ME340B – Elasticity of Microscopic Structures – Wei Cai – Stanford University – Winter 2004

Problem Set 4. Eshelby's Inclusion

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Problem 4.1 (15') Spherical inclusion.

(a) Derive the expressions for the auxiliary tensor $\mathcal{D}_{ijkl}$ for a spherical inclusion in an isotropic medium with shear modulus μ and Poisson's ratio ν.
[Hint: many components of $\mathcal{D}_{ijkl}$ are zero, unless there are repeated indices.]

(b) Derive the corresponding expressions for Eshelby's tensor $\mathcal{S}_{ijkl}$.

Solution

(a) The auxiliary tensor inside the inclusion can be expressed in terms of the following surface integral (see lecture notes),

$$\mathcal{D}_{ijkl} = -\frac{abc}{4\pi}\int_0^{\pi}\int_0^{2\pi}(zz)^{-1}_{ij}z_k\,z_l\,\frac{\sin\Phi}{\beta^3}\,\mathrm{d}\Theta\,\mathrm{d}\Phi \tag{1}$$

where

$$\beta \;=\; \lambda/k = \sqrt{(a^2\cos^2\Theta + b^2\sin^2\Theta)\sin^2\Phi + c^2\cos\Phi} \tag{2}$$

$$\mathbf{z} \;=\; \mathbf{k}/k = (\sin\Phi\cos\Theta, \sin\Phi\sin\Theta, \cos\Theta) \tag{3}$$

For a sphere $a = b = c$ and $\beta = a$. Therefore,

$$\mathcal{D}_{ijkl} = -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}(zz)^{-1}_{ij}z_k\,z_l\,\sin\Phi\,\mathrm{d}\Theta\,\mathrm{d}\Phi \tag{4}$$

For isotropic material, $(zz)^{-1}_{ij}$ has an explicit form, so that,

$$\begin{aligned}
\mathcal{D}_{ijkl} &= -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{1}{\mu}\left(\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu}\,z_i\,z_j\right)z_k\,z_l\,\sin\Phi\,\mathrm{d}\Theta\,\mathrm{d}\Phi \\
&= -\frac{1}{4\pi\mu}\int_0^{\pi}\int_0^{2\pi}\left(\delta_{ij}\,z_k\,z_l - \frac{\lambda+\mu}{\lambda+2\mu}z_i\,z_j z_k\,z_l\right)\sin\Phi\,\mathrm{d}\Theta\,\mathrm{d}\Phi
\end{aligned}$$

We can write the three components of $\mathbf{z}$ in terms of Φ and Θ

$$z_1 = \sin\Phi\cos\Theta \qquad z_2 = \sin\Phi\sin\Theta \qquad z_3 = \cos\Phi \tag{5}$$

Let us consider the first integral

$$H_{kl} \equiv \int_0^{\pi} \int_0^{2\pi} z_k\, z_l \,\sin\Phi \,\mathrm{d}\Theta \,\mathrm{d}\Phi$$

This integral will be zero if $k \neq l$. By symmetry, $H_{11} = H_{22} = H_{33}$. H_{33} is the simplest to evaluate in the present form, i.e.,

$$\begin{aligned} H_{33} &= \int_0^{\pi} \int_0^{2\pi} \cos^2\Phi \sin\Phi d\Theta d\Phi \\ &= 2\pi \int_{-1}^{1} s^2 ds \\ &= \frac{4\pi}{3} \end{aligned} \tag{6}$$

Therefore,

$$H_{kl} = \frac{4\pi}{3}\delta_{kl} \tag{7}$$

Now let us consider the second integral,

$$J_{ijkl} \equiv \int_0^{\pi} \int_0^{2\pi} z_i\, z_j z_k\, z_l \,\sin\Phi \,\mathrm{d}\Theta \,\mathrm{d}\Phi$$

The J_{ijkl} is non-zero only if the indices are all the same, or come in pairs. There are two types of terms,

$$J_{1111} = J_{2222} = J_{3333} = \int_0^{\pi} \int_0^{2\pi} \cos^4\Phi \sin\Phi \,\mathrm{d}\Theta \,\mathrm{d}\Phi = \frac{4\pi}{5} \tag{8}$$

and $J_{1122} = J_{1133} = J_{2233} = J_{1212} = \ldots$

$$J_{1133} = \int_0^{\pi} \int_0^{2\pi} \cos^4\Phi \sin^3\Phi \,\mathrm{d}\Theta \,\mathrm{d}\Phi = \frac{4\pi}{15}$$

