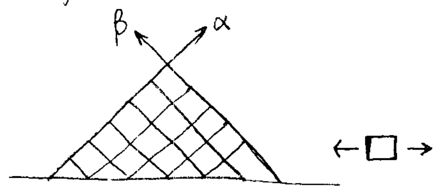
A lot of solutions can be constructed by combining a few slip line fields obtained before.

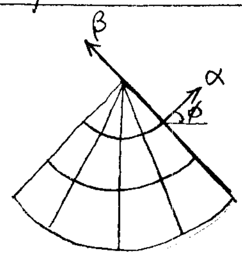
§1. Elementary Solutions

1. Uniform stress state



especially useful near flat free surfaces where the only non-zero stress is normal stress parallel to surface.

2. Simple stress state, especially with one family as concentric arcs.

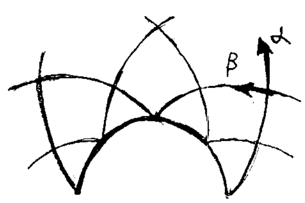


Here β-lines are straight, so

$$\frac{\bar{\sigma}}{2k} - \phi = \xi = \xi_0 \quad (\text{const})$$

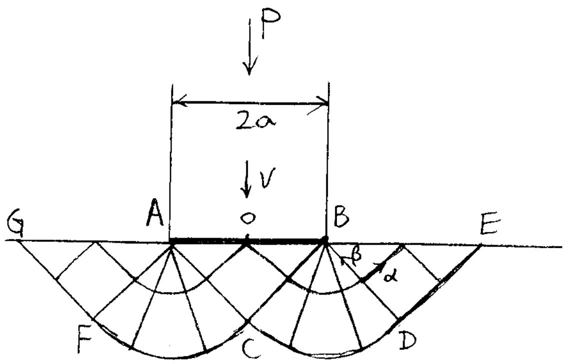
$$\bar{\sigma} = 2k(\phi + \xi_0)$$

3. Logarithmic spiral



especially useful near cylindrical free surfaces

§2. Flat indentor (rigid)

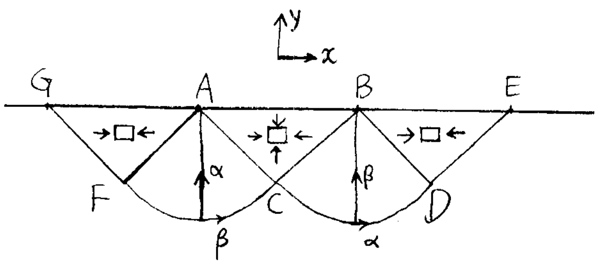


uniform stress state: BDE, GFA, ACB

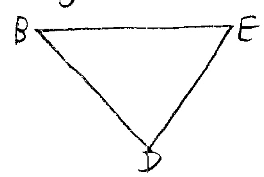
simple stress state: AFC, BCD

Prandtl's solution

(Kachanov, P. 218)



Region BDE

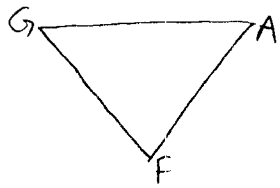


$\sigma_{yy} = 0$

$\sigma_{xx} = 2k$

$\bar{\sigma} = k$

Region GFA

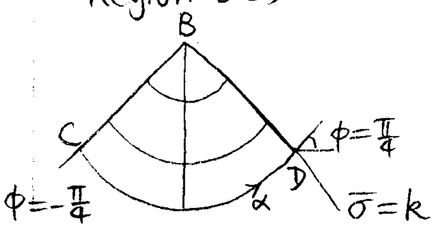


$\sigma_{yy} = 0$

$\sigma_{xx} = 2k$

$\bar{\sigma} = k$

Region BCD



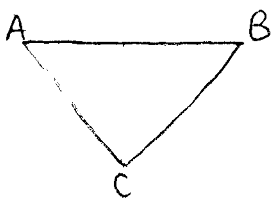
Along BD, $\phi = \frac{\pi}{4}$, $\bar{\sigma} = k$

Along α-line, $\bar{\sigma} - 2k\phi = \text{const}$

Hence along BC $\bar{\sigma} = -k + 2k \cdot \left(-\frac{\pi}{2}\right)$

$= -k(\pi + 1)$

Region ABC



$\bar{\sigma} = -k(\pi+1)$

$\sigma_{yy} = \bar{\sigma} - k = -k(\pi+2)$

$\sigma_{xx} = \bar{\sigma} + k = -k\pi$

(assume lubricated contact, no shear stress on AB)

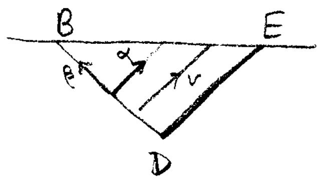
So the limit load is $P = -2a \cdot \sigma_{yy}$

$$\underline{P = 2ak(\pi+2)}$$

We now determine the velocity field using the compatibility condition
$$\begin{cases} dv_\alpha - v_\beta\, d\phi = 0 & \text{along } \alpha\text{-line} \\ dv_\beta + v_\alpha\, d\phi = 0 & \text{along } \beta\text{-line} \end{cases}$$

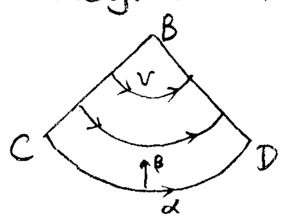
and boundary condition.

