The solution to the rigid-plastic model can be used to construct upper and lower bounds for the load on a elastic-perfectly plastic material reaching a critical state.

§1. Theorems of Limit Analysis (Prager-Hodge, §33, p.213)

Assume surface traction $T_j(\underset{\sim}{x})$ increases monotonically with time by an overall multiplicative constant. A state of impending plastic flow will eventually be reached, where plastic strain increases without increasing $T_j(\underset{\sim}{x})$ (i.e. unrestricted plastic flow).

Vanishing of stress rate at impending plastic flow: $\dot{\sigma}_{ij} = 0$ everywhere

This can be proved using Druker's inequality postulate

$$\dot{\sigma}_{ij}\, \dot{\varepsilon}_{ij}^{p} \geq 0 \qquad \text{(to be discussed later)}$$

which is valid for both work-hardening and perfectly-plastic (here) materials.

The statement $\dot{\sigma}_{ij} = 0$ is needed to prove the two theorems below.

Lower Bound Theorem

Any statically admissible (stress field) solution provides a lower bound of the load at incipient plasticity.

A statically admissible stress field σ^0_{ij} satisfies the following conditions:

① equilibrium condition everywhere: $\sigma^0_{ij,i} + F_j = 0$

② boundary condition: $\sigma^0_{ij} n_i = T_j$

(assume only two types of B.C.: S_T where T_j is specified and S_u where $u_i = 0$)

③ yield inequality everywhere: $(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 - 4k^2 \leq 0$

Since $T_j(x)$ on S_T is specified up to an overall constant, the largest constant for which a statically admissible stress field σ^0_{ij} can still be found is still a lower bound to this constant at impending plastic flow.

Upper Bound Theorem

Any kinematically admissible (velocity field) solution provides an upper bound of the load at incipient plasticity.

A kinematically admissible velocity field v_i^* satisfies the following conditions:

① incompressibility $v^*_{x,x} + v^*_{y,y} = 0$
everywhere, or throughout each finite subregions separated by lines of discontinuity (where only tangential velocity can be discontinuous)

② boundary condition: $v_i^* = 0$ at S_u where $u = 0$.

③ positive work rate: $\int_{S_T} T_j(x)\, v_j^*(x)\, dS > 0$

The overall multiplicative constant in $T_j(x)$ is obtained from the condition (to give an upper bound)

$$\dot{W} \equiv \int_{S_T} T_j(x)\, v_j^*(x)\, dS = k\left[\int_V \sqrt{\dot{\varepsilon}^*_{ij}\, \dot{\varepsilon}^*_{ij}}\, dV + \int_{S_J} |\Delta v_T|\, dS\right] \equiv \dot{D}$$

where $\dot{\varepsilon}^*_{ij} = \frac{1}{2}\left(v^*_{i,j} + v^*_{j,i}\right)$.

S_J: lines of discontinuity

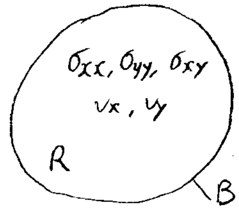
$|\Delta v_T|$: jump in tangential velocity

v_i^* provides a possible scenario of failure (incipient plastic flow). The above equation states that the work done by the external load must be sufficient to balance the internal dissipation of the failure scenario

Note that a true solution to the elastic-plastic material at incipient plasticity must has a stress field that is statically admissible and a velocity field that is kinematically admissible, simultaneously.

§2 Principle of virtual work (Prager-Hodge, §32 p.209)

The derivation of the lower and upper bound theorems require the principle of virtual work, which is stated here (in 2D plane strain), even though we are not going to derive the lower and upper bound theorems.



consider a stress field $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ and a velocity field v_x, v_y in region R that are continuous and have continuous first derivatives.

The stress field and the velocity field can be chosen <u>independently of each other</u>

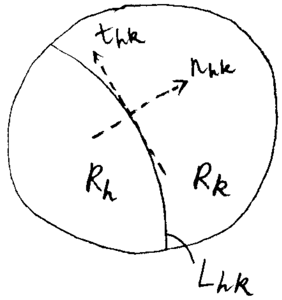
Define: traction force at boundary: $T_j = \sigma_{ij} n_i$

strain rate: $\dot{\varepsilon}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$

<u>Principle of virtual work:</u>

$$\int_R (\sigma_{xx}\dot{\varepsilon}_{xx} + \sigma_{yy}\dot{\varepsilon}_{yy} + 2\sigma_{xy}\dot{\varepsilon}_{xy})\, dA = \int_B (T_x v_x + T_y v_y)\, dS$$

as long as σ_{ij} satisfies equilibrium condition in R.



