The purpose of this lecture is to gather all equations dealing with a <u>perfectly plastic, isotropic</u> medium, with brief derivations and discussions.

The types of relations include:

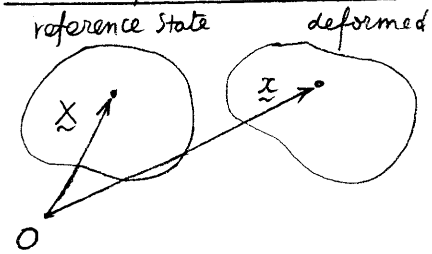
- equilibrium condition for stress (§4)
- compatibility condition for total strain (§5)
- Stress-Strain relation in Elastic Regime (§7)
- Yield condition (§8)
- stress-strain relation (flow rule) in Plastic Regime (§9)

A summary of key formulas is given in §10.
The goal is to provide a global view of a (simplest) plasticity theory.
More discussions on yield condition and flow rule will be given, with examples, in subsequent lectures.

A perfectly plastic medium (i.e. no hardening) is assumed here to keep the discussion simple. Various hardening rules will be discussed later.

Similar to the theory of elasticity, in the theory of plasticity we will also be dealing with stress, strain and displacement field. The main difference is that the stress-strain relationship is non-linear (and history-dependent) in the theory of plasticity.

§1. Displacement Field



Displacement $\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X}$

displacement field $\underset{\sim}{u}(\underset{\sim}{x})$

In this class, we will assume $|\underset{\sim}{u}| \ll 1$ and ignore the difference between $\underset{\sim}{u}(\underset{\sim}{x})$ and $\underset{\sim}{u}(\underset{\sim}{X})$.

displacement field in component form:

$u_x(x, y, z)$
$u_y(x, y, z)$
$u_z(x, y, z)$

§2. Strain Field

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

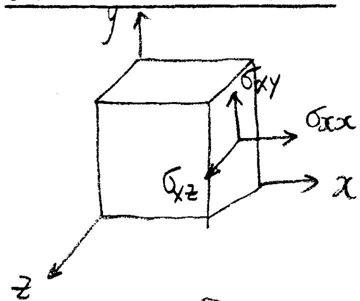
$i, j = 1, 2, 3$
$x_1 = x,$
$x_2 = y,$
$x_3 = z$

e.g. $\varepsilon_{xx} = u_{x,x}$

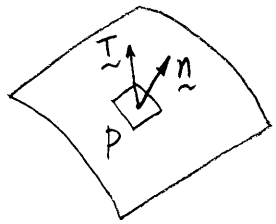
$\varepsilon_{xy} = \frac{1}{2}(u_{x,y} + u_{y,x})$

The engineering strain, e.g. $\gamma_{xy} = u_{x,y} + u_{y,x}$, is not a component of the strain tensor.

§3. Stress Field



σ_{ij} : force per unit area on i-th face in j-th direction



Given the stress at point P, the traction force $\underset{\sim}{T}$ per unit area on any surface element with normal vector $\underset{\sim}{n}$ is

$$T_j = \sigma_{ij} \, n_i$$

u_i is a vector

ε_{ij}, σ_{ij} are (symmetric) second order tensors.

The components change due to transformation of coordinate system as

$$u_i' = Q_{ip} \, u_p$$

$$\varepsilon_{ij}' = Q_{ip} \, Q_{jq} \, \varepsilon_{pq}$$

$$\sigma_{ij}' = Q_{ip} \, Q_{jq} \, \sigma_{pq}$$

where $Q_{ip} \equiv (\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_p)$

unit vector of new coordinate system ($\underset{\sim}{e}_i'$)

unit vector of original coordinate system ($\underset{\sim}{e}_p$)

(For more details, see ME340 Lecture Notes Winter 2013.)

§4. Equilibrium Condition for Stress

$$\sigma_{ij,i} + F_j = 0$$

F_j : body force per unit volume.

in tensor notation $\nabla \cdot \underset{\approx}{\sigma} + \underset{\sim}{F} = 0$

in component form:

$$\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{yx} + \frac{\partial}{\partial z} \sigma_{zx} + F_x = 0$$

$$\frac{\partial}{\partial x} \sigma_{xy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \sigma_{zy} + F_y = 0$$

$$\frac{\partial}{\partial x} \sigma_{xz} + \frac{\partial}{\partial y} \sigma_{yz} + \frac{\partial}{\partial z} \sigma_{zz} + F_z = 0$$

Note: The equilibrium condition is satisfied regardless of whether the material is elastic or plastic.

