In this lecture, we give more discussions on the yield conditions, (Von Mises and Tresca), including their graphical representation and experimental verification.

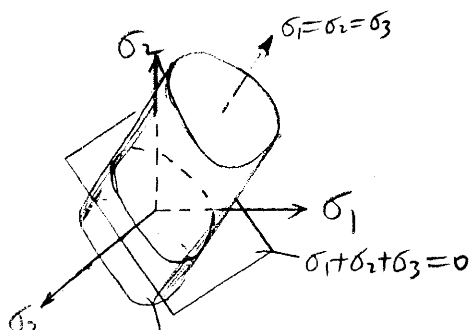
§1 Yield Surface in the space of Principal Stresses

If the material is isotropic, the three pricipal stress values contains all the information about whether the material has reached the yield condition or not.

In other words, the yield condition is reached on a 2D surface, i.e. the yield surface, in the 3D space spanned by σ_1, σ_2, σ_3

$$f(\sigma_1, \sigma_2, \sigma_3) = 0$$

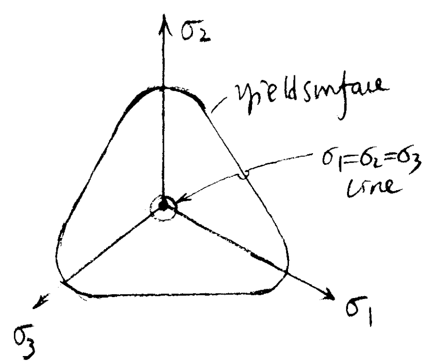
Further more, experiments have shown that hydrostatic stress $\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ plays no role in yield.



yield surface $f(\sigma_1,\sigma_2,\sigma_3)=0$

This means that the yield surface must have a prismatic shape, with the axes along the line defined by $\sigma_1=\sigma_2=\sigma_3$ (i.e. [111] direction)

In other words, the intersection between the yield surface $f(\sigma_1,\sigma_2,\sigma_3)=0$ and any plane defined by $\sigma_1+\sigma_2+\sigma_3=c$ gives a curve of the same shape



$\sigma_1+\sigma_2+\sigma_3=0$ plane

Imagine looking down on the $\sigma_1+\sigma_2+\sigma_3=0$ then the entire yield surface looks like a curve.

In other words, all we need to do is to specify the shape of this curve. The yield surface can be constructed by translating this curve along [111].

Material isotropy require the yield surface be symmetric with respect to 120° rotation around the $\sigma_1=\sigma_2=\sigma_3$ line,

i.e. $\sigma_1 \to \sigma_2$, $\sigma_2 \to \sigma_3$, $\sigma_3 \to \sigma_1$,

as well as mirror reflection, e.g. $\sigma_1 \to \sigma_3$, $\sigma_3 \to \sigma_1$

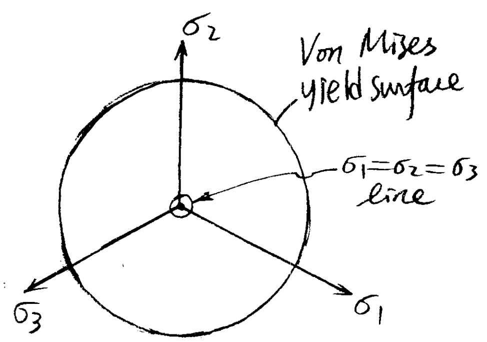
Other restrictions can be imposed on the yield surface. For example, the yield surface is usually assumed to be convex, which we will discuss later.