Region BDE



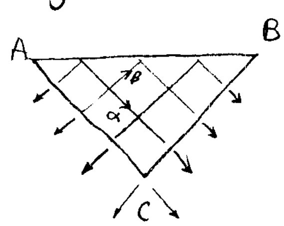
DE is elastic-plastic boundary.
Because we assume elastic region is rigid.
$v_\beta = 0$ on DE
Because $d\phi = 0$ on all β-lines (straight)
$v_\beta = 0$ on entire BDE region.
So the velocity is in the DE direction
(Still need to find magnitude distribution)

Region BCD

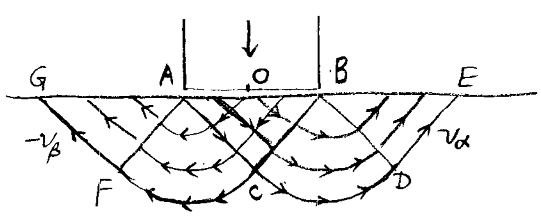


CD is elastic-plastic boundary
$v_\beta = 0$ on CD
Because β lines are straight,
$v_\beta = 0$ on the entire BCD region
So the velocity direction follows circular arcs

Region A B C



Because both α-lines and β-lines are straight,
v_α remain constant on α-lines
v_β remain constant on β-lines



Arrows indicate velocity stream lines

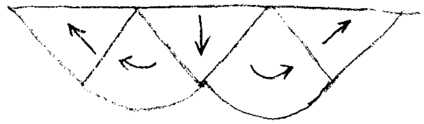
Velocities are discontinuous at AC and CB

However, the magnitude of the velocity must be conserved along each stream line, but not necessarily the same everywhere.

All that is required is v_y = const (negative) on AB

That means $$-\frac{v_\alpha}{\sqrt{2}} + \frac{v_\beta}{\sqrt{2}} = v_y \quad \text{on } AB.$$

Prandtl proposed $v_\alpha = -\frac{v_y}{\sqrt{2}}$, $v_\beta = \frac{v_y}{\sqrt{2}}$ everywhere,

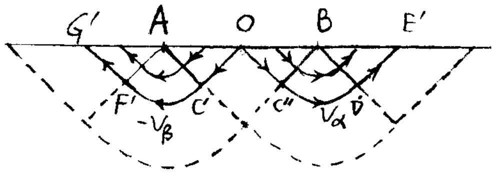


This means all triangular blocks move as rigid bodies.

But this solution is not unique.

Hill proposed an alternative solution, which essentially corresponds to letting

$$v_\alpha = -\sqrt{2}\, v_y \quad \text{on } OB$$
$$v_\beta = \sqrt{2}\, v_y \quad \text{on } AO$$



Hill's solution leads to similar stress fields.

BD′E′, G′F′A: $\sigma_{yy} = 0$, $\sigma_{xx} = -2k$, $\bar{\sigma} = -k$

BC″D′: $\bar{\sigma} = -k$ on BD′, $\bar{\sigma} = -k(\pi+1)$ on BC″

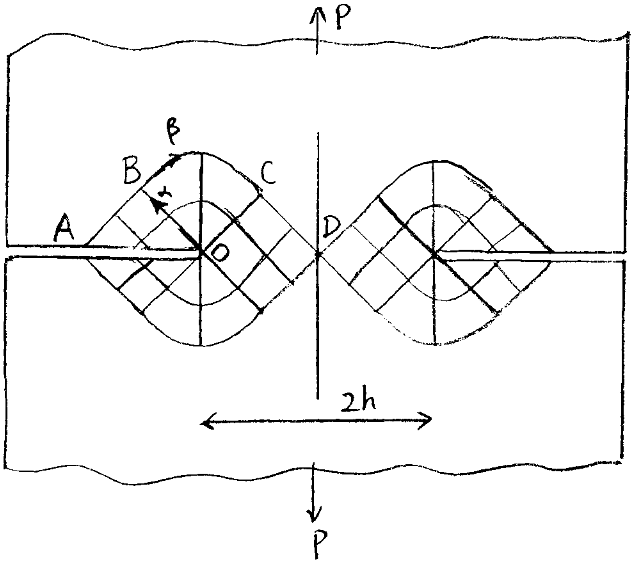
OC″B′, AC′O: $\bar{\sigma} = -k(\pi+1)$, $\sigma_{yy} = -k(\pi+2)$, $\sigma_{xx} = -k\pi$

So Hill's solution leads to the same limit load $P = 2ak(\pi+2)$

But the plastic zone is smaller, and the rigid blocks are moving at twice the speed of Prandtl's solution

Hill argued that plasticity should start to develop near points A, B (singular stress field points in elasticity), so the Hill's solution is reached first and Prandtl's solution may never be reached. We shall make further use of Hill's solution in the following.

