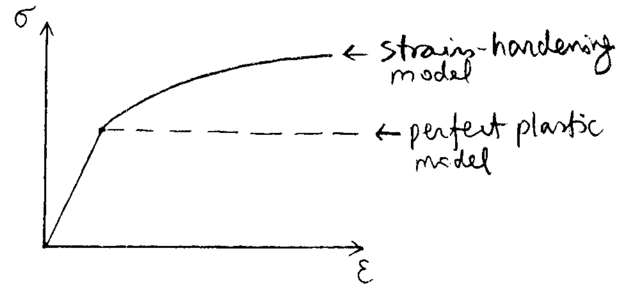
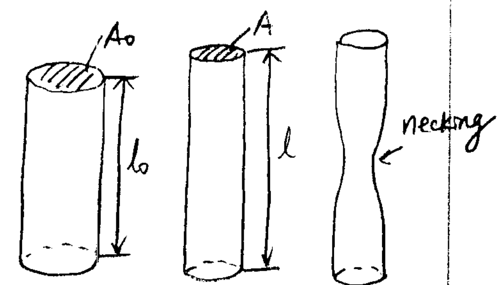
So far we have considered elastic-perfect plastic models in which the yield surface remain unchanged, i.e. no strain hardening.

We now consider various ways to include strain hardening into the constitutive equation.

§1. Plastic Instability in Tension (R. Hill, p.11)





We will show that a perfect plastic material (no hardening) is unstable in tension, i.e. necking will immediately develop after yielding.

In fact, strain hardening ($\frac{d\sigma}{d\varepsilon}$) need to be sufficiently strong for stability, i.e. to prevent necking.

Most ductile materials loaded in tension eventually fail by necking, due to violation of stability criterion (i.e. insufficient hardening).

Let A_0, l_0 be the cross sectional area and lengths of the bar prior to deformation (i.e. reference state).

Let A, l be the values at current state (before necking)

Neglect elastic strains, because plastic strain conserves volume, we have

$$A \cdot l = A_0 \cdot l_0$$

$$d(A \cdot l) = A\,dl + l\,dA = 0$$

$$\frac{dA}{A} = -\frac{dl}{l}$$

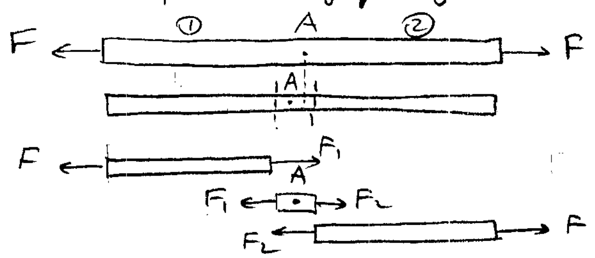
define: ε: <u>engineering strain</u> $d\varepsilon = \frac{dl}{l_0}$, $\varepsilon = \frac{l-l_0}{l_0}$, $l = l_0(1+\varepsilon)$

ε^t: <u>true strain</u> $d\varepsilon^t = \frac{dl}{l}$, $\varepsilon^t = \ln\frac{l}{l_0} = \ln(1+\varepsilon)$

$$\left(\begin{array}{l}\text{note: if } l=0,\ \varepsilon=-1,\ \varepsilon^t=-\infty \\ \qquad\ \text{if } l=2l_0,\ \varepsilon=1,\ \varepsilon^t=\ln 2\end{array}\right)$$

Stability requires that the total force F on the cross section increases with positive dl, i.e. $\frac{dF}{dl} > 0$

If this is not the case, the system can spontaneously lower the free energy by strain <u>localization</u>.



Imagine two sections of the rod.

Imagine a virtual deformation where point A moves left

i.e. $dl^{①} < 0$, $dl^{②} = -dl^{①} > 0$

Suppose $\frac{dF}{dl} < 0$, then $F_1 > F$, $F_2 < F$

then point A will move further to the left.