§5. Compatibility condition for total strains.

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$ is the total strain

$$\varepsilon_{ij} = \underset{\substack{\uparrow \\ \text{elastic} \\ \text{strain}}}{\varepsilon_{ij}^{el}} + \underset{\substack{\uparrow \\ \text{plastic} \\ \text{strain}}}{\varepsilon_{ij}^{pl}}$$

ε_{ij} has to satisfy the compatibility condition if the material is not ruptured.

compatibility condition: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$

The compatibility condition gaurentees that ε_{ij} (6 DOF at every point) can be written in terms of spatial derivatives of u_i (3 DOF).

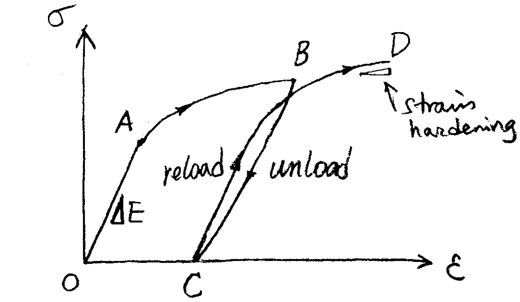
The compatibility condition is automatically satisfied if we start from u_i and obtain ε_{ij} as $\frac{1}{2}(u_{i,j} + u_{j,i})$.

(For more details, see ME340 Lecture Notes Winter 2013).

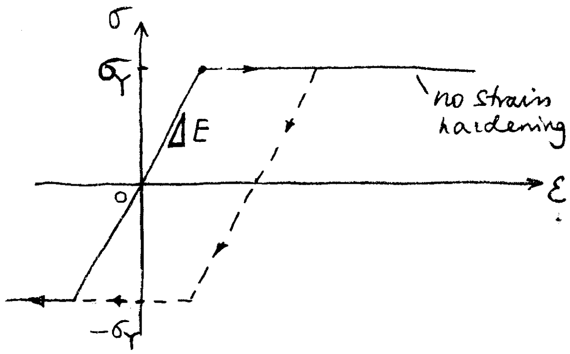
Note: The elastic strain field ε_{ij}^{el} does not have to satisfy compatibility condition if $\varepsilon_{ij}^{pl} \neq 0$.

* up to this point, every equation is the same as in elasticity theory (provided $\varepsilon_{ij}^{pl} = 0$).

§6. Tensile Stress-Strain Curve



Stress-strain curve of a ductile material in a tensile test



Stress-strain curve of a perfectly plastic material (idealization)

We will discuss more complex (realistic) models of plastic material later.

σ_Y: yield stress

Note: After the yield point is reached, the stress-strain curve becomes history dependent.

§7 Constitutive (Stress-Strain) Relation in the Elastic Regime

If the stress never reaches the yield condition (i.e. $\sigma < \sigma_Y$ in tensile test) then the stress-strain relation is linear and history-independent.

Generalized Hooke's Law: $\sigma_{ij} = C_{ijkl}\, \varepsilon^{el}_{kl}$

$\uparrow$ elastic stiffness tensor

$$\varepsilon_{ij} = \varepsilon^{el}_{ij}, \qquad \varepsilon^{pl}_{ij} = 0$$

Isotropic elasticity: $C_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}, \quad \lambda = \frac{2\mu\nu}{1-2\nu}$$

μ: shear modulus

ν: Poisson's ratio

In component form

$$\sigma_{xx} = (\lambda+2\mu)\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \qquad \sigma_{xy} = 2\mu\,\varepsilon_{xy}$$

$$\sigma_{yy} = \lambda\,\varepsilon_{xx} + (\lambda+2\mu)\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \qquad \sigma_{yz} = 2\mu\,\varepsilon_{yz}$$

$$\sigma_{zz} = \lambda\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + (\lambda+2\mu)\,\varepsilon_{zz} \qquad \sigma_{zx} = 2\mu\,\varepsilon_{zx}$$

notice factor of 2

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \qquad \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \qquad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} \qquad \varepsilon_{zx} = \frac{1}{2\mu}\sigma_{zx}$$

$E = 2\mu(1+\nu)$ — Young's modulus

Define hydrostatic stress (mean normal stress), hydrostatic strain (mean normal strain)

$$\bar{\sigma} \equiv \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) \qquad \bar{\varepsilon} \equiv \frac{1}{3}\varepsilon_{ii} = \frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})$$

$$\sigma_{xx}+\sigma_{yy}+\sigma_{zz} = (3\lambda+2\mu)(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})$$