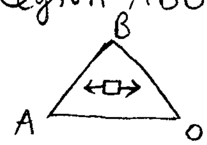
§3. Double Cracks



Turning the flat indentor in tension (with glue applied) and then replace the indentor by the mirror reflection of the sample itself, we get the double crack configuration.

Not surprisingly, the solution look a lot like Hill's solution (plus mirror reflection)

Region ABO

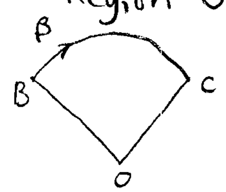


$\sigma_{xx} = 2k$

$\sigma_{yy} = 0$

$\bar{\sigma} = k$

Region OBC

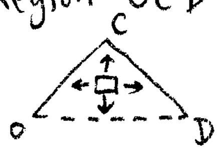


$\bar{\sigma} = k$ on OB

$\bar{\sigma} + 2k\phi = \text{const}$ on β-line

$\bar{\sigma} = (\pi+1)k$ on OC

Region OCD

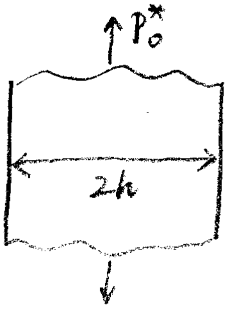


$\bar{\sigma} = (\pi+1)k$

$\sigma_{yy} = (\pi+2)k$

$\sigma_{xx} = \pi k$

The limit load is $P^* = 2(\pi+2)hk$



compared with uniform strip with width 2h whose limit load is $P_0^* = 2h \cdot 2k$

$$\frac{P^*}{P_0^*} = 1 + \frac{\pi}{2}$$

(This solution is only valid for sufficiently deep notches.)

Other Slip Line Field Solutions for Notches

L. M. Kachanov, Fundamentals of the Theory of Plasticity, Dover 2004

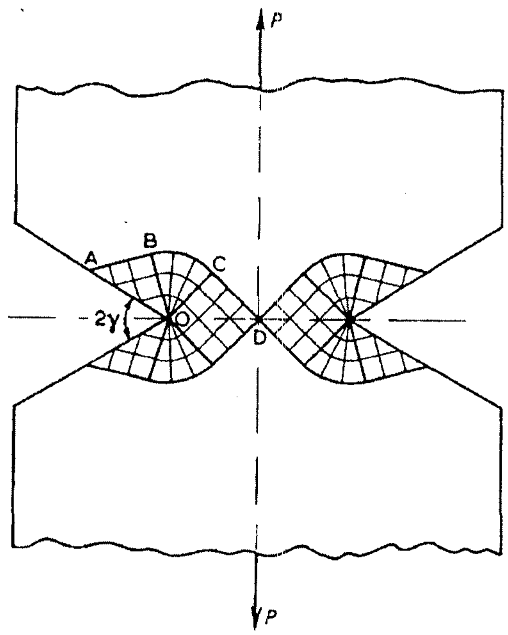


Fig. 106.

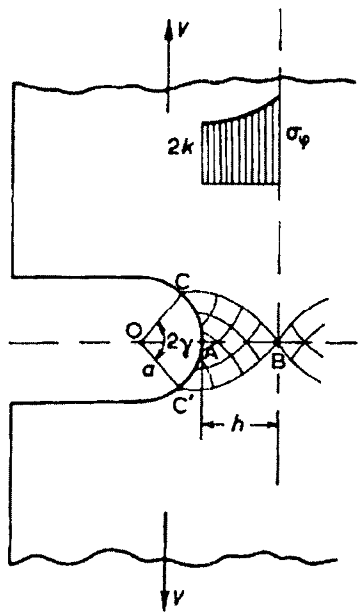


Fig. 107.