$\rightarrow$ unstable.

$$F = A \cdot \sigma$$

$$dF = d(A\cdot\sigma) = A\,d\sigma + \sigma\,dA$$

$$= A\left(d\sigma + \frac{dA}{A}\sigma\right) = A\left(d\sigma - \frac{dl}{l}\sigma\right)$$

$$\frac{dl}{l} = \frac{l_0\,d\varepsilon}{l_0(1+\varepsilon)} = \frac{d\varepsilon}{1+\varepsilon}$$

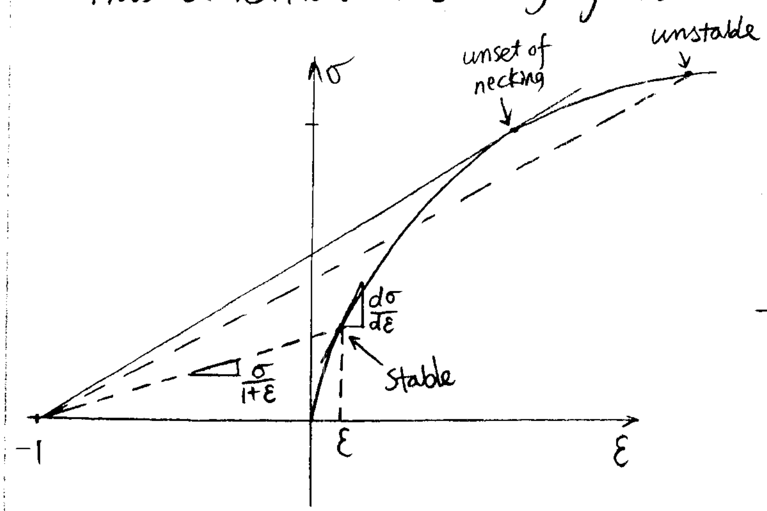
$$dF = A\left(d\sigma - \sigma\frac{d\varepsilon}{1+\varepsilon}\right) = A\,d\varepsilon\left(\frac{d\sigma}{d\varepsilon} - \frac{\sigma}{1+\varepsilon}\right)$$ in terms of engineering strain

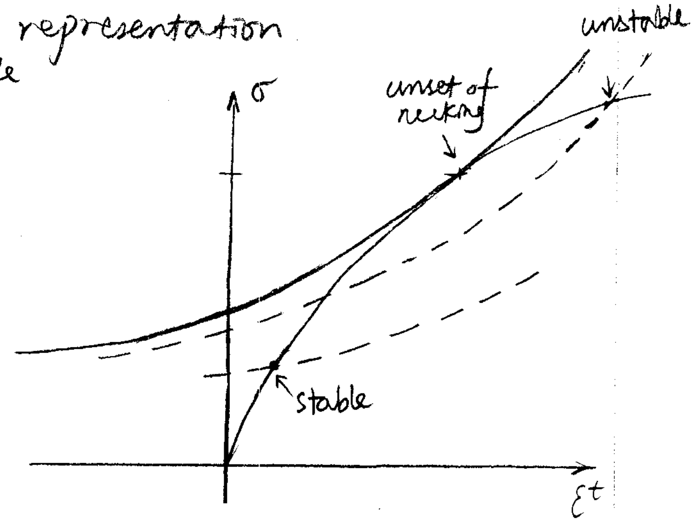
$$dF = A\left(d\sigma - \sigma\,d\varepsilon^t\right) = A\,d\varepsilon^t\left(\frac{d\sigma}{d\varepsilon^t} - \sigma\right)$$ in terms of true strain

Stability criterion $\frac{dF}{dl} > 0$, i.e. $\frac{dF}{d\varepsilon} > 0$, i.e. $\frac{dF}{d\varepsilon^t} > 0$

requires $\boxed{\frac{d\sigma}{d\varepsilon} > \frac{\sigma}{1+\varepsilon}}$ or equivalently $\boxed{\frac{d\sigma}{d\varepsilon^t} > \sigma}$

This condition has a graphical representation





draw a family of curves (dashed line) for $\sigma = \alpha(1+\varepsilon)$

i.e. straight lines passing $(\varepsilon=-1, \sigma=0)$

regardless of α, all these curves satisfy the condition

$$\frac{d\sigma}{d\varepsilon} = \alpha = \frac{\sigma}{1+\varepsilon}$$

At the intersection point between the stress strain curve $\sigma(\varepsilon)$ and any dashed line, if $\frac{d\sigma}{d\varepsilon}$ of the stress strain curve is larger than that of the dashed line, then the point is stable.