§2. Von Mises Yield Condition

$$J_2 \equiv \frac{1}{2}(S_1^2+S_2^2+S_3^2) = k^2$$

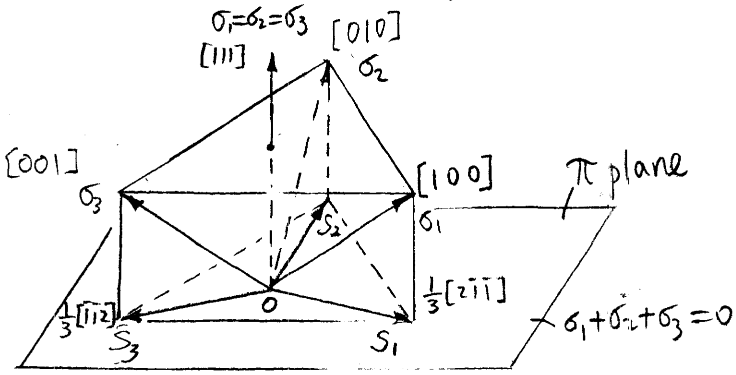
Note $\bar{\sigma} = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$, $S_1 = \sigma_1-\bar{\sigma}$, $S_2 = \sigma_2-\bar{\sigma}$, $S_3 = \sigma_3-\bar{\sigma}$

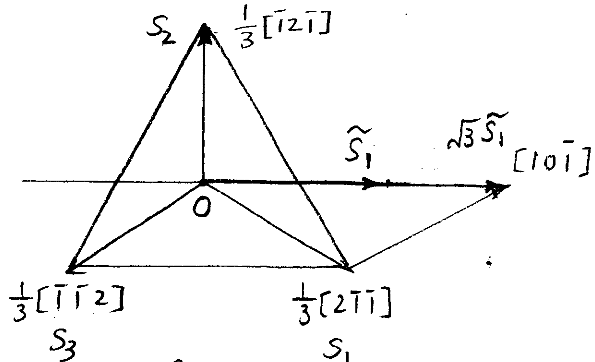
$$\begin{aligned} J_2 &= \frac{1}{2}\left[(\sigma_1-\bar{\sigma})^2+(\sigma_2-\bar{\sigma})^2+(\sigma_3-\bar{\sigma})^2\right] \\ &= \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-3\bar{\sigma}^2\right) \\ &= \frac{1}{3}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] \\ &= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \end{aligned}$$

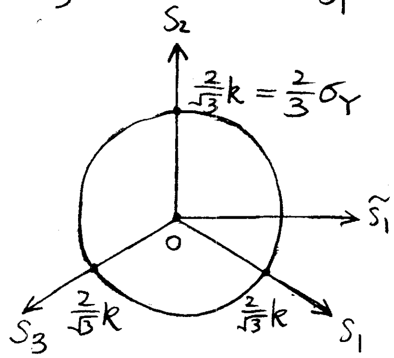


The Von Mises yield surface is a cylinder with a circular cross section.

The proof is given in the next page.







To be more precise, we shall express J in terms of vectors on the π-plane, i.e.

$S_1,\ S_2,\ S_3,$

Note: $S_1+S_2+S_3=0$

$$S_1 = \frac{1}{3}(2\ \ {-1}\ \ {-1})\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \frac{2\sigma_1-\sigma_2-\sigma_3}{3}$$

$$S_2 = \frac{1}{3}({-1}\ \ 2\ \ {-1})\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \frac{-\sigma_1+2\sigma_2-\sigma_3}{3}$$

$$S_3 = \frac{1}{3}({-1}\ \ {-1}\ \ 2)\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \frac{-\sigma_1-\sigma_2+2\sigma_3}{3}$$

Note that S_1, S_2, S_3 are not independent.

The axis of S_1, S_2, S_3 are not perpendicular to each other.

Introduce $\tilde{S}_1$ whose axis is perpendicular to the axis of S_2.

$$\tilde{S}_1 = \frac{1}{\sqrt{3}}(1\ \ 0\ \ {-1})\cdot\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \frac{\sigma_1-\sigma_3}{\sqrt{3}}$$

Von Mises yield condition is a circle of radius

$$\frac{2}{\sqrt{3}}k = \frac{2}{3}\sigma_Y$$

in the plane of S_1, S_2, S_3.