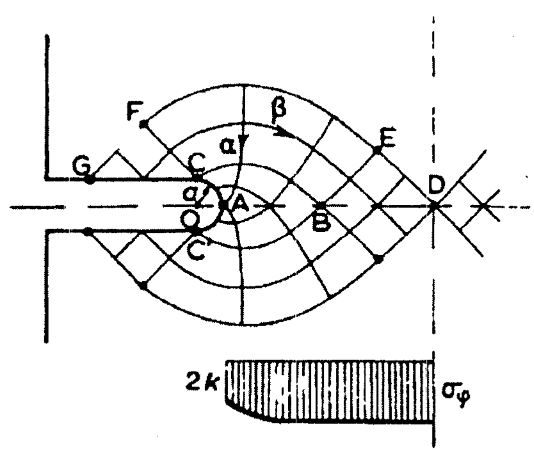
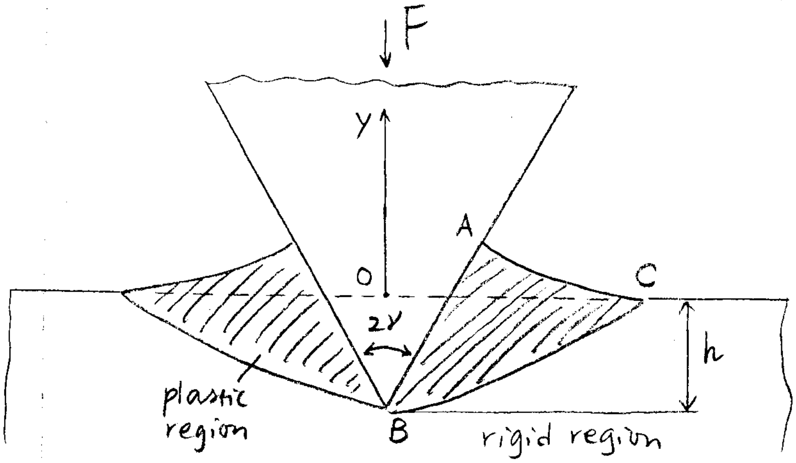


Fig. 108.

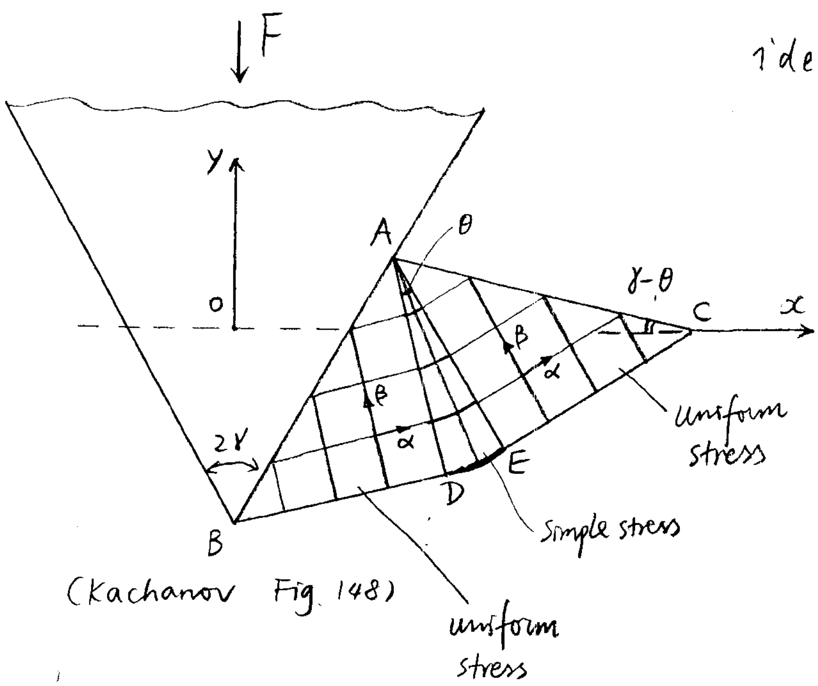
§4. Wedge Indentation

(Kachanov, p.246)



(Kachanov Fig.147)

idealization



(Kachanov Fig. 148)

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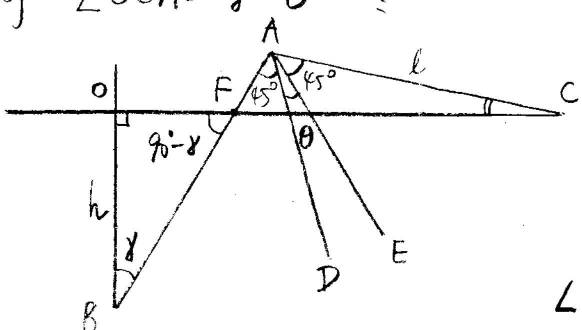
proof of $\angle OCA = \gamma - \theta$:



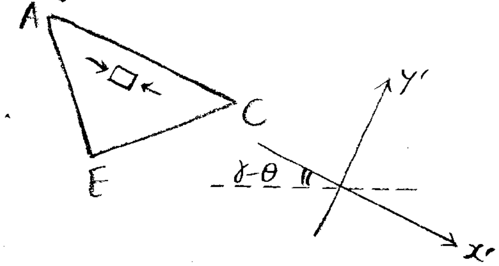
$AB = AC = l$
$OB = h$

$\angle OBA = \gamma \rightarrow \angle OFB = 90° - \gamma$
$\angle AFC = 90° - \gamma$

$$\left.\begin{aligned} \angle BDA &= 45° \\ \angle DAE &= \theta \\ \angle EAC &= 45° \end{aligned}\right\} \rightarrow \angle CAF = 90° + \theta$$

$$\begin{aligned} \angle OCA &= 180° - \angle CAF - \angle AFC \\ &= 180° - (90° + \theta) - (90° - \gamma) \\ &= \gamma - \theta \end{aligned}$$