$\Theta \equiv \frac{d\sigma}{d\varepsilon}$ is called strain hardening rate

draw a family of curves (dashed line) for

$$\sigma = \alpha e^{\varepsilon^t}$$

(recall $\varepsilon^t = \ln(1+\varepsilon)$)

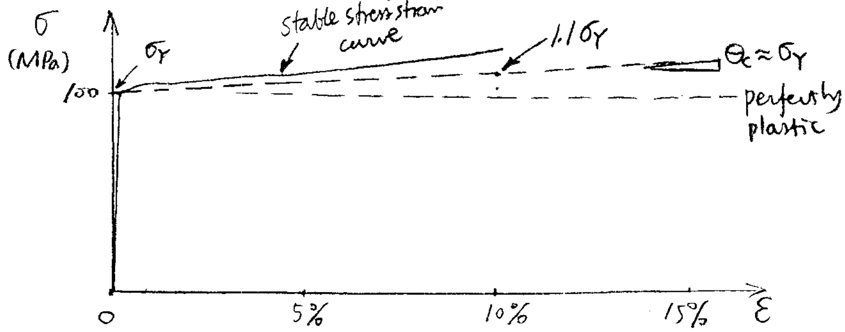
Regardless of α, all these curves satisfy the condition

$$\frac{d\sigma}{d\varepsilon^t} = \alpha e^{\varepsilon^t} = \sigma$$

At the intersection point, if $\frac{d\sigma}{d\varepsilon^t}$ of the stress-strain curve is larger than that of the dashed curve, the point is stable.

(note the point $\varepsilon = -1$ is equivalent to $\varepsilon^t = -\infty$)

How much hardening is required for stability in tension?



Suppose

$\sigma_Y = 100$ MPa

$E = 200$ GPa

Strain at $\sigma = \sigma_Y$

$\varepsilon_Y = \frac{\sigma_Y}{E} = 0.5 \times 10^{-3} = 0.05\%$

$\varepsilon_Y \approx 0$ at this scale

The required hardening rate is $\theta_c \approx \sigma_Y = 100$ MPa.

This means at $\varepsilon = 10\%$, $\sigma > 1.1\sigma_Y$

i.e. only 10% increase in flow stress for 10% plastic strain

This explains why material may satisfy the tensile stability criterion and yet can still be well approximated by the elastic-perfectly plastic model.

§2 Isotropic Hardening Model

The tensile stress-strain curve is not sufficient to fully describe the plastic behavior. We need to know how the entire yield surface change with plastic deformation.

The isotropic hardening model and the kinematic hardening model are two commonly used approximations to the complex behaviors of real materials.

In isotropic hardening model, we write the yield condition as

$$f(s_{ij}) = \varphi(q) \qquad \text{(Kachanov P.80)}$$

where s_{ij} is the deviatoric part of the stress σ_{ij} $\quad (s_{ij} = \sigma_{ij} - \bar{\sigma}\,\delta_{ij})$

(since yield should not depend on average stress $\bar{\sigma}$)

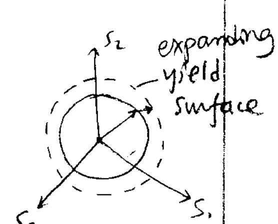
$q > 0$ is some measure of isotropic hardening.