$K = \frac{1}{3}(3\lambda+2\mu) = \frac{2\mu(1+\nu)}{3(1-2\nu)} = \frac{E}{3(1-2\nu)}$ — Bulk modulus

$$\boxed{\bar{\sigma} = 3K\,\bar{\varepsilon}}$$

Define deviatoric stress tensor, deviatoric strain tensor

$$s_{ij} \equiv \sigma_{ij} - \bar{\sigma}\,\delta_{ij} \qquad e_{ij} \equiv \varepsilon_{ij} - \bar{\varepsilon}\,\delta_{ij}$$

$s_{xx} = \sigma_{xx}-\bar{\sigma}$, $s_{yy} = \sigma_{yy}-\bar{\sigma}$, $s_{zz} = \sigma_{zz}-\bar{\sigma}$ $\qquad$ $e_{xx} = \varepsilon_{xx}-\bar{\varepsilon}$, $e_{yy} = \varepsilon_{yy}-\bar{\varepsilon}$, $e_{zz} = \varepsilon_{zz}-\bar{\varepsilon}$

$s_{xy} = \sigma_{xy}$, $s_{yz} = \sigma_{yz}$, $s_{zx} = \sigma_{zx}$ $\qquad$ $e_{xy} = \varepsilon_{xy}$, $e_{yz} = \varepsilon_{yz}$, $e_{zx} = \varepsilon_{zx}$

The relation between deviatoric stress and deviatoric strain is very simple

$$\boxed{s_{ij} = 2\mu\, e_{ij}} \qquad \text{for all } i,j = 1,2,3$$

* obviously, the equations here are also the same as in elasticity theory. An emphasis on deviatoric stress-strain is given here to better compare with plastic strain (which is all deviatoric).

§8 Yield Condition

We postulate that the yield condition (i.e. onset of plastic deformation) can be described by a function of stress tensor

$$f(\{\sigma_{ij}\}) = 0 \quad , \quad \text{i.e.} \quad f(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}) = 0$$

At this point, f is a general function.

Note that time and rates ($\dot{\sigma}_{ij} \equiv \frac{d}{dt}\sigma_{ij}$, $\dot{\varepsilon}_{ij} \equiv \frac{d}{dt}\varepsilon_{ij}$) do not enter f.

So this is still an idealization, which is well justified by experimental observations of ductile metals.

We shall assume that the material is isotropic, then the value of f should not change by a coordinate transformation, i.e.

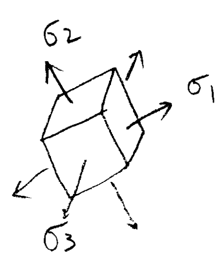
$$f(\{\sigma_{ij}\}) = f(\{\sigma_{ij}'\}), \quad \text{where} \quad \sigma_{ij}' = Q_{ip}\, Q_{jq}\, \sigma_{pq}$$

This means that we must be able to write f in terms stress invariants — functions of σ_{ij} that does not change by coordinate transformation.

For example, the mean normal stress $\bar{\sigma} \equiv \frac{1}{3}\sigma_{ii}$ is a stress invariant,

i.e. $\bar{\sigma} = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\sigma_{ii}'$

Other stress invariants can be defined in terms of the principal stress values : σ_1, σ_2, σ_3



They are the normal stress values in a special coordinately system such that all shear stresses vanish.

Mathematically, $\sigma_1, \sigma_2, \sigma_3$ are the eigenvalues of the stress matrix

$$\{\sigma_{ij}\} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

i.e. $\sigma_1, \sigma_2, \sigma_3$ are the three solutions of the eigen-equation

$$\det\left[\underset{\approx}{\sigma} - \hat{\lambda}\, \underset{\approx}{I}\right] = \begin{vmatrix} \sigma_{xx}-\hat{\lambda} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-\hat{\lambda} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\hat{\lambda} \end{vmatrix} = 0 \qquad \text{------- (1)}$$

The eigen-equation is a polynomial equation of $\hat{\lambda}$:

$$\hat{\lambda}^3 - I_1 \hat{\lambda}^2 - I_2 \hat{\lambda} - I_3 = 0 \qquad \text{------- (2)}$$

where I_1, I_2, I_3 are stress invariants.

