right)} = \frac{1}{\cosh \left( \sqrt{3} \mu \frac{\dot{\varepsilon}_{xx}(t)}{k} \right)} \]
So along path BD,

\[
\frac{\sigma_{xx}}{k} = \sqrt{3} \tanh \left( \sqrt{3} \mu \frac{\varepsilon_{xx}}{k} \right)
\]

\[
\frac{\sigma_{xy}}{k} = \frac{1}{\cosh \left( \sqrt{3} \mu \frac{\varepsilon_{xx}}{k} \right)}
\]

compared with path AD, the roles of \( \sigma_{xx} \) and \( \sigma_{xy} \) are reversed.

(*exercise: plot plastic strain \( \varepsilon_{xx}^p \), \( \varepsilon_{xy}^p \) as functions of \( \varepsilon_{xx} \).)

(*exercise: consider a strain path in which \( \varepsilon_{xx} \) goes from 0 to \( \frac{\sigma_{xy}}{E} \), and then \( \varepsilon_{yy} \) goes from 0 to \( \frac{\sigma_{xy}}{E} \), find \( \sigma_{xx} \) and \( \sigma_{xy} \).)

At point B, \( \sigma_{xx} = 0 \) and \( \sigma_{xy} = k \).

With increasing normal strain, the shear stress \( \sigma_{xy} \) decreases while the normal stress increases.

At point D, \( \sigma_{xx} = 1.31 k \) and \( \sigma_{xy} = 0.64 k \).

If normal strain keeps on increasing beyond D, eventually

\( \sigma_{xx} \rightarrow \sqrt{3} k \) (yield stress in tension), \( \sigma_{xy} \rightarrow 0 \).