A common measure of q is the work of plastic deformation

$$q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl}$$

Another, less frequent, measure is charaterisitic of accumulated plastic strain

$$q = \int \sqrt{2\, d\varepsilon_{ij}^{pl}\, d\varepsilon_{ij}^{pl}}$$



We can still use the von Mises criterion for function f.

$$f(s_{ij}) = \frac{1}{2} s_{ij}\, s_{ij}$$

(The special case of perfect-plasticity (no hardening) corresponds to
$\varphi(q) = k^2$ (constant))

For hardening model, we can choose $\varphi(q)$ to reproduce the tensile stress-strain curve.

As an example, let $f(S_{ij}) = \frac{1}{2} S_{ij} S_{ij}$, $q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl}$

and $\varphi(q) = k^2 + \beta q$

and let's find out how the tensile stress-strain curve looks like.

In uniaxial tension $\sigma_{xx} > 0$, $\sigma_{yy} = \sigma_{zz} = 0$, $\bar{\sigma} = \frac{1}{3}\sigma_{xx}$

$$S_{xx} = \sigma_{xx} - \bar{\sigma} = \frac{2}{3}\sigma_{xx}, \quad S_{yy} = -\frac{1}{3}\sigma_{xx}, \quad S_{zz} = -\frac{1}{3}\sigma_{yy}$$

$$f(S_{ij}) = \frac{1}{2} S_{ij} S_{ij} = \frac{1}{2}\left(\frac{4}{9}\sigma_{xx}^2 + \frac{1}{9}\sigma_{xx}^2 + \frac{1}{9}\sigma_{xx}^2\right) = \frac{1}{3}\sigma_{xx}^2$$

plastic flow rule:

$$d\varepsilon_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu} S_{ij}\, dt \qquad \underline{d\varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}}{2\mu}\frac{2}{3}\sigma_{xx}\, dt}$$

$$q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl} = \int S_{ij} \frac{\tilde{\lambda}}{2\mu} S_{ij}\, dt$$

$$dq = \frac{\tilde{\lambda}}{2\mu} S_{ij} S_{ij}\, dt = \frac{\tilde{\lambda}}{2\mu}\frac{1}{3}\sigma_{xx}^2\, dt$$

hardening law:

$$f(S_{ij}) \equiv \frac{1}{2} S_{ij} S_{ij} = \varphi(q) \equiv k^2 + \beta q$$

$$\frac{1}{3}\sigma_{xx}^2 = k^2 + \beta q$$

$$\frac{2}{3}\sigma_{xx}\, d\sigma_{xx} = \beta\, dq = \beta \cdot \frac{\tilde{\lambda}}{2\mu}\cdot\frac{1}{3}\sigma_{xx}^2\, dt$$

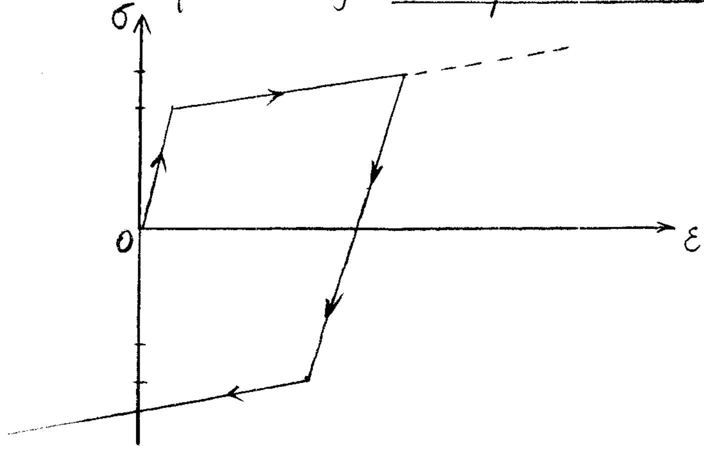
$$\underline{d\sigma_{xx} = \frac{\tilde{\lambda}}{2\mu}\cdot\frac{\beta}{2}\sigma_{xx}\, dt}$$