Because $\sigma_1, \sigma_2, \sigma_3$ are solutions of the eigen-equation, we must have

$$(\hat{\lambda}-\sigma_1)(\hat{\lambda}-\sigma_2)(\hat{\lambda}-\sigma_3) = 0$$

$$\hat{\lambda}^3 - (\sigma_1+\sigma_2+\sigma_3)\hat{\lambda} - (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\hat{\lambda} - \sigma_1\sigma_2\sigma_3 = 0 \quad \text{---- (3)}$$

Comparing Eq (2) and Eq (3), we get

$$\begin{cases} I_1 = \sigma_1+\sigma_2+\sigma_3 = 3\bar{\sigma} \\ I_2 = -(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) \\ I_3 = \sigma_1\sigma_2\sigma_3 \end{cases}$$

Three stress invariants in terms of the principal stress values.

(Note: some books choose opposite sign in the definition of I_2)

The stress invariants can also be computed from $\{\sigma_{ij}\}$ of an arbitrary coordinate system (for which shear stresses do not vanish).

Comparing Eq (1) and Eq (2), we get

$$\begin{cases} I_1 = \sigma_{xx}+\sigma_{yy}+\sigma_{zz} = 3\bar{\sigma} \\ -I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} = (\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}) - (\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2) \\ \qquad = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \\ I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} \end{cases}$$

Because the yield condition is experimentally found to be independent of pressure (i.e. hydrostatic stress $\bar{\sigma}$), it is more convenient to express function f in terms of the invariants of deviatoric stress S_{ij}.

Let S_1, S_2, S_3 be the principal values of $\{S_{ij}\}$.
i.e. they are the solution of the eigen-equation

$$\det(\underset{\approx}{S} - \hat{\lambda}\underset{\approx}{I}) = \begin{vmatrix} S_{xx}-\hat{\lambda} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy}-\hat{\lambda} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz}-\hat{\lambda} \end{vmatrix} = 0 \qquad \text{-----} (4)$$

$$\hat{\lambda}^3 - J_1\hat{\lambda}^2 + J_2\hat{\lambda} + J_3 = 0 \qquad \text{-----} (5)$$

where J_1, J_2, J_3 are invariants of deviatoric stress $\{S_{ij}\}$.

$$(\hat{\lambda} - S_1)(\hat{\lambda} - S_2)(\hat{\lambda} - S_3) = 0$$

$$\hat{\lambda}^3 - (S_1+S_2+S_3)\hat{\lambda}^2 + (S_1S_2+S_2S_3+S_3S_1)\hat{\lambda} - S_1S_2S_3 = 0 \qquad \text{-----} (6)$$

Comparing Eq (4) and Eq (5), we get

(because $S_1+S_2+S_3=0$
$0=(S_1+S_2+S_3)^2 = S_1^2+S_2^2+S_3^2+2S_1S_2+2S_2S_3+$

$$\begin{cases} J_1 = S_1+S_2+S_3 = 0 \\ J_2 = -(S_1S_2+S_2S_3+S_3S_1) = \frac{1}{2}(S_1^2+S_2^2+S_3^2) \\ J_3 = S_1S_2S_3 = \frac{1}{3}(S_1^3+S_2^3+S_3^3) \end{cases}$$

(* sign convention for J_2 consistent with Prager-Hodge

($S_1^3+S_2^3+S_3^3 = S_1^3+S_2^3-(S_1+S_2)^3$
$= -3S_1S_2^2 - 3S_1^2S_2 = -3S_1S_2(S_1+S_2)$
$= 3S_1S_2S_3$)

Comparing Eq (4) and Eq (6), we get

(because $S_{xx}+S_{yy}+S_{zz}=0$
$0=(S_{xx}+S_{yy}+S_{zz})^2 = S_{xx}^2+S_{yy}^2+S_{zz}^2$
$+2S_{xx}S_{yy}+2S_{yy}S_{zz}+2S_{zz}$

$$\begin{cases} J_1 = S_{xx}+S_{yy}+S_{zz} = 0 \\ J_2 = -(S_{xx}S_{yy}+S_{yy}S_{zz}+S_{zz}S_{xx}) + (S_{yz}^2+S_{zx}^2+S_{xy}^2) \\ \quad = \frac{1}{2}(S_{xx}^2+S_{yy}^2+S_{zz}^2) + (S_{yz}^2+S_{zx}^2+S_{xy}^2) \\ \quad = \frac{1}{2}S_{ij}S_{ij} \\ J_3 = \begin{vmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{vmatrix} \end{cases}$$

($J_2 = \frac{1}{2}S_{ij}S_{ij}$ is a measure of the magnitude of the deviatoric stress, similar to the norm of a vector squared.)