Region AEC



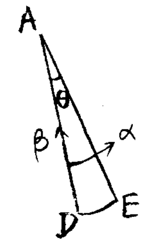
$\sigma_{x'x'} = -2k$

$\sigma_{y'y'} = 0$

$\sigma_{x'y'} = 0$

$\bar{\sigma} = -k$

Region DAE



β lines are straight, so

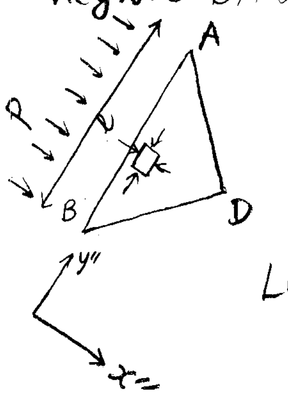
$$\frac{\bar{\sigma}}{2k} - \phi = \xi = \xi_0 \quad \text{(const)}$$

Along AE, $\bar{\sigma} = -k$

Along AD, $\bar{\sigma} = -k + 2k(-\theta)$

$$= -k(2\theta + 1)$$

Region BAD



$\bar{\sigma} = -k(2\theta+1)$

$\sigma_{x''x''} = \bar{\sigma} - k = -2k(\theta+1)$

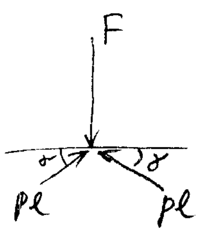
$\sigma_{y''y''} = \bar{\sigma} + k = -2k\theta$

Let length of AB $= l$, pressure on AB $= p$

$$p = 2k(\theta+1)$$

Total indentor force

$$F = 2pl\sin\gamma = 4kl(\theta+1)\sin\gamma$$

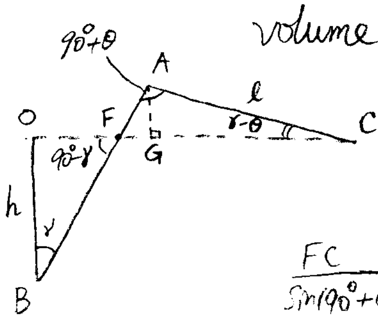


We still need to determine θ given γ.

volume conservation: area(OFB) = area(AFC)

area(OFB) $= \frac{1}{2}(OB)\cdot(OF)$

area(AFC) $= \frac{1}{2}(AG)\cdot(FC)$



$$\frac{FC}{\sin(90^\circ+\theta)} = \frac{AF}{\sin(\gamma-\theta)} = \frac{l}{\sin(90^\circ-\gamma)}$$

$$\frac{FC}{\cos\theta} = \frac{AF}{\sin(\gamma-\theta)} = \frac{l}{\cos\gamma}$$

$$FC = \frac{l\cos\theta}{\cos\gamma}$$

$$AG = l\sin(\gamma-\theta)$$

$$\text{area}(AFC) = \frac{1}{2} l\sin(\gamma-\theta) \cdot \frac{l\cos\theta}{\cos\gamma} = \frac{l^2\sin(\gamma-\theta)\cos\theta}{2\cos\gamma}$$

$$AF = \frac{l\sin(\gamma-\theta)}{\cos\gamma}$$

$$FB = AB - AF = l - \frac{l\sin(\gamma-\theta)}{\cos\gamma}$$

$$OF = FB\cdot\sin\gamma = l\left(1-\frac{\sin(\gamma-\theta)}{\cos\gamma}\right)\sin\gamma$$

$$\boxed{h = OB = FB\cdot\cos\gamma = l(\cos\gamma - \sin(\gamma-\theta))}$$

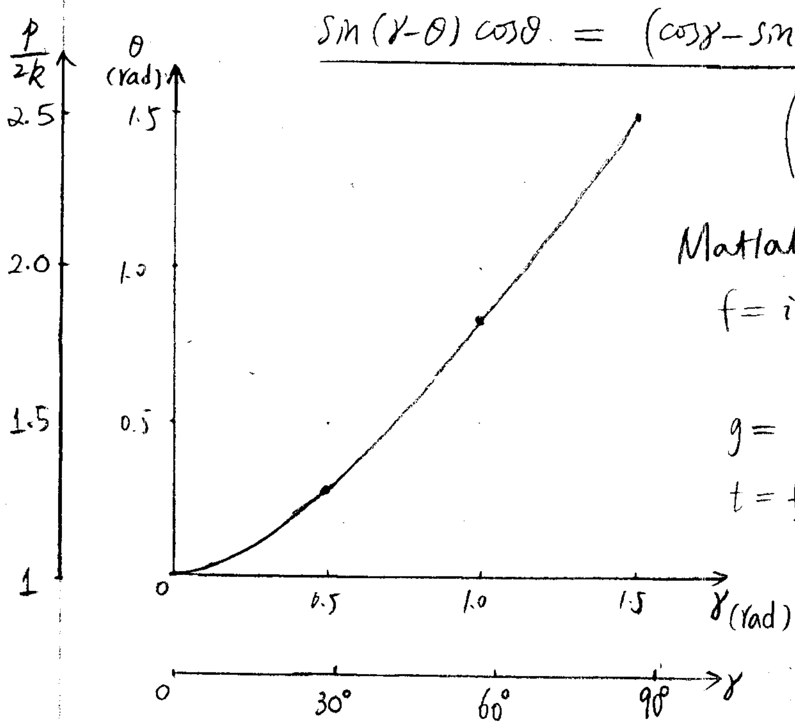
$$\text{area}(OFB) = \frac{1}{2} l\left(1-\frac{\sin(\gamma-\theta)}{\cos\gamma}\right)\sin\gamma \cdot l(\cos\gamma-\sin(\gamma-\theta))$$

