

Problem Set 3. Elastic Green function

Chris Weinberger and Wei Cai

© All rights reserved

Problem 3.1 (10') Numerical calculation of Green's function.

(a) Write a Matlab program that returns C_{ijkl} given C_{11} , C_{12} , and C_{44} of an anisotropic elastic medium with cubic symmetry.

Solution:

```
c=zeros(3,3,3,3);
c(1,1,1,1) = c11; c(2,2,2,2) = c11; c(3,3,3,3) = c11;
c(1,1,2,2) = c12; c(1,1,3,3) = c12; c(2,2,1,1) = c12;
c(2,2,3,3) = c12; c(3,3,1,1) = c12; c(3,3,2,2) = c12;
c(1,2,1,2) = c44; c(2,1,2,1) = c44; c(1,3,1,3) = c44;
c(3,1,3,1) = c44; c(2,3,2,3) = c44; c(3,2,3,2) = c44;
c(1,2,2,1) = c44; c(2,1,1,2) = c44; c(1,3,3,1) = c44;
c(3,1,1,3) = c44; c(2,3,3,2) = c44; c(3,2,2,3) = c44;
```

(b) Write a Matlab program that computes $(zz)_{ij}$ and $(zz)_{ij}^{-1}$ given C_{ijkl} and z_i . The elastic constants of Silicon are $C_{11} = 161.6\text{GPa}$, $C_{12} = 81.6\text{GPa}$, $C_{44} = 60.3\text{GPa}$. What are the values for all components of $g_{ij}(\mathbf{k})$ for $\mathbf{k} = [112]$ (\mathbf{k} in unit of μm^{-1})?

Solution:

```
K = [1 1 2];
z = K/norm(K);
zz=zeros(3,3);
for i=1:3
    for j=1:3
        for k=1:3
            for l=1:3
                zz(j,k) = c(i,j,k,l)*z(i)*z(l) + zz(j,k);
            end
        end
    end
end
zzinv = inv(zz);
```

```
g = zzinv/norm(K)^2;
```

The resulting $g_{ij}(\mathbf{k})$ is in unit of

$$\frac{\text{GPa}^{-1}}{\mu\text{m}^{-2}} = 10^{-21}\text{Pa}^{-1} \cdot \text{m}^2 = 10^{-21}\text{m}^4 \cdot \text{N}^{-1} \quad (1)$$

For $\mathbf{k} = [112]$,

$\mathbf{g}_{ij} =$

2.822547473790110e-003	-2.907775573431395e-004	-9.367878778196223e-004
-2.907775573431395e-004	2.822547473790110e-003	-9.367878778196223e-004
-9.367878778196223e-004	-9.367878778196223e-004	1.997028421708497e-003

(c) Write a Matlab program that computes $G_{ij}(\mathbf{x})$ given C_{ijkl} and \mathbf{x} . What are the values for all components of $G_{ij}(\mathbf{x})$ for $\mathbf{x} = [112]$ (\mathbf{x} in unit of μm)? Plot $G_{33}(x, y)$ on plane $z = 1$.

Include a print out of your source code in your report. You may feel free to use other softwares (e.g. Mathematica) instead of Matlab if you prefer to do so.

Solution:

Matlab source code in `green.m` (see appendix).

The resulting $G_{ij}(\mathbf{k})$ is in unit of

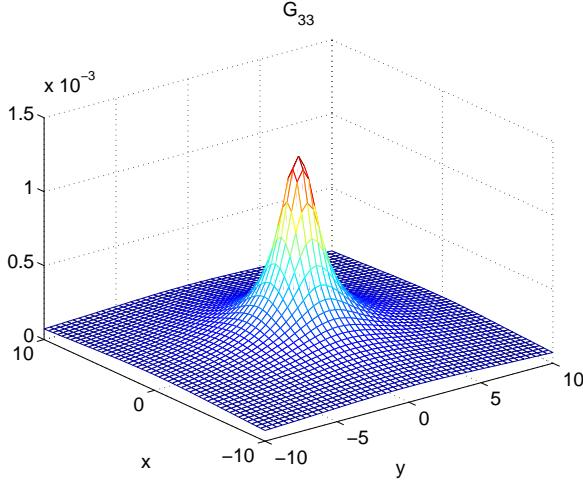
$$\frac{\text{GPa}^{-1}}{\mu\text{m}} = 10^{-3}\text{Pa}^{-1} \cdot \text{m}^{-1} = 10^{-3}\text{m} \cdot \text{N}^{-1} \quad (2)$$

For $\mathbf{x} = [112]$,

$\mathbf{G}_{ij} =$

4.633362938780277e-004	5.456834158880396e-005	9.290941326219717e-005
5.456834158880396e-005	4.633362938780278e-004	9.290941326219712e-005
9.290941326219714e-005	9.290941326219717e-005	5.459857627368208e-004

Plot of $G_{33}(x, y)$ on plane $z = 1$.



Problem 3.2 (10') Reciprocal Theorem.

Use Betti's theorem (under zero body force),

$$\int_S \mathbf{t}^{(1)} \cdot \mathbf{u}^{(2)} dS = \int_S \mathbf{t}^{(2)} \cdot \mathbf{u}^{(1)} dS \quad (3)$$

to show that, the volume change of an isotropic medium with Young's modulus E and Possion's ratio ν under surface traction $\mathbf{t}^{(1)}$ is,

$$\delta V_1 = \int_S \frac{1-2\nu}{E} x_i t_i^{(1)} dS \quad (4)$$

Notice that the traction force satisfies,

$$\int_S t_i^{(1)} dS = 0 \quad (5)$$

$$\int_S \epsilon_{ijk} x_j t_k^{(1)} dS = 0 \quad (6)$$

[Hint: use auxiliary solution $\sigma_{ij}^{(2)} = \delta_{ij}$, i.e. the medium under unit hydrostatic tension.]

