Problem Set 6. Cracks

Chris Weinberger and Wei Cai

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Problem 6.1 (15') Plane strain and plain stress equivalence.
Let the elastic stiffness tensor of a homogeneous solid be C_{ijkl} and its inverse (compliance tensor) be S_{ijkl}. In the plane strain problem, $e_{13} = e_{23} = e_{33} = 0$. Let the 2-dimensional elastic stiffness tensor be c_{ijkl}, i.e.,

$$\sigma_{ij} = c_{ijkl} e_{kl} \quad \text{for } i, j, k, l, = 1, 2 \quad \text{(plane strain)} \tag{1}$$

Obviously, $c_{ijkl} = C_{ijkl}$ for $i, j, k, l = 1, 2$.

For a plain stress problem, $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. Let the 2-dimensional elastic compliance tensor be $\tilde{s}_{ijkl}$, i.e.,

$$e_{ij} = \tilde{s}_{ijkl} \sigma_{kl} \quad \text{for } i, j, k, l, = 1, 2 \tag{2}$$

Obviously, $\tilde{s}_{ijkl} = S_{ijkl}$ for $i, j, k, l = 1, 2$. The inverse of $\tilde{s}_{ijkl}$ (in 2-dimension) is the effective elastic stiffness tensor in plain stress, $\tilde{c}_{ijkl}$.

(a) For isotropic elasticity, write down the explicit expression for c_{ijkl} and $\tilde{c}_{ijkl}$.

(b) The Kolosov's constant is defined as

$$\kappa = \begin{cases} 3 - 4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases}$$

Express c_{ijkl} and $\tilde{c}_{ijkl}$ in terms of μ and κ. (They should have the same expression now.)

Solution

(a)

$$\begin{aligned} c_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &= \frac{2\mu\nu}{1-2\nu} \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \tag{3}$$

For plane stress

$$\sigma_{ij} = C_{ijkl}e_{kl} \qquad \text{for i,j,k,l =1..3} \tag{4}$$

Writing out the three normal components

$$\begin{aligned}
\sigma_{11} &= (\lambda + 2\mu)e_{11} + \lambda e_{22} + \lambda e_{33} \\
\sigma_{22} &= (\lambda + 2\mu)e_{22} + \lambda e_{11} + \lambda e_{33} \\
\sigma_{33} &= 0 = (\lambda + 2\mu)e_{33} + \lambda e_{22} + \lambda e_{11}
\end{aligned}$$

solving for e_{33} in terms of e_{11} and e_{22}

$$\begin{aligned}
e_{33} &= -\frac{\lambda}{\lambda + 2\nu}(e_{11} + e_{22}) \\
&= -\frac{\nu}{1-\nu}(e_{11} + e_{22})
\end{aligned}$$

Thus

$$\begin{aligned}
\sigma_{11} &= (\lambda + 2\mu)e_{11} + \lambda e_{22} + -\lambda\frac{\nu}{1-\nu}(e_{11} + e_{22}) \\
&= \frac{2\mu(1-\nu)}{1-2\nu}e_{11} + \frac{2\mu\nu}{1-2\nu}e_{22} - \frac{\nu}{1-\nu}\frac{2\mu\nu}{1-2\nu}(e_{11} + e_{22}) \\
&= \frac{2\mu}{1-\nu}e_{11} + \frac{2\mu\nu}{1-\nu}e_{22}
\end{aligned}$$

Similarly

$$\sigma_{22} = \frac{2\mu}{1-\nu}e_{22} + \frac{2\mu\nu}{1-\nu}e_{11}$$

and there is no change to the shear relationship

$$\sigma_{12} = \mu e_{12} \tag{5}$$

Thus

$$\tilde{c}_{ijkl} = \frac{2\mu\nu}{1-\nu}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{6}$$

(b) In plane strain, $\kappa = 3 - 4\nu$, $\nu = (3-\kappa)/4$.

$$\begin{aligned}
c_{ijkl} &= \frac{2\mu(3-\kappa)/4}{1-2(3-\kappa)/4}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&= \frac{\mu(3-\kappa)}{(\kappa-1)}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\end{aligned}$$