$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}^{pl}} = \frac{3}{4}\beta, \quad \frac{d\varepsilon_{xx}^{tot}}{d\sigma_{xx}} = \frac{d\varepsilon_{xx}^{el}}{d\sigma_{xx}} + \frac{d\varepsilon_{xx}^{pl}}{d\sigma_{xx}} = \frac{1}{E} + \frac{4}{3\beta}, \quad \text{Hardening rate: } \theta \equiv \frac{d\sigma_{xx}}{d\varepsilon_{xx}^{tot}} = \left(\frac{1}{E} + \frac{4}{3\beta}\right)^{-1}$$

Note that when hardening is present, $\tilde{\lambda}$ can be determined from $d\sigma_{xx}/dt$, which was not possible in perfectly plastic material.

Q: Find the hardening rate in uniaxial tension if $q = \int \sqrt{2\, d\varepsilon_{ij}^{pl}\, d\varepsilon_{ij}^{pl}}$ is used instead.

consequence of <u>isotropic hardening</u> model in reverse loading.

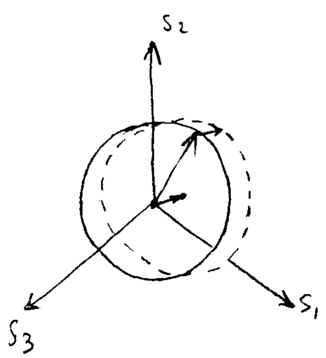


yield stress in reverse loading also increases

no Bauschinger effect.

§3 Kinematic Hardening Model

(Translational hardening in Kachanov)



$$f(s_{ij} - \alpha_{ij}) = k^2$$

$$\alpha_{ij} = c\, \varepsilon_{ij}^{pl}$$

Let's find out its consequence in uniaxial tension

$$s_{xx} = \tfrac{2}{3}\sigma_{xx}, \quad s_{yy} = -\tfrac{1}{3}\sigma_{xx}, \quad s_{zz} = -\tfrac{1}{3}\sigma_{xx}.$$

$$d\varepsilon_{ij}^{pl} = \frac{\hat{\lambda}}{2\mu} s_{ij}\, dt$$

$$\therefore \quad d\varepsilon_{yy}^{pl} = d\varepsilon_{zz}^{pl} = -\tfrac{1}{2}\, d\varepsilon_{xx}^{pl}$$

$$\varepsilon_{yy}^{pl} = \varepsilon_{zz}^{pl} = -\tfrac{1}{2}\, \varepsilon_{xx}^{pl}$$

$$\alpha_{xx} = c\,\varepsilon_{xx}^{pl}, \quad \alpha_{yy} = -\tfrac{1}{2} c\,\varepsilon_{xx}^{pl}, \quad \alpha_{zz} = -\tfrac{1}{2} c\,\varepsilon_{xx}^{pl}$$

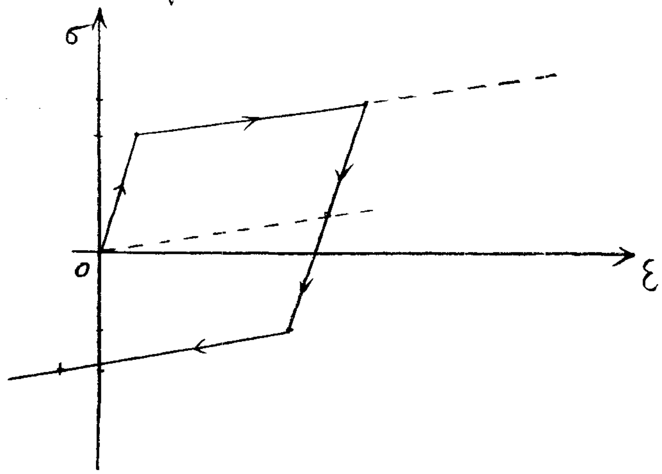
$$f(s_{ij} - \alpha_{ij}) = \frac{1}{2}\left[\left(\tfrac{2}{3}\sigma_{xx} - c\,\varepsilon_{xx}^{pl}\right)^2 + \left(-\tfrac{1}{3}\sigma_{xx} + \tfrac{1}{2}c\,\varepsilon_{xx}^{pl}\right)^2 + \left(-\tfrac{1}{3}\sigma_{xx} + \tfrac{1}{2}c\,\varepsilon_{xx}^{pl}\right)^2\right]$$