Here are the stress invariants we have obtained

$$I_1 = 3\bar{\sigma}, \quad I_2 = -\frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}), \quad I_3 = \det(\sigma_{ij})$$

$$\bar{\sigma}, \quad J_1 = 0, \quad J_2 = \frac{1}{2} S_{ij} S_{ij}, \quad J_3 = \det(S_{ij})$$

The yield condition can be written as either

$$f(I_1, I_2, I_3) = 0$$

or

$$f(\bar{\sigma}, J_2, J_3) = 0$$

Experimentally, it was found that $\bar{\sigma}$ has no significant effect on the yield condition (and plastic deformation in general, which conserves volume).

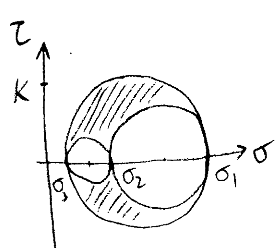
So the yield condition of isotropic medium can be written as

$$f(J_2, J_3) = 0$$

In the most widely used yield condition (J_2-plasticity) the dependence on J_3 is also dropped.

$$f(J_2) = J_2 - k^2 = 0 \qquad \text{— Von Mises criterion}$$

In another widely used yield condition, k_T is the greatest shear stress.



3D Mohr's circle

If $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then $\sigma_1 - \sigma_3 = 2k_T$ i.e. $S_1 - S_2 = 2k_T$ — Tresca criterion

In general, $[(S_1-S_2)^2 - 4k_T^2][(S_3-S_1)^2 - 4k_T^2][(S_1-S_2)^2 - 4k_T^2] = 0$

$$\rightarrow f(J_2, J_3) = 4J_2^3 - 27J_3^2 - 36k_T^2 J_2^2 + 96 k_T^4 J_2 - 64 k_T^6 = 0$$

Mathematically much more complex.
More difficult to use if principal stress not already known.

In practice, the difference between Von Mises and Tresca criteria is less than 15%, within typical experimental error.

The simple form of von Mises criterion makes it most popular.
(For experimental verification of Von Mises criterion, see Lecture Note 2)

We now relate the parameter k to the yield stress σ_Y in tensile test.

In tensile test, $\sigma_{xx} > 0$, $\sigma_{yy} = \sigma_{zz} = 0$

$$\bar{\sigma} = \frac{1}{3}\sigma_{xx}$$

$$S_{xx} = \frac{2}{3}\sigma_{xx}, \quad S_{yy} = -\frac{1}{3}\sigma_{xx}, \quad S_{zz} = -\frac{1}{3}\sigma_{xx}$$

$$J_2 = \frac{1}{2}\left(S_{xx}^2 + S_{yy}^2 + S_{zz}^2\right) = \frac{1}{2}\cdot\frac{6}{9}\,\sigma_{xx}^2 = \frac{1}{3}\sigma_{xx}^2$$

At yield point $\quad J_2 = \frac{1}{3}\sigma_Y^2 = k^2$

$$\therefore \sigma_Y = \sqrt{3}\,k, \qquad k = \frac{\sigma_Y}{\sqrt{3}} \qquad \text{(Von Mises)}$$

(For Tresca yield condition, $k_T = \frac{\sigma_Y}{2}$)

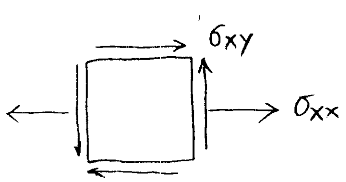
If the sample is subjected to pure shear stress σ_{xy}. (all other stress component=0)

then $J_2 = \sigma_{xy}$

At onset of yield, $\sigma_{xy} = \tau_c$, $\quad J_2 = \tau_c^2 = k^2$

$\therefore \tau_c = k$, i.e. k is the critical shear stress when a pure shear stress is applied. (Von Mises)

(For Tresca yield condition, $k_T = \tau_c$ also)