$$= \frac{l^2}{2}\frac{(\cos\gamma-\sin(\gamma-\theta))^2\sin\gamma}{\cos\gamma}$$

$$\therefore \frac{l^2\sin(\gamma-\theta)\cos\theta}{2\cos\gamma} = \frac{l^2(\cos\gamma-\sin(\gamma-\theta))^2\sin\gamma}{2\cos\gamma}$$

$$\underline{\sin(\gamma-\theta)\cos\theta = (\cos\gamma-\sin(\gamma-\theta))^2\sin\gamma}$$

$\left(2\gamma = \theta + \operatorname{acos}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right.$ Kachanov, p. 247 $\left.\right)$



Matlab:

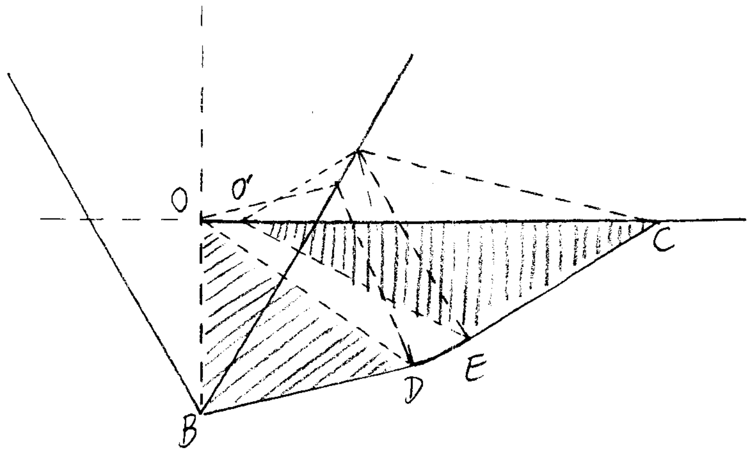
```
f= inline('sin(g-t).*cos(t)-(cos(g)-sin(g-t)).^2
        .*sin(g)', 't', 'g');
g= [0:0.02:1.5];
t= fsolve(@(x) f(x,g), g);
```