$$= \frac{1}{3}\left(\sigma_{xx} - \tfrac{3}{2}c\,\varepsilon_{xx}^{pl}\right)^2 = k^2$$

$$\sigma_{xx} = \sqrt{3}\,k + \tfrac{2}{3}c\,\varepsilon_{xx}^{pl}$$

$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}^{pl}} = \frac{2}{3}c \qquad \text{Hardening rate: } \Theta \equiv \frac{d\sigma_{xx}}{d\varepsilon_{xx}^{tot}} = \left(\frac{1}{E} + \frac{3}{2c}\right)^{-1}$$

consequence of kinematic hardening model in reverse loading.



Bauschinger effect

of course, a combination of isotropic and kinematic hardening model can be desired, e.g.

$$f(S_{ij}-\alpha_{ij}) = k^2 + \beta q$$

$$\alpha_{ij} = C\,\varepsilon_{ij}^{pl}, \qquad q = \int \sigma_{ij}\, d\varepsilon_{ij}^{pl}$$

Q: work out its consequence in uniaxial tension test

§4 Associated Flow Law

The flow rule we have been using: $\dot{\varepsilon}_{ij}^{pl} = \frac{\hat{\lambda}}{2\mu} s_{ij}$

is an associated flow law with the von Mises yield condition.

$$f(s_{ij}) \equiv \frac{1}{2} s_{ij} s_{ij} = \varphi(q)$$

On the other hand, it is not associative with the Tresca yield condition: $|\sigma_1 - \sigma_3| = \varphi(q)$, $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$

To understand what it means, we shall introduce a function $g(\sigma_{ij})$ (plastic potential) such that the flow rule can be written as

$$\dot{\varepsilon}_{ij}^{pl} = h \frac{\partial g}{\partial \sigma_{ij}}$$ where h is a scalar function.

If $g = f$, then the flow rule is associative (with the yield condition)

for flow — for yield surface

— plastic flow is always normal to yield surface

For example, for the von Mises yield condition, $f = \frac{1}{2} s_{ij} s_{ij}$.

the associated flow rule is

$$\dot{\varepsilon}_{ij}^{pl} = h \cdot \frac{\partial f}{\partial \sigma_{ij}} = h \frac{\partial}{\partial s_{ij}} \left(\frac{1}{2} s_{ij} s_{ij}\right) = h \cdot s_{ij}$$

This is the same as $\dot{\varepsilon}_{ij}^{pl} = \frac{\hat{\lambda}}{2\mu} s_{ij}$ if we identify $h = \frac{\hat{\lambda}}{2\mu}$.

Q: what is the associative flow rule for Tresca's yield condition?

§5. Drucker's Postulate

(Kachanov p86)

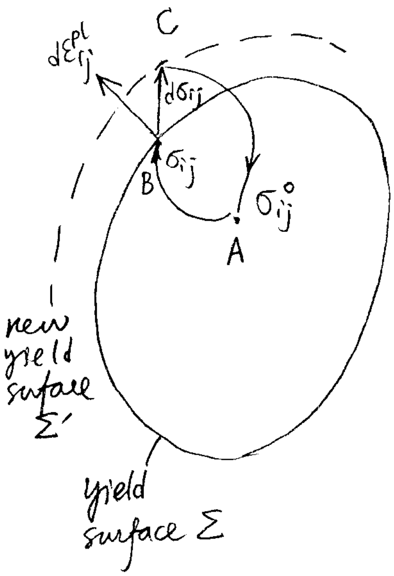
Consider a hardening medium with original stress σ_{ij}°, on which we apply additional stress $d\sigma_{ij}$ and then remove it.