Because

$$\frac{1}{2}\left\{(1\ \ 0\ \ {-1}) + \frac{1}{3}(1\ \ {-2}\ \ 1)\right\} = \frac{1}{3}(2\ \ {-1}\ \ {-1})$$

$$\frac{1}{2}\left\{({-1}\ \ 0\ \ 1) + \frac{1}{3}(1\ \ {-2}\ \ 1)\right\} = \frac{1}{3}({-1}\ \ {-1}\ \ 2)$$

We have

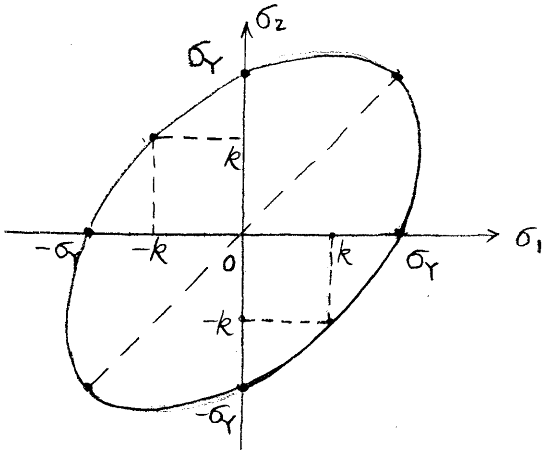
$$S_1 = \frac{\sqrt{3}}{2}\tilde{S}_1 - \frac{1}{2}S_2$$

$$S_3 = -\frac{\sqrt{3}}{2}\tilde{S}_1 - \frac{1}{2}S_2$$

$$J_2 = \frac{1}{2}(S_1^2+S_2^2+S_3^2) = \frac{3}{4}(\tilde{S}_1^2+S_2^2) = k^2$$

$$\tilde{S}_1^2+S_2^2 = \left(\frac{2}{\sqrt{3}}k\right)^2 = \left(\frac{2}{3}\sigma_Y\right)^2$$

Von Mises Criterion in plane stress



$\sigma_3 = 0$

$$J_2 = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

$$= \frac{1}{6}\left(\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_2^2 + \sigma_1^2\right)$$

$$= \frac{1}{3}\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right) = k^2$$

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = (\sqrt{3}\,k)^2 = \sigma_Y^2$$

The intersection of yield surface cylinder with $\sigma_3=0$ plane is an ellipse

several special points on yield surface

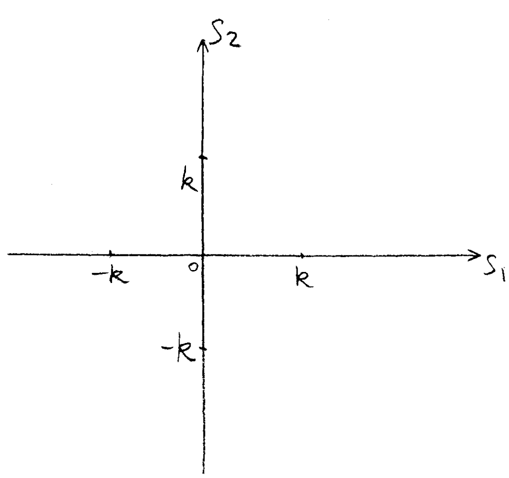
$\sigma_1 = \sigma_Y \quad \sigma_2 = 0$

$\sigma_1 = 0 \quad \sigma_2 = \sigma_Y$

$\sigma_1 = \sigma_Y \quad \sigma_2 = \sigma_Y$

$\sigma_1 = k \quad \sigma_2 = -k$

Von Mises Criterion in the plane of S_1-S_2



by definition. $S_1 + S_2 + S_3 = 0$

$$S_3 = -(S_1 + S_2)$$

$$J_2 = \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right)$$

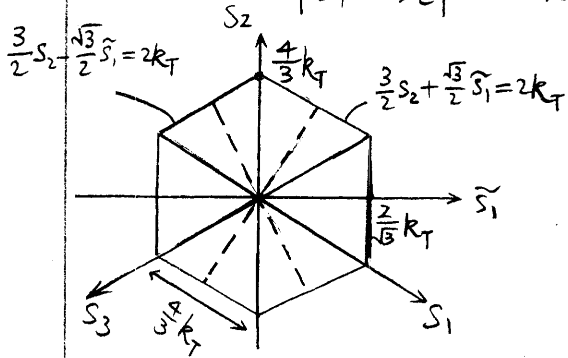
$$= \frac{1}{2}\left(S_1^2 + S_2^2 + (S_1+S_2)^2\right)$$

$$= S_1^2 + S_2^2 + S_1 S_2$$

Sketch the yield surface on S_1-S_2 plane

§3. Tresca Yield Condition

$$|s_1 - s_2| = 2k_T, \quad \text{or} \quad |s_2 - s_3| = 2k_T, \text{ or } |s_3 - s_1| = 2k_T$$