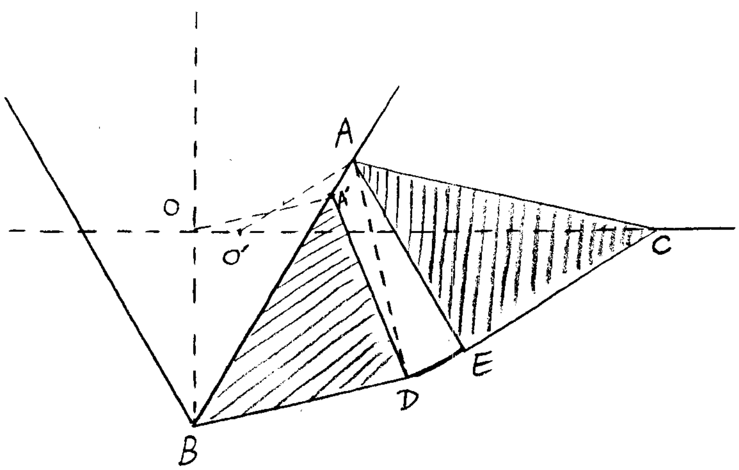
pressure at indentor face: p

$$\boxed{\frac{p}{2k} = \theta + 1}$$

total indenting force: F

$$\frac{F}{kl} = 4(\theta+1)\sin\gamma$$



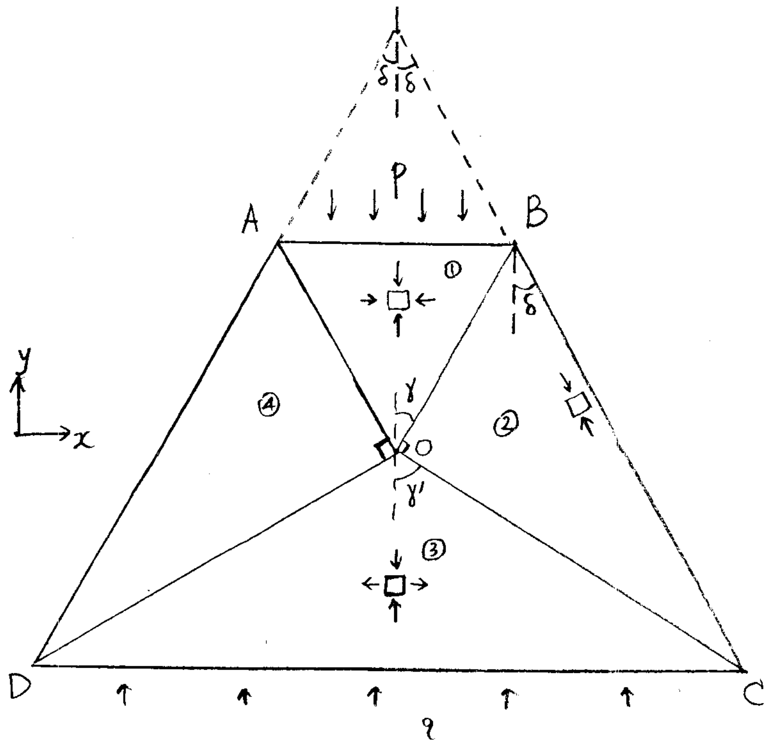


deformation pattern:

original domains		final domains		
OBD	$\rightarrow$	A'BD	simple shear	$O \rightarrow A'$
O'EC	$\rightarrow$	AEC	simple shear	$O' \rightarrow A$
ODEO'	$\rightarrow$	A'DEA	complex deformation	

§5 Truncated Wedge in compression

(Prager & Hodge, p.161)



all regions ①, ②, ③, ④ are in uniform stress.

Lines OA, OB, OC, OD are lines of discontinuity

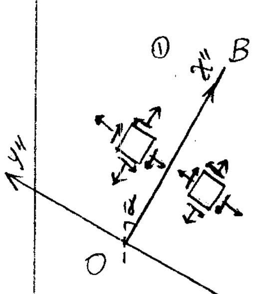
Geometrical relations:

$\gamma = \frac{\pi}{4} - \frac{\delta}{2}$ (to be proved below)

$\gamma' = \frac{\pi}{4} + \frac{\delta}{2}$

$\gamma + \gamma' = \frac{\pi}{2}$

$\therefore \angle AOD = \angle BOC = \frac{\pi}{2}$

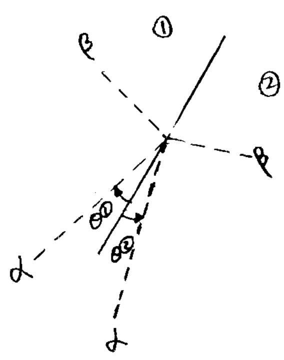


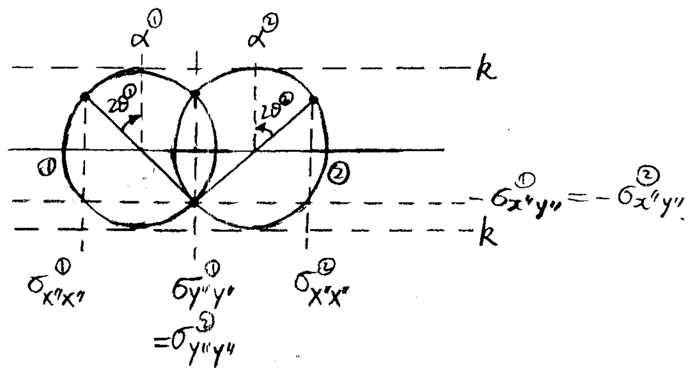
Across line of discontinuity, e.g. OB

$\sigma_{y''y''}$, $\sigma_{x''y''}$ still have to be continuous

only $\sigma_{x''x''}$ can be discontinuous, so $\bar{\sigma}$ may also be discontinuous

At the same time, $\underset{\approx}{\sigma}^{①}$ have to be on Mohr's circle of radius k. so is $\underset{\approx}{\sigma}^{②}$