In plane stress, $\kappa = (3-\nu)/(1+\nu)$, $\nu = (3-\kappa)/(\kappa+1)$.

$$\begin{aligned}
\tilde{c}_{ijkl} &= \frac{2\mu(3-\kappa)/(\kappa+1)}{1-(3-\kappa)/(\kappa+1)}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&= \frac{\mu(3-\kappa)}{(\kappa-1)}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\end{aligned}$$

Problem 6.2 (15') Mode II crack
(a) Derive the eigenstrain of equivalent inclusion for a slit-like crack (width $2a$) under uniform shear σ^A_{12} in plane strain.

(b) Derive the stress distribution in front of the crack tip. What is the stress intensity factor $K_{II} = \lim_{r\to 0} \sigma_{12}(r)\sqrt{2\pi r}$, where $r = x - a$ is the distance from the crack tip?

Solution

(a) Since this problem is in isotropic elasticity, we know that the shear terms will be completely decoupled, thus we only have to solve for e^*_{12}. The stress inside the equivalent inclusion is

$$
\begin{aligned}
\sigma^I_{12} &= \sigma^c_{12} - \sigma^*_{12} \\
&= 2C_{1212}\mathcal{S}_{1212}2e^*_{12} - 2C_{1212}e^*_{12} \\
&= 2\frac{\mu}{1-\nu}\left(\frac{a^2+b^2}{2(a+b)^2} + \frac{1-2\nu}{2}\right)e^*_{12} - 2\mu e^*_{12} \\
&= 2\frac{\mu}{1-\nu}\left(\frac{a^2+b^2}{2(a+b)^2} - \frac{1}{2}\right)e^*_{12} \\
&= -\frac{2\mu}{1-\nu}\frac{ab}{(a+b)^2}
\end{aligned}
$$

Now, for there to be no stress inside the crack $\sigma^I_{12} = -\sigma^A_{12}$ and solving for e^*_{12}

$$e^*_{12} = \frac{(a+b)^2}{ab}\frac{1-\nu}{2\mu}\sigma^A_{12}$$

Now, defining $e^* \equiv \lim_{b\to 0} e^*_{12}b$ then

$$e^* = a\frac{1-\nu}{2\mu}\sigma^A_{12}$$

(b) In class we derived the expression for the Eshelby tensor outside the ellipsoid along the x-axis. The $\mathcal{S}_{1212}$ term is

$$\mathcal{S}_{1212} = -\frac{1}{1-\nu}\frac{\Delta}{2}$$

where

$$\Delta \equiv \frac{b}{a}\left(1 - \frac{|x|}{\sqrt{x^2-a^2}}\right)$$

and from above

$$e^*_{12} = \frac{a}{b}\frac{1-\nu}{2\mu}\sigma^A_{12}$$

The constrained stress field is

$$\begin{aligned}
\sigma_{12}^c &= 4C_{1212}\mathcal{S}_{1212}e_{12}^* \\
&= 4\mu\left(-\frac{1}{1-\nu}\frac{\Delta}{2}\right)e_{12}^* \\
&= -\frac{2\mu}{1-\nu}\frac{b}{a}\left(1-\frac{|x|}{\sqrt{x^2-a^2}}\right)\frac{a}{b}\frac{1-\nu}{2\mu}\sigma_{12}^A \\
&= \left(\frac{|x|}{\sqrt{x^2-a^2}}-1\right)\sigma_{12}^A
\end{aligned}$$

The total stress is

$$\begin{aligned}
\sigma_{12}^{TOT} &= \sigma_{12}^A+\left(\frac{|x|}{\sqrt{x^2-a^2}}-1\right)\sigma_{12}^A \\
&= \frac{|x|}{\sqrt{x^2-a^2}}\sigma_{12}^A
\end{aligned}$$

The total stress can be written as a function of r as $r \to 0$

$$\sigma_{12}^{TOT} = \sqrt{\frac{a}{2r}}\sigma_{12}^A$$

and thus the stress intensity factor is

$$K_{II} = \sqrt{\pi a}\sigma_{12}^A$$