Thus we can write J_{ijkl} as

$$J_{ijkl} = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{9}$$

The auxiliary tensor $\mathcal{D}_{ijkl}$ is

$$\mathcal{D}_{ijkl} = -\frac{1}{4\pi\mu}\left(\delta_{ij}H_{kl} - \frac{\lambda+\mu}{\lambda+2\mu}J_{ijkl}\right) \tag{10}$$

Substituting the solutions for H_{kl} and J_{ijkl}, we have

$$\begin{aligned} \mathcal{D}_{ijkl} &= -\frac{1}{15\mu}\left(5\delta_{ij}\delta_{kl} - \frac{\lambda+\mu}{\lambda+2\mu}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right) \\ &= -\frac{1}{30\mu}\left(10\delta_{ij}\delta_{kl} - \frac{1}{1-\nu}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right) \\ &= -\frac{1}{30\mu(1-\nu)}\left[(9-10\nu)\delta_{ij}\delta_{kl} - (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] \end{aligned} \tag{11}$$

using the fact that $\frac{\lambda+\mu}{\lambda+2\mu} = \frac{1}{2(1-\nu)}$.

(b) The Eshelby's tensor $\mathcal{S}_{ijkl}$ is related to the auxiliary tensor $\mathcal{D}_{ijkl}$ through,

$$\mathcal{S}_{ijmn} = -\frac{1}{2}C_{lkmn}(\mathcal{D}_{iklj} + \mathcal{D}_{jkli})$$

For isotropic medium,

$$C_{lkmn} = \lambda\delta_{lk}\delta_{mn} + \mu(\delta_{lm}\delta_{kn} + \delta_{ln}\delta_{km}) \tag{12}$$

thus,

$$\mathcal{S}_{ijmn} = -\frac{\lambda}{2}(\mathcal{D}_{ikkj} + D_{jkki})\delta_{mn} - \frac{\mu}{2}(\mathcal{D}_{inmj} + \mathcal{D}_{jnmi} + \mathcal{D}_{imnj} + \mathcal{D}_{jmni})$$

When $\mathcal{D}_{ijkl}$ is given in Eq. (11), it satisfies both major and minor symmetries, i.e. $\mathcal{D}_{ijkl} = \mathcal{D}_{klij} = \mathcal{D}_{jikl} = \mathcal{D}_{ijlk}$. Thus,

$$\mathcal{S}_{ijmn} = -\lambda\mathcal{D}_{ikkj}\delta_{mn} - \mu(\mathcal{D}_{inmj} + \mathcal{D}_{imnj}) \tag{13}$$

Noticing that

$$\mathcal{D}_{ikkj} = -\frac{5-10\nu}{30\mu(1-\nu)}\delta_{ij} \tag{14}$$

$$\lambda = \frac{2\mu\nu}{1-2\nu} \tag{15}$$

$$\lambda\mathcal{D}_{ikkj} = -\frac{\nu}{3(1-\nu)}\delta_{ij} \tag{16}$$

Thus

$$\begin{aligned}
\mathcal{S}_{ijmn} &= \frac{\nu}{3(1-\nu)}\delta_{ij}\delta_{mn} + \frac{1}{30(1-\nu)}\left[(9-10\nu)\delta_{ij}\delta_{kl} - (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] \\
&= \frac{\nu}{3(1-\nu)}\delta_{ij}\delta_{mn} + \frac{1}{30(1-\nu)}[(9-10\nu)(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn}) \\
&\quad -(\delta_{im}\delta_{jn} + \delta_{ij}\delta_{mn} + \delta_{in}\delta_{jm} + \delta_{ij}\delta_{mn})] \\
&= \frac{1}{30(1-\nu)}[(10\nu-2)\delta_{ij}\delta_{mn} + (9-10\nu-1)(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn})] \\
&= \frac{5\nu-1}{15(1-\nu)}\delta_{ij}\delta_{mn} + \frac{4-5\nu}{15(1-\nu)}(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn})
\end{aligned} \tag{17}$$

Problem 4.2 (15') Dilation field.

