Problem Set 5. Eshelby's Inclusion

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Problem 5.1 (15') Use work method to derive the energy inside the inclusion E^I and inside the matrix E^M for an ellipsoidal inclusion in an infinite matrix. Follow the Eshelby's 4 steps to construct the inclusion.

(a) What are the forces applied to the inclusion and to the matrix in all 4 steps?

(b) What are the work done to the inclusion and to the matrix in all 4 steps?

(c) What is the elastic energy inside the inclusion E^I, and what is the elastic energy inside the matrix E^M at the end of step 4?

Solution

(a) The forces on the inclusion and matrix at the end of each step is

step	matrix	inclusion
1	0	0
2	0	$-\sigma^*_{ij}n_i$
3	0	$-\sigma^*_{ij}n_i$
4	$-(\sigma^c_{ij}-\sigma^*_{ij})n_i$	$(\sigma^c_{ij}-\sigma^*_{ij})n_i$

(b) For the Inclusion:

$$\begin{aligned}
E_1 &= 0 \\
\Delta W^I_{12} &= \frac{1}{2}\int_{S_0}(-\sigma^*_{ij}n_i)(-e^*_{jk}x_k)\,\mathrm{d}S \\
&= \frac{1}{2}\sigma^*_{ij}e^*_{ij}V_0 \\
\Delta W^I_{23} &= 0 \qquad \text{(because } u_j = 0\text{)} \\
\Delta W^I_{34} &= \frac{1}{2}\int_{S_0}(\sigma^c_{ij}-2\sigma^*_{ij})n_i e^c_{jk}x_k\,\mathrm{d}V \\
&= \frac{1}{2}(\sigma^c_{ij}-2\sigma^*_{ij})e^c_{ij}V_0
\end{aligned} \tag{1}$$

For the matrix:

$$\begin{aligned}
E_1 &= 0 \\
\Delta W_{12}^I &= 0 \\
\Delta W_{23}^I &= 0 \\
\Delta W_{34}^I &= -\frac{1}{2}\int_{S_0} (\sigma_{ij}^c - \sigma_{ij}^*) n_i e_{jk}^c x_k \, \mathrm{d}V \\
&= -\frac{1}{2}(\sigma_{ij}^c - \sigma_{ij}^*) e_{ij}^c V_0
\end{aligned}$$

(c) For the inclusion

$$\begin{aligned}
E^I &= \frac{1}{2}\sigma_{ij}^* e_{ij}^* V_0 + \frac{1}{2}(\sigma_{ij}^c - 2\sigma_{ij}^*) e_{ij}^c V_0 \\
&= \frac{1}{2}(\sigma_{ij}^* e_{ij}^* + \sigma_{ij}^c e_{ij}^c - 2\sigma_{ij}^* e_{ij}^c) V_0 \\
&= \frac{1}{2}\sigma_{ij}^I e_{ij}^I V_0
\end{aligned}$$

where $\sigma_{ij}^I = \sigma_{ij}^c - \sigma_{ij}^*$, and $e_{ij}^I = e_{ij}^c - e_{ij}^*$.

For the matrix

$$\begin{aligned}
E^M &= -\frac{1}{2}(\sigma_{ij}^c - \sigma_{ij}^*) e_{ij}^c V_0 \\
&= -\frac{1}{2}\sigma_{ij}^I e_{ij}^c V_0
\end{aligned}$$

The total energy is

$$\begin{aligned}
E &= E^I + E^M \\
&= \frac{1}{2}\sigma_{ij}^I (e_{ij}^I - e_{ij}^c) V_0 \\
&= -\frac{1}{2}\sigma_{ij}^I e_{ij}^* V_0 \qquad (2)
\end{aligned}$$

Problem 5.2 (15') Spherical inclusion. The Eshelby's tensor of a spherical inclusion inside an infinite medium is (see Lecture Note 2),

$$\mathcal{S}_{ijkl} = \frac{5\nu - 1}{15(1-\nu)}\delta_{ij}\delta_{kl} + \frac{4 - 5\nu}{15(1-\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad (3)$$

Consider a spherical inclusion of radius R with a pure shear eigenstrain $e_{12}^* = \varepsilon$ (other components of $e_{ij}^* = 0$).