Now consider R containing lines of discontinuity that divide R into a finite number of subregions: $R_1, R_2, \cdots, R_n$. Such that $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are continuous and have continuous first derivatives in each subregion.

We also allow tangential velocity to be discontinuous at line of discontinuity

<u>Principle of virtual work:</u>

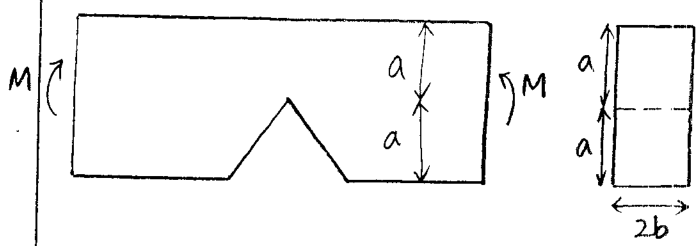
$$\int_R (\sigma_{xx}\dot{\varepsilon}_{xx} + \sigma_{yy}\dot{\varepsilon}_{yy} + 2\sigma_{xy}\dot{\varepsilon}_{xy})\, dA = \int_B (T_x v_x + T_y v_y)\, dS + \sum \int_{L_{hk}} \overbrace{T^{(hk)} (v_T^h - v_T^k)}^{\text{negative}}\, dl_{hk}$$

as long as σ_{ij} satisfies equilibrium condition in R.

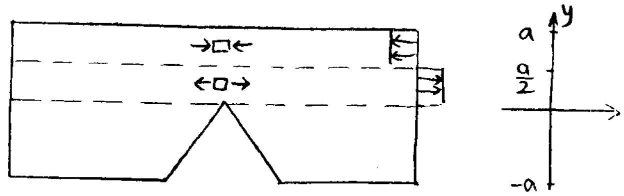
$T^{(hk)}$ is shear stress transmitted across L_{hk} from R_k to R_h

$(v_T^h - v_T^k)$ is the discontinuity in tangential velocity across L_{hk}.

§3. Notched Bar in Bending



Find M at impending plasticity



lower bound estimate

construct a statically admissible stress field

$$\sigma^{\circ}_{xx} = -2k \qquad \frac{a}{2} < x \leq a$$

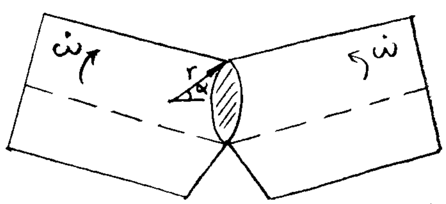
$$\sigma^{\circ}_{xx} = 2k \qquad 0 < x < \frac{a}{2}$$

$$\sigma^{\circ}_{xx} = 0 \qquad -a \leq x < 0$$

all other stress components zero

$$\left(\because (\sigma^{\circ}_{xx} - \sigma^{\circ}_{yy})^2 + 4\,\sigma^{\circ 2}_{xy} - 4k^2 \leq 0\right)$$

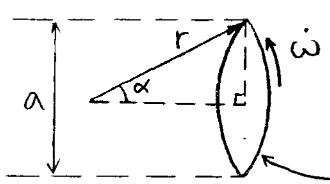
$$M = 2 \cdot 2k \cdot \frac{a}{2} \cdot \frac{a}{4} \cdot 2b = k a^2 b$$



Upper bound estimate

construct a kinematically admissible velocity field

assume a "mechanism" in which two rigid segment rotate around a hinge



rate of external work: $\dot{W} = 2M\dot{\omega}$

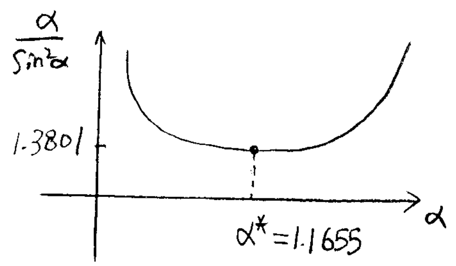
tangential velocity jump at interface

$$|\Delta V_T| = r\dot{\omega} = \frac{a/2}{\sin\alpha}\dot{\omega} \qquad \left(r\sin\alpha = \frac{a}{2}\right)$$

length of discontinuity line: $2 \cdot 2\alpha \cdot r = \frac{2\alpha a}{\sin\alpha}$

total internal dissipation:

$$\dot{D} \equiv k\left(\frac{a/2}{\sin\alpha}\right)\dot{\omega}\,\frac{2\alpha a}{\sin\alpha} \cdot 2b = 2ka^2b\dot{\omega}\frac{\alpha}{\sin^2\alpha}$$