The "constrained" dilation of a transformed inclusion (not necessarily ellipsoidal) is,

$$\begin{aligned}
u^{\mathrm{c}}_{i,i} &= \int_{S_0} \sigma^*_{kj} n_k(\mathbf{x}') G_{ij,i}(\mathbf{x}-\mathbf{x}') dS(\mathbf{x}') \\
&= -\int_{V_0} \sigma^*_{kj} G_{ij,ik}(\mathbf{x}-\mathbf{x}') dV(\mathbf{x}')
\end{aligned} \tag{18}$$

(a) Show that if $e^*_{ij} = \varepsilon\delta_{ij}$ (pure dilational eigenstrain), then in isotropic elasticity the constrained dilation is constant inside the inclusion and independent of inclusion shape.

(b) What is $u^c_{i,i}$ inside the inclusion in terms of ε?

Hint: The Green's function $G_{ij}(\mathbf{x})$ can be expressed in terms of second derivatives of $R = |\mathbf{x}|$.

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi\mu}\left[\delta_{ij}\nabla^2 R - \frac{1}{2(1-\nu)}\partial_i\partial_j R\right] \tag{19}$$

Notice that

$$\nabla^2 R = \frac{2}{R} \tag{20}$$

$$\nabla^2 \frac{1}{R} = -4\pi\delta(\mathbf{x}) \tag{21}$$

Solution

In the case of

$$e^*_{ij} = \varepsilon\delta_{ij} \tag{22}$$

the eigenstress is

$$\sigma^*_{kj} = C_{kjmn}e^*_{mn} = \varepsilon\, C_{kjmm} \tag{23}$$

where

$$C_{kjmm} = \lambda\delta_{kj}\delta_{mm} + 2\mu\delta_{km}\delta jm = (3\lambda + 2\mu)\delta_{kj} \tag{24}$$

Hence,

$$\sigma^*_{kj} = \varepsilon(3\lambda + 2\mu)\delta_{kj} \tag{25}$$

Substituting this into the equation for constrained dilatation

$$\begin{aligned} u^c_{i,i} &= -\int_{V_0} \sigma^*_{kj}G_{ij,ik}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\ &= -\int_{V_0} \varepsilon(3\lambda+2\mu)G_{ij,ij}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \end{aligned}$$

Notice that

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi\mu}\left[\delta_{ij}R_{,kk} - \frac{1}{2(1-\nu)}R_{,ij}\right] \tag{26}$$

Therefore,

$$\begin{aligned}
G_{ij,ij}(\mathbf{x}) &= \frac{1}{8\pi\mu}\left[\delta_{ij}R_{,kkij} - \frac{1}{2(1-\nu)}R_{,ijij}\right] \\
&= \frac{1}{8\pi\mu}\left[R_{,kkii} - \frac{1}{2(1-\nu)}R_{,ijij}\right] \\
&= \frac{1-2\nu}{16\pi\mu(1-\nu)}R_{,iijj} \\
&= \frac{1-2\nu}{16\pi\mu(1-\nu)}\nabla^2\frac{2}{R} \\
&= \frac{1-2\nu}{16\pi\mu(1-\nu)}[-8\pi\delta(\mathbf{x})] \\
&= -\frac{1-2\nu}{2\mu(1-\nu)}\delta(\mathbf{x}) \qquad (27)
\end{aligned}$$

In other words,

$$G_{ij,ij}(\mathbf{x}-\mathbf{x}') = -\frac{1-2\nu}{2\mu(1-\nu)}\delta(\mathbf{x}-\mathbf{x}') \qquad (28)$$

Hence when $\mathbf{x}$ is inside V_0,

$$\begin{aligned}
u^c_{i,i} &= \int_{V_0}\varepsilon(3\lambda+2\mu)\frac{1-2\nu}{2\mu(1-\nu)}\delta(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\
&= \varepsilon(3\lambda+2\mu)\frac{1-2\nu}{2\mu(1-\nu)} \\
&= \varepsilon\frac{1+\nu}{1-\nu} \qquad (29)
\end{aligned}$$

When $\mathbf{x}$ is outside V_0, the constrained dilation field must be zero.