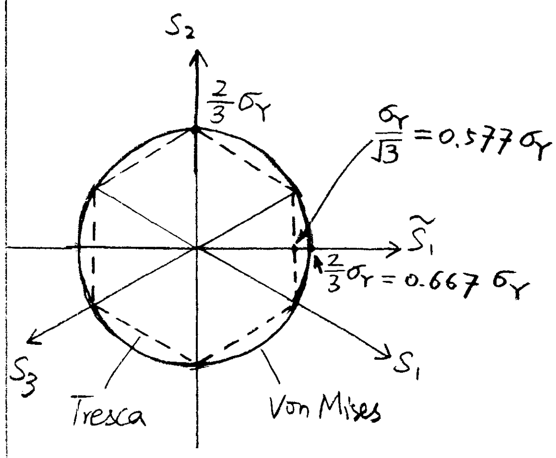
In terms of $\tilde{s}_1$ and s_2,

$$\left|\frac{\sqrt{3}}{2}\tilde{s}_1 - \frac{3}{2}s_2\right| = 2k_T$$

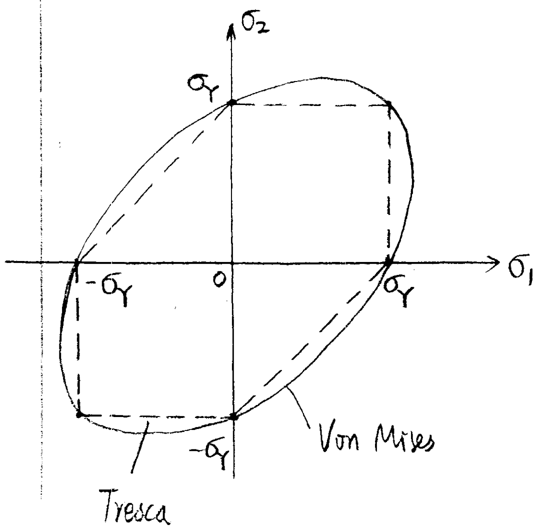
$$\left|\frac{\sqrt{3}}{2}\tilde{s}_1 + \frac{3}{2}s_2\right| = 2k_T$$

$$\left|\sqrt{3}\,\tilde{s}_1\right| = 2k_T$$

Tresca yield condition is a **hexagon** of side length $\frac{4}{3}k_T = \frac{2}{3}\sigma_Y$ in the plane of s_1, s_2, s_3



Tresca yield condition is a hexagon inscribed in the circle corresponding to the Von Mises condition.



Tresca yield condition in plane stress ($\sigma_3 = 0$): $|\sigma_1 - \sigma_2| = \sigma_Y$, $|\sigma_1| = \sigma_Y$, $|\sigma_2| = \sigma_Y$ is a polygon inscribed in the ellipse corresponding to the von Mises condition.

§4. Flow Rule for Perfectly Plastic Material (no hardening)