(a) What is the total elastic energy of the system E as a function of R?

(b) Now apply a uniform stress field $\sigma_{12}^A = \tau$ to the solid (other stress components are zero). What is the total elastic energy $E(R)$?

(c) What is the enthalpy of the system $H(R)$? What is the driving force for inclusion growth, i.e. $f(R) = -dH(R)/dR$?

[Hint: Consider the solid has a finite but very large volume V. The external stress is applied at the external surface. Volume V is so large that the Eshelby's solution in infinite solid remains valid.]

Solution
(a) When $e^*_{12} = e^*_{21} = \varepsilon$, the total energy is

$$\begin{aligned} E &= -\frac{1}{2}\sigma^I_{ij}e^*_{ij}V_0 \\ &= -\frac{1}{2}(\sigma^c_{ij} - \sigma^*_{ij})e^*_{ij}V_0 \\ &= -(\sigma^c_{12} - \sigma^*_{12})\varepsilon V_0 \end{aligned}$$

The relevant component of Eshelby's tensor is,

$$\mathcal{S}_{1212} = \mathcal{S}_{1221} = \frac{4-5\nu}{15(1-\nu)} \tag{4}$$

Therefore,

$$e^c_{12} = \frac{4-5\nu}{15(1-\nu)}(e^*_{12} + e^*_{21}) = \frac{4-5\nu}{15(1-\nu)}2\varepsilon \tag{5}$$

For isotropic material

$$\sigma_{12} = 2\mu e_{12}$$

Thus,

$$\sigma^c_{12} = \frac{4-5\nu}{15(1-\nu)}4\mu\varepsilon \tag{6}$$

$$\sigma^*_{12} = 2\mu e^*_{12} = 2\mu\varepsilon \tag{7}$$

$$\begin{aligned} E(R) &= -\frac{1}{2}\sigma^I_{ij}e^*_{ij}V_0 \\ &= -\frac{1}{2}(\sigma^c_{ij} - \sigma^*_{ij})e^*_{ij}V_0 \\ &= -(\sigma^c_{12} - \sigma^*_{12})e^*_{12}V_0 \\ &= -\left[\frac{4-5\nu}{15(1-\nu)}4\mu\varepsilon - 2\mu\varepsilon\right]\varepsilon\frac{4}{3}\pi R^3 \\ &= \frac{8}{3}\mu\varepsilon^2\pi R^3\frac{7-5\nu}{15(1-\nu)} \end{aligned}$$

(b) From Colonetti's theorem, let E^∞ be the energy of the solid under zero applied stress,

$$
\begin{aligned}
E(R) &= E^\infty + E^A \\
E^A &= \frac{1}{2}\sigma_{12}^A e_{12}^A V + \frac{1}{2}\sigma_{21}^A e_{21}^A V = \sigma_{12}^A e_{12}^A V \\
\sigma_{12}^A &= \tau = 2\mu e_{12}^A \\
e_{12}^A &= \frac{\tau}{2\mu} \\
E^A &= \frac{\tau^2}{2\mu} V \\
E(R) &= \frac{8}{3}\mu\varepsilon^2\pi R^3 \frac{7-5\nu}{15(1-\nu)} + \frac{\tau^2}{2\mu} V
\end{aligned}
$$

(c)

$$
\begin{aligned}
H &= E^\infty - E^A - \sigma_{ij}^A e_{ij}^* V_0 \\
&= \frac{8}{3}\mu\varepsilon^2\pi R^3 \frac{7-5\nu}{15(1-\nu)} - \frac{\tau^2}{2\mu} V - \frac{8}{3}\tau\varepsilon\pi R^3
\end{aligned}
$$

Now, we can caculate the force for inclusion growth

$$
\begin{aligned}
f(R) = -\frac{\partial H}{\partial R} &= -8\mu\varepsilon^2\pi R^3 \frac{7-5\nu}{15(1-\nu)} + 8\tau\varepsilon\pi R^3 \\
&= 8\pi R^2 \left[\tau\epsilon - \frac{7-5\nu}{15(1-\nu)}\mu\varepsilon^2\right]
\end{aligned}
$$