To obtain the tightest upper bound, we choose α^* that minimizes $\frac{\alpha}{\sin^2\alpha}$

$$\min_{0<\alpha<\pi} \frac{\alpha}{\sin^2\alpha} = 1.3801 \quad \text{at} \quad \alpha^* = 1.1655$$

Matlab:
```
f=inline('x./sin(x).^2');
x0 = fminsearch(f, 1)
f(x0)
```

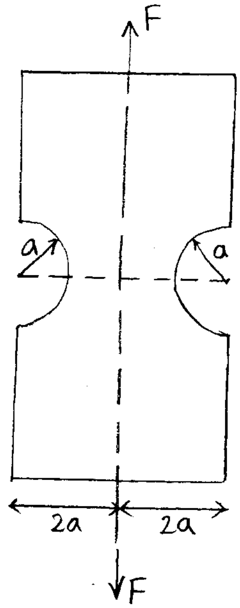
$$\dot{W} = 2M\dot{\omega} = 2ka^2b\,\dot{\omega}\frac{\alpha^*}{\sin^2\alpha^*}$$

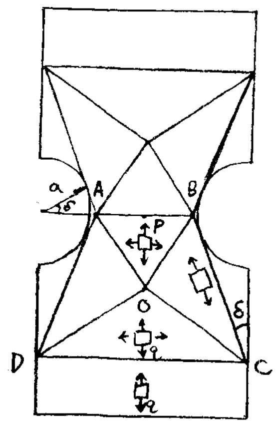
$$M = \frac{\alpha^*}{\sin^2\alpha^*}ka^2b = 1.38\,ka^2b$$

Therefore, the moment M at impending plastic flow:

$$1.0ka^2b \leq M \leq 1.38\,ka^2b$$

§4. Groved Rectangular Block





Find F at impending plasticity

lower bound estimate

construct a statically admissible stress field

recall the geometric condition

$$\frac{AB}{CD} = \frac{1-\sin\delta}{1+\sin\delta}$$

$$AB = 2\left(2a - \frac{a}{\cos\delta}\right)$$

$$CD = 4a$$

$$1 - \frac{1}{2\cos\delta} = \frac{1-\sin\delta}{1+\sin\delta} \rightarrow 2\sin 2\delta - \sin\delta - 1 = 0$$

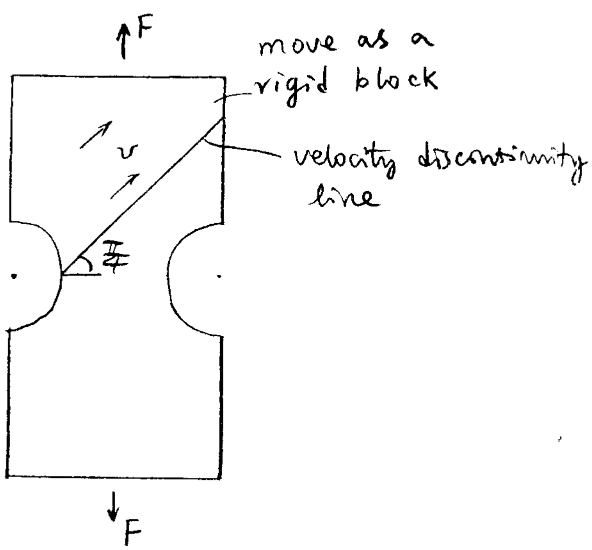
[Matlab: f=inline('2*sin(2*x)-sin(x)-1'); fzero(f, 0.1)]

$$\delta = 0.3765 \,, \quad \frac{AB}{CD} = \frac{1-\sin\delta}{1+\sin\delta} = 0.4623$$

$$q = 2k(1-\sin\delta)$$

$$F = 4a \cdot q = 8(1-\sin\delta)\, ka$$

$$F = 5.06\, ka$$



Upper bound estimate

construct a kinematically admissible velocity field

assume a rigid block sliding off

rate of external work: $\dot{W} = F \cdot V \cdot \frac{\sqrt{2}}{2}$

tangential velocity jump: $|\Delta V_T| = V$

Length of discontinuity line: $3a \cdot \sqrt{2}$

total internal dissipation:

$$\dot{D} = k \cdot V \cdot 3a\sqrt{2}$$

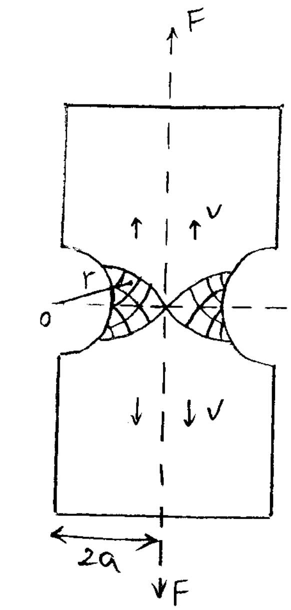
$$\dot{W} = F \cdot V \cdot \frac{\sqrt{2}}{2} = k \cdot V \cdot 3a\sqrt{2} = \dot{D}$$

$$F = 6\, ka$$

Therefore, the force at impending plastic flow: $5.06\, ka \le F \le 6\, ka$

a tighter upper bound can be obtained using the <u>slip line solution</u>

Logarithmic spirals



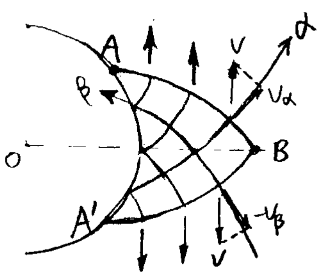
(Image on Coursework front page)

Stress field: $\sigma_{rr} = 0$ at $r = a$

$$\sigma_{rr} = 2k \ln \frac{r}{a} \qquad a \le r \le 2a$$

$$\sigma_{\theta\theta} = 2k + 2k \ln \frac{r}{a}$$

velocity field:



The constant velocity v of the rigid regions specifies boundary conditions for

V_α along spiral AB

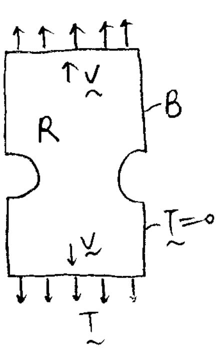
and for

V_β along spiral A'B

(tangential velocity can be discontinuous at the rigid-plastic boundary)

using: $dV_\alpha - V_\beta \, d\phi = 0$ along α-line

$dV_\beta - V_\alpha \, d\phi = 0$ along β-line



The entire velocity field V_x, V_y can be constructed for region ABA' (left as exercise)

AB and A'B are lines of discontinuity

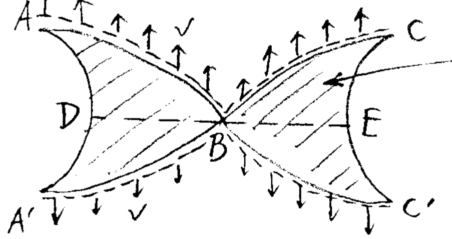
The <u>slip line velocity field</u> is <u>kinematically admissible</u> because:

① incompressibility $V_{x,x} + V_{y,y} = 0$

② there are no boundary where $\underset{\sim}{u} = 0$

③ $\int_B T_j v_j \, ds > 0$

To compute the dissipation rate $\dot{D}$ for this velocity field, we use the principle of virtual work, with the slip line stress field σ_{ij} (any stress field satisfying equilibrium and is ok)



in region $R = ABA' + CBC'$

lines of discontinuity $S_J = AB + A'B + CB + C'B$

$$F \cdot V = \dot{D} \equiv k\left[\int_R \sqrt{\dot{\varepsilon}^*_{ij}\, \dot{\varepsilon}^*_{ij}}\, dA + \int_{S_J} |\Delta V_T|\, ds \right]$$

$$= \int_{\text{Boundary}} (T_x V_x + T_y V_y)\, ds$$

dashed line in figure above → Boundary

where $T_j = \sigma_{ij} n_i$, σ_{ij} is the slip line field solution

$$= 2V \int_{AB+BC} T_y\, dS$$

$$= 2V \int_{DE} T_y\, dS$$

because σ_{ij} satisfies equilibrium

$$= 2V \int_{DE} \sigma_{yy}\, ds$$

$$= 4V \int_{DB} \sigma_{\theta\theta}\, ds$$

$$= 4V \int_a^{2a} 2k\left(1 + \ln\frac{r}{a}\right) dr$$

$$= 4k \cdot v \cdot \left[r \ln\frac{r}{a} \right]_a^{2a}$$

$$= 4k\, v \cdot (2a \ln 2)$$

$$F = 8 \ln 2 \cdot ka = 5.55\, ka \quad \text{(upper bound)}$$

$$\therefore \quad 5.06\, ka \le F \le 5.55\, ka$$

↑ a tighter upper bound than before ($6ka$).