elastic response: $e_{ij}^{el} = \frac{1}{2\mu} S_{ij}$

e_{ij}^{el}: deviatoric elastic strain

S_{ij}: deviatoric stress

$$d e_{ij}^{el} = \frac{1}{2\mu} dS_{ij}$$

$d e_{ij}^{el}$: incremental deviatoric elastic strain

dS_{ij}: incremental deviatoric stress

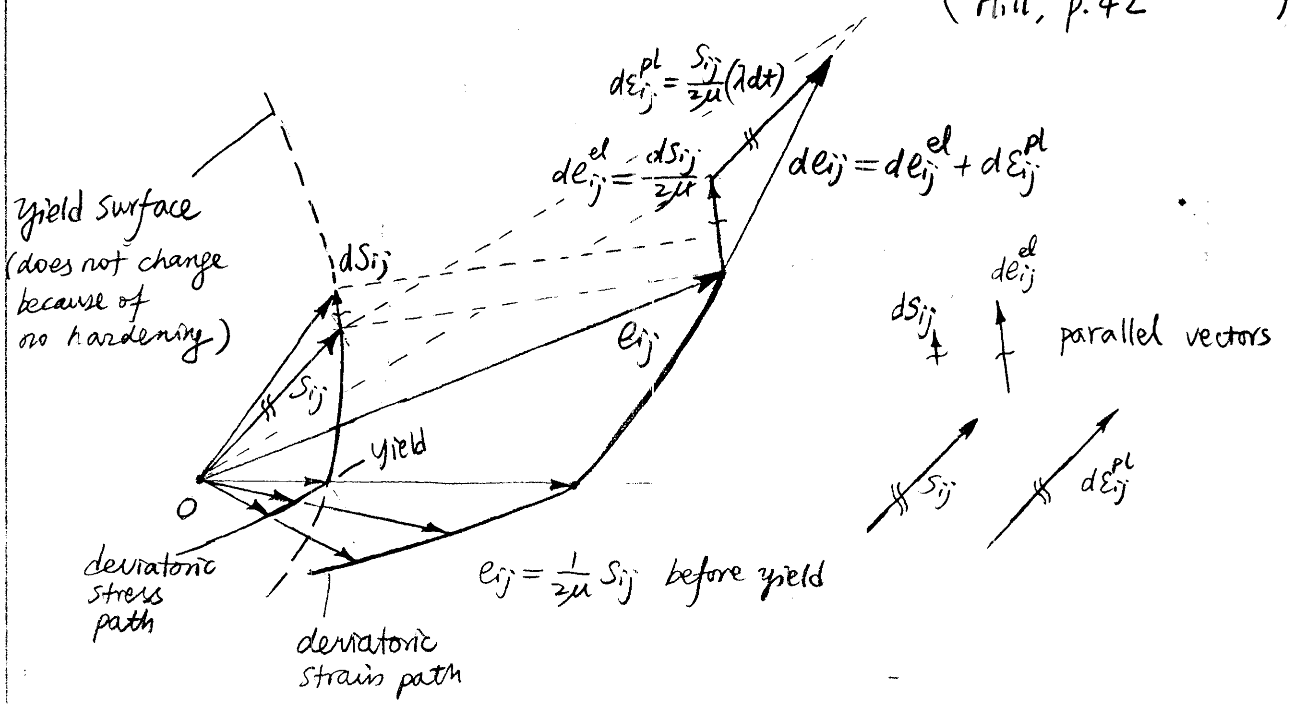
plastic response: $\dot{\varepsilon}_{ij}^{pl} = \frac{1}{2\mu} \lambda S_{ij}$

$\dot{\varepsilon}_{ij}^{pl}$: plastic strain rate (automatically deviatoric)

$$d\varepsilon_{ij}^{pl} = \frac{1}{2\mu} S_{ij} \cdot (\lambda dt)$$

$d\varepsilon_{ij}^{pl}$: incremental plastic strain

(analogus to Fig 7 of Hill, p.42)



- before S_{ij} reaches yield surface, e_{ij} is parallel to S_{ij}, $de_{ij} \parallel dS_{ij}$
- after yield condition is reached, e_{ij} is no longer parallel to S_{ij}.
  de_{ij} is no longer parallel to dS_{ij}
- $de_{ij} = de_{ij}^{el} + d\varepsilon_{ij}^{pl}$, $de_{ij}^{el} \parallel dS_{ij}$, $d\varepsilon_{ij}^{pl} \parallel S_{ij}$

In the plastic regime,

given S_{ij}, e_{ij}, de_{ij} (change of total deviatoric strain) ← elastic + plastic

find dS_{ij} (change of deviatoric stress)

- draw line SQ' parallel to S_{ij}
- Construct the patch of yield surface at point P
- Draw a parallel surface patch at point Q
- Find the intersection point R between line SQ' and surface patch
- $RQ' = d\varepsilon_{ij}^{pl}$, $QR = de_{ij}^{el}$, $\underline{PP' = dS_{ij} = 2\mu \cdot QR}$