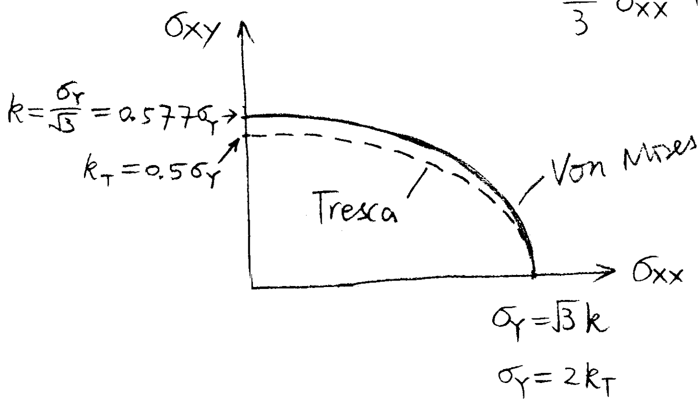


when σ_{xx} and σ_{xy} are the two non-zero stress components,

$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2$$

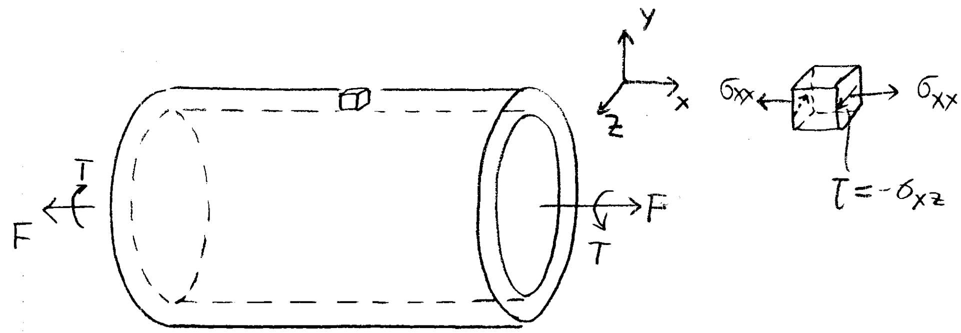
Von Mises yield condition

$$\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$$



(exercise: draw the curve corresponding to the Tresca yield condition on the $\sigma_{xx} - \sigma_{xy}$ plane using Matlab.

Testing Yield Criterion by Experiments



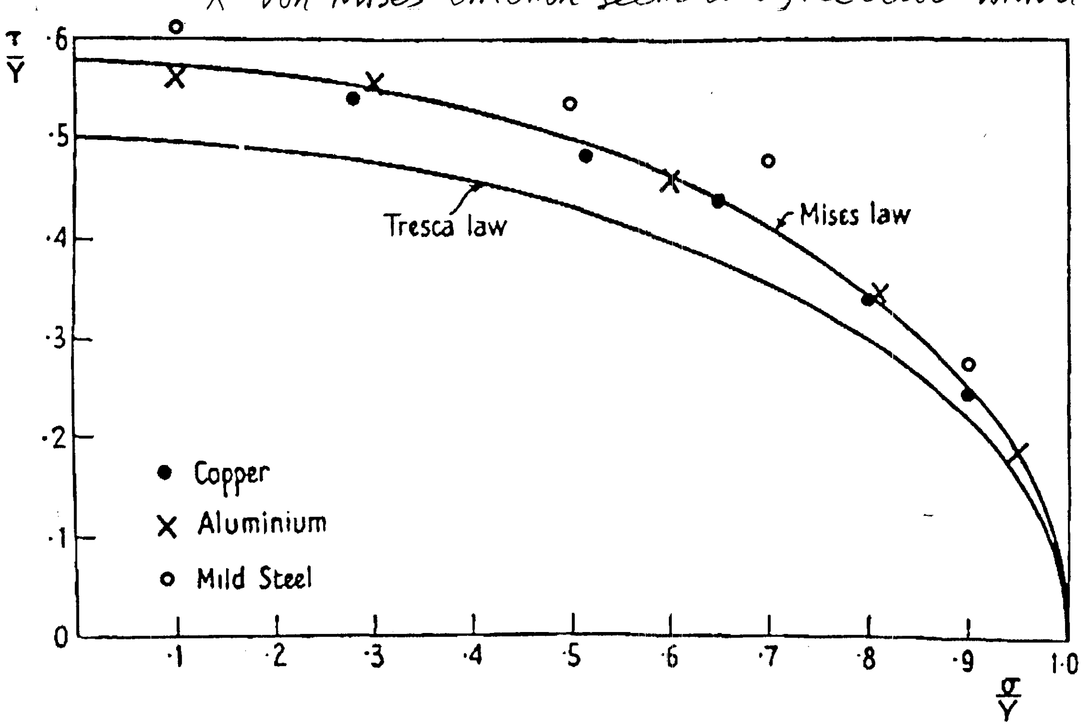
consider a tube subjected to both tension force F and torque T.