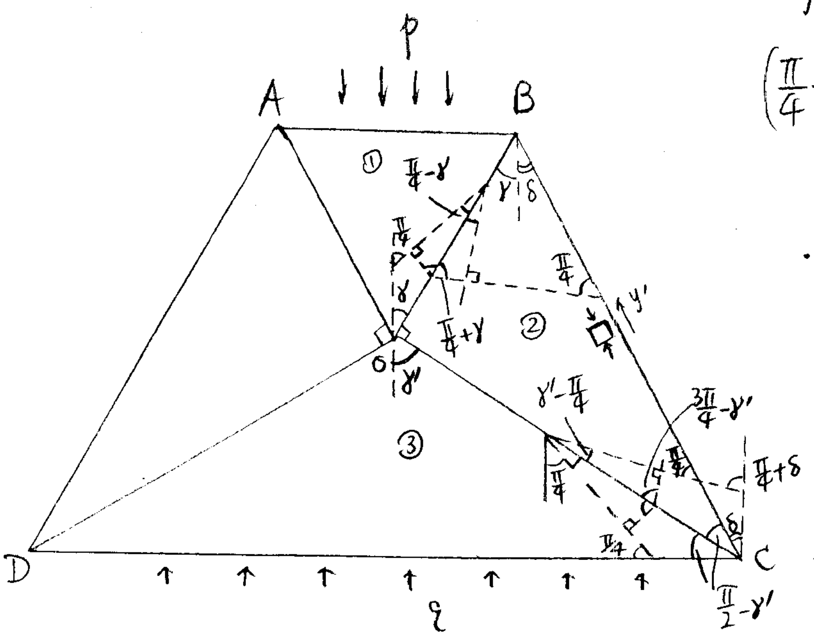




The angle of rotation to maximum shear orientation $\theta^{①}, \theta^{②}$ must be equal in magnitude and opposite in sign.

Hence the line of discontinuity must bisect slip lines of the same kind.

Proof of Geometrical relations:



$$\left(\frac{\pi}{4}+\gamma\right)+\left(\gamma+\delta\right)+\frac{\pi}{4}=\pi$$

$$2\gamma+\delta=\frac{\pi}{2}$$

$$\therefore\ \gamma=\frac{\pi}{4}-\frac{\delta}{2}$$

$$\left(\frac{\pi}{2}-\gamma'\right)+\left(\frac{\pi}{2}-\gamma'\right)+\delta=\frac{\pi}{2}$$

$$-2\gamma'+\delta+\frac{\pi}{2}=0$$

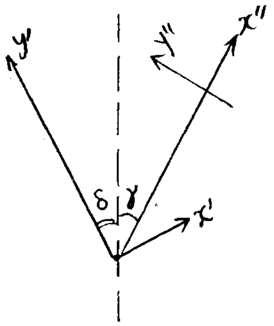
$$\therefore\ \gamma'=\frac{\pi}{4}+\frac{\delta}{2}$$

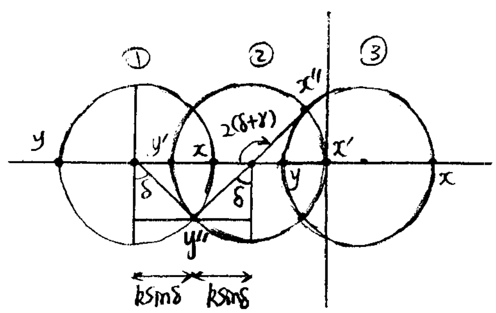
$$OB=\frac{AB}{2\sin\gamma},\quad OC=\frac{CD}{2\sin\gamma'}$$

$$\frac{OB}{OC}=\tan\left(\frac{\pi}{2}-\gamma'\right)=\frac{\cos\gamma'}{\sin\gamma'}=\frac{\sin\gamma}{\sin\gamma'}$$

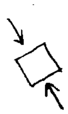
$$\Longrightarrow\ \boxed{\frac{AB}{CD}=\frac{\sin\gamma}{\sin\gamma'}\frac{OB}{OC}=\frac{\sin^2\gamma}{\sin^2\gamma'}=\frac{1-\sin\delta}{1+\sin\delta}}$$

(Q: what if $\frac{CD}{AB}>\frac{1+\sin\delta}{1-\sin\delta}$?)





Region ② BOC

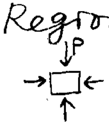


$$\sigma_{y'y'}=-2k$$
$$\sigma_{x'x'}=0$$
$$\sigma_{x'y'}=0$$
$$\bar{\sigma}=-k$$

rotation angle from y' to x'' on Mohr's circle $2(\delta+\gamma)=\frac{\pi}{2}+\delta$

$$\sigma_{y''y''}=-k-k\sin\delta$$
$$\sigma_{x''y''}=k\cos\delta$$

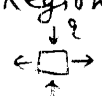
Region ①



$$\sigma_{y''y''}=-k-k\sin\delta$$
$$\sigma_{x''y''}=k\cos\delta$$
$$\bar{\sigma}=-k-2k\sin\delta$$

$$\sigma_{xx}=\bar{\sigma}+k=-2k\sin\delta$$
$$\sigma_{yy}=\bar{\sigma}-k=-2k(1+\sin\delta)=-p$$
$$\boxed{p=2k(1+\sin\delta)}$$

Region ③



$$\bar{\sigma}=-k+2k\sin\delta$$

$$\sigma_{xx}=\bar{\sigma}+k=2k\sin\delta$$
$$\sigma_{yy}=\bar{\sigma}-k=-2k(1-\sin\delta)=-q$$
$$\boxed{q=2k(1-\sin\delta)}$$