The loading is slow enough to be considered isothermal.

Drucker's Postulate:

1. In the loading process, additional stress $d\sigma_{ij}$ does positive work
2. For a complete cycle of loading and unloading, additional stress does positive work if plastic deformation takes place.

   (For a hardening material, the work will be zero only for purely elastic deformation)



Consider loading path $A \to B \to C$

unloading path $C \to A$

Drucker's Postulate (2): $\oint (\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij} > 0$

Given that work done on elastic strain over the complete cycle is zero:

$$\oint (\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij}^{el} = 0$$

We have

$$\oint (\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij}^{pl} = 0$$

But plastic strain only occurs during $B \to C$

$$\therefore \quad (\sigma_{ij} - \sigma_{ij}^{\circ})\, d\varepsilon_{ij}^{pl} > 0 \qquad \text{(local maximum principle)}$$

This is true for any σ_{ij}° inside the yield surface Σ.

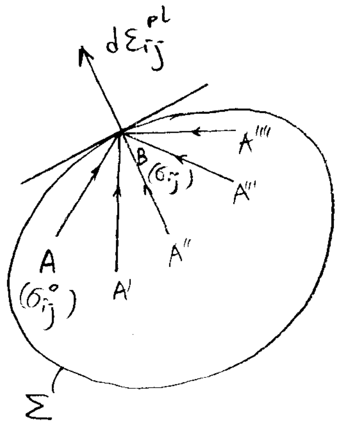
Hence it is a pretty strong requirement.

If we let the point A coincide with point B, we have

during $B \to C$: $d\sigma_{ij} \, d\varepsilon_{ij} > 0$ Drucker's postulate (1)

considering the cycle $B \to C \to B$: $d\sigma_{ij} \, d\varepsilon_{ij}^{pl} > 0$ Drucker's postulate (2)

Associative Flow as a consequence



Assume yield surface is convex

consider point B (σ_{ij}) on the yield surface.

Note the requirement

$$(\sigma_{ij} - \sigma_{ij}^{\circ}) \, d\varepsilon_{ij}^{pl} > 0$$

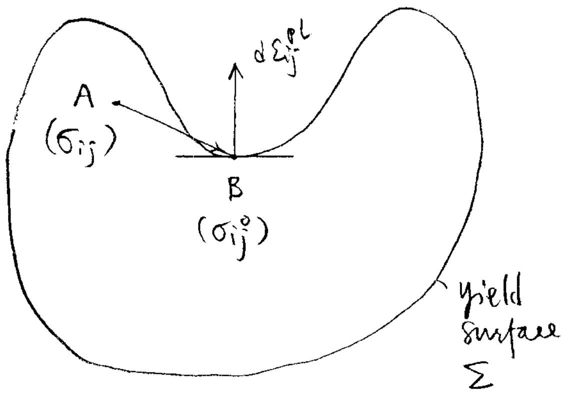
for all σ_{ij}° inside Σ, i.e. for all points A, A', A'', A''', ...

This is only possible if $d\varepsilon_{ij}^{pl}$ is normal to the yield surface Σ at point B.

$\Longrightarrow$ Associative flow !

Hence we see that Drucker's postulate implies associative flow rule ($g = f$).

convexity of yield surface



If the yield surface Σ is not convex, then we can always find a point $A(\sigma_{ij})$ such that

$$(\sigma_{ij} - \sigma_{ij}^{o})\, d\varepsilon_{ij}^{pl} < 0$$

violating Drucker's postulate

(physically, it means that work is extracted in a complete cycle $A \to B \to C \to A$ that produces plastic deformation, which is possible if there is internal stress in the material.)

For perfect plasticity (no hardening), Drucker's postulate is modified to: $\quad d\sigma_{ij}\, d\varepsilon_{ij}^{p} = 0$.

(Valid for every loading process $B \to C$ and for every cycle $B \to C \to B$).