The element shown above will be subjected to both normal stress σ_{xx} and shear stress σ_{xz}.

Increasing F and T together while keeping $\frac{T}{F}$ at constant allows us to increase σ_{xx} and σ_{xz} together with $\frac{\sigma_{xz}}{\sigma_{xx}} = \text{const}$.

Measuring the critical values of σ_{xx}, σ_{xz} for various $\frac{\sigma_{xz}}{\sigma_{xx}}$ allows us to map the yield surface and compare with theoretical models.

* Von Mises criterion seem to agree better with experiments than Tresca.



From: R. Hill, The Mathematical Theory of Plasticity, Oxford Univ. Press 1950 P.22

Taylor and Quinney Phil. Trans. Roy. Soc. A 230, 323 (1931)

Fig. 4. Experimental results of Taylor and Quinney from combined torsion and tension tests, each metal being work-hardened to the same state for all tests. The Mises law is $\sigma^2 + 3\tau^2 = Y^2$, while the Tresca law is $\sigma^2 + 4\tau^2 = Y^2$, where σ = tensile stress, τ = shear stress, Y = tensile yield stress.

§9 Constitutive (Stress-strain) Relation in the Plastic Regime

For simplicity, here we consider a perfectly plastic material (no strain hardening).

Then in the plastic regime $f(\{\sigma_{ij}\}) = J_2 - k^2 = 0$, k remains constant

i.e. $J_2 = k^2$, $\dot{J}_2 = 0$

However, this does not mean that all stress components stays constant.

$J_2 = k^2$ defines a 'yield surface' in the stress space.

The stress components can vary within the yield surface

Recall $J_2 = \frac{1}{2}(S_{xx}^2 + S_{yy}^2 + S_{zz}^2) + (S_{yz}^2 + S_{zx}^2 + S_{xy}^2)$

$$\dot{J}_2 = S_{xx}\dot{S}_{xx} + S_{yy}\dot{S}_{yy} + S_{zz}\dot{S}_{zz} + 2S_{yz}\dot{S}_{yz} + 2S_{zx}\dot{S}_{zx} + 2S_{xy}\dot{S}_{xy} = S_{ij}\dot{S}_{ij} = 0$$

This is a constraint on the stress rate $\dot{S}_{ij}$.

While the stress stays on the yield surface, the material will undergo plastic strain ε_{ij}^{pl}, so that $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$

ε_{ij}^{el} remain proportional to stress σ_{ij} : $\sigma_{ij} = \lambda\,\varepsilon_{kk}^{el}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}^{el}$

$$\boxed{S_{ij} = 2\mu\, e_{ij}^{el}}$$

Q: How to determine ε_{ij}^{pl}?

It took a long time for scientists to find out the correct way to compute ε_{ij}^{pl} (after many wrong starts).

It became know that ε_{ij}^{pl} is history dependent, i.e. it cannot be determined by the current stress state (σ_{ij}).

Instead, ε_{ij}^{pl} must be determined by accumulating small increments.

$$\varepsilon_{ij}^{pl} = \int_0^t \dot{\varepsilon}_{ij}^{pl}(t)\, dt$$

In addition, it has been found that plastic deformation produces negligible volume change: $\varepsilon_{ii}^{pl} = 0$, $\dot{\varepsilon}_{ii}^{pl} = 0$

$\therefore$ deviatoric plastic strain is the same as plastic strain

Assumption of "associative flow".

when yield condition is reached, $\dot{\varepsilon}_{ij}^{pl}$ follow the direction of s_{ij}

$$\boxed{2\mu\,\dot{\varepsilon}_{ij}^{pl} = \tilde{\lambda}\, s_{ij}}$$

where λ is a positive scalar factor to be determined.

compare this with $2\mu\, e_{ij}^{el} = s_{ij}$, $2\mu\, \dot{e}_{ij}^{el} = \dot{s}_{ij}$

Note both the deviatoric elastic strain (e_{ij}^{el})
and the plastic strain rate ($\dot{\varepsilon}_{ij}^{pl}$)
follow the deviatoric stress (s_{ij}).

Total deviatoric strain rate

$$2\mu\,\dot{e}_{ij} = 2\mu\left(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}\right) = \dot{s}_{ij} + \lambda s_{ij}$$

To determine the factor $\tilde{\lambda}$, we need to introduce

$$\dot{W} = s_{ij}\,\dot{e}_{ij}$$

$$= s_{ij}\left(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}\right)$$

← rate of work done associated with shape change
part of which goes to stored elastic energy
part of which dissipated as heat

(work done associated with volume change is entirely stored as elastic energy)

$$\dot{W} = s_{xx}\dot{e}_{xx} + s_{yy}\dot{e}_{yy} + s_{zz}\dot{e}_{zz} + 2\, s_{yz}\dot{e}_{yz} + 2\, s_{zx}\dot{e}_{zx} + 2\, s_{xy}\dot{e}_{xy}$$

$$2\mu\,\dot{W} = s_{ij}\left(\dot{s}_{ij} + \tilde{\lambda} s_{ij}\right)$$

Recall $\dot{J}_2 = s_{ij}\dot{s}_{ij} = 0$ for perfectly plastic material

$$2\mu\dot{W} = \tilde{\lambda} s_{ij} s_{ij} = 2\tilde{\lambda} J_2 = 2\tilde{\lambda} k^2$$

$$\therefore \boxed{\tilde{\lambda} = \frac{2\mu}{2k^2}\dot{W}}$$

Q: How to determine $\dot{\sigma}_{ij}$?

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2}\cdot s_{ij} \quad , \quad \dot{s}_{ij} = 2\mu\,\dot{e}_{ij}^{el} = 2\mu\left(\dot{e}_{ij} - \dot{\varepsilon}_{ij}^{pl}\right)$$

$$\dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \frac{\dot{W}}{2k^2} s_{ij}\right)$$

describes how stress responds to imposed strain rate $\dot{\varepsilon}_{ij}$

The hydrostatic stress-strain response is purely elastic

$$\bar{\sigma} = 3K\bar{\varepsilon} \quad , \quad \dot{\bar{\sigma}} = 3K\dot{\bar{\varepsilon}}$$

Summary: How stress changes under a specified (total) strain rate $\dot{\varepsilon}_{ij}$ in the plastic regime?

Given $\dot{\varepsilon}_{ij}$ and current stress σ_{ij}, to find $\dot{\sigma}_{ij}$ (imagine a metal forming process)

$$\dot{\varepsilon}_{ij} \to \dot{\bar{\varepsilon}} = \frac{1}{3}\dot{\varepsilon}_{ii} \to \dot{\bar{\sigma}} = 3K\dot{\bar{\varepsilon}}$$

$$\dot{\varepsilon}_{ij} \to \dot{e}_{ij} = \dot{\varepsilon}_{ij} - \dot{\bar{\varepsilon}}\,\delta_{ij}$$

$$\sigma_{ij} \to \bar{\sigma} = \frac{1}{3}\sigma_{ii}$$

$$\sigma_{ij} \to s_{ij} = \sigma_{ij} - \bar{\sigma}\,\delta_{ij}$$

$$\dot{W} = s_{ij}\dot{e}_{ij} \to \dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \frac{\dot{W}}{2k^2}s_{ij}\right)$$

$$\dot{\sigma}_{ij} = \dot{s}_{ij} + \dot{\bar{\sigma}}\,\delta_{ij}$$

§10 Summary of All Formulas

Elastic Regime:

$$\bar{\sigma} = 3K\,\bar{\varepsilon} \quad \text{(hydrostatic part)}$$
$$s_{ij} = 2\mu\, e_{ij} \quad \text{(deviatoric part)}$$

Yield condition:

$$J_2 = \frac{1}{2}s_{ij}s_{ij} = k^2 \quad \text{(Von Mises)}$$
$$\sigma_Y = \sqrt{3}\,k \quad \text{(yield stress in tension)}$$

Plastic Regime:

$$\dot{e}^{el}_{ij} = \frac{1}{2\mu}\dot{s}_{ij} \quad \text{(deviatoric elastic strain rate)}$$
$$\dot{\varepsilon}^{pl}_{ij} = \frac{\dot{W}}{2k^2}s_{ij} \quad \text{(plastic strain rate)}$$
$$\dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \frac{\dot{W}}{2k^2}s_{ij}\right) \quad \text{(deviatoric stress rate)}$$
$$\dot{\bar{\sigma}} = 3K\,\dot{\bar{\varepsilon}} \quad \text{(hydrostatic stress rate)}$$