Solution

Using the hint that $\sigma_{ij}^{(2)} = \delta_{ij}$ we want to write a relationship between stress and strain in field 2. It is easy to show that

$$e_{ij}^{(2)} = \frac{1}{3K} \delta_{ij} \quad (7)$$

where K is the bulk modulus, which is

$$K \equiv \frac{E}{3(1-2\nu)} \quad (8)$$

Now, we can write the displacements as

$$u_i^{(2)} = \frac{1}{3K} x_i = \frac{1-2\nu}{E} x_i \quad (9)$$

The left hand side of Betti's theorem is,

$$\int_S t_i^{(1)} u_i^{(2)} dS = \frac{1-2\nu}{E} \int_S t_i^{(1)} x_i dS$$

The right hand side of Betti's theorem is,

$$\begin{aligned} \int_S t_i^{(2)} u_i^{(1)} dS &= \int_V (\sigma_{ij}^{(2)} u_j)_{,i} dV \\ &= \int_V \sigma_{ij,i}^{(2)} u_j + \sigma_{ij}^{(2)} u_{j,i} dV \\ &= 0 + \int_V \delta_{ij} u_{j,i} dV \\ &= \int_V u_{i,i} dV \\ &= \int_V e_{ii} dV \\ &= \delta V_1 \end{aligned} \tag{10}$$

Using Betti's theorem, we get

$$\delta V_1 = \frac{(1-2\nu)}{E} \int_S t_i^{(1)} x_i dS \tag{11}$$

Problem 3.3 (10') Contact problem.

Consider a semi-infinite isotropic elastic medium filling the half space $x_3 \geq 0$. Let the shear modulus be μ and Poisson's ratio be ν . The Green's function for the half space is $G_{ij}^h(\mathbf{x}, \mathbf{x}')$. If the force is only applied to the surface, i.e. $x'_3 = 0$, then the Green's function can be written as,

$$G_{ij}^h(\mathbf{x}, \mathbf{x}') = G_{ij}^h(\mathbf{x} - \mathbf{x}') \tag{12}$$

Introduce function $F(\mathbf{x}) = x_3 \ln(x_3 + R) - R$ where $R = |\mathbf{x}|$. Then the surface Green's function can be expressed as (when the surface force is applied at $\mathbf{x}' = 0$),

$$G_{ij}^h(\mathbf{x}) = \frac{1}{4\pi\mu} [\delta_{ij} \nabla^2 R - \partial_i \partial_j R - (-1)^{\delta_{i3}} (1-2\nu) \partial_i \partial_j F] \tag{13}$$

(a) What is the explicit form of $G_{33}^h(\mathbf{x})$, i.e. the normal displacement in response to a normal surface force? What is the normal displacement $G_{33}^h(x, y)$ on the surface ($x_3 = 0$)?

(b) Consider a spherical indentor with radius of curvature ρ punching on the surface along the x_3 axis. Let a be the radius of the contact area. The indentor is much stiffer than the substrate so that we can assume the substrate conforms to the shape of the indentor in the contact area, i.e.,

$$u_3(x, y) = u_3(r) = d - \frac{r^2}{2\rho} \tag{14}$$

where d is the maximum displacement on the surface and $r \equiv \sqrt{x^2 + y^2}$. What is the pressure distribution on the surface $p(r)$?

(c) For self consistency, the pressure must vanish at $r = a$. What is the relationship between d , ρ and a ?

(d) What is the total indenting force as a function of ρ and d ?

Solution:

(a) It can be shown that

$$\begin{aligned}\nabla^2 R &= \frac{2}{R} \\ R_{,33} &= \frac{1}{R} - \frac{x_3^2}{R^3} \\ F_{,33} &= \frac{1}{R} \\ G_{33}^h(\mathbf{x}) &= \frac{1}{4\pi\mu} \left(\frac{2(1-\nu)}{R} + \frac{x_3^2}{R^3} \right)\end{aligned}\tag{15}$$

and with $x_3 = 0$

$$G_{33}^h(x, y) = \frac{1-\nu}{2\pi\mu R}\tag{16}$$

(b) The total force F is just the integral of the pressure over the area. Thus

$$\begin{aligned}F &= \int_A p(x, y) dA \\ &= 2\pi B \int_0^a \sqrt{1 - \frac{r^2}{a^2}} r dr \\ &= \frac{2}{3}\pi a^2 B\end{aligned}$$

Thus

$$B = \frac{3}{2} \frac{F}{\pi a^2}\tag{17}$$

Thus, $p(x, y)$ is

$$p(x, y) = \frac{3}{2} \frac{F}{\pi a^2} \sqrt{1 - (x/a)^2 - (y/a)^2}\tag{18}$$

Now, using the Green's function we can write u_3 as

$$\begin{aligned}
u_3(x, y) &= \int_A \frac{1-\nu}{2\pi\mu} \frac{3F}{2\pi a^2} \frac{\sqrt{1-(x'/a)^2-(y'/a)^2} dx' dy'}{\sqrt{(x-x')^2+(y-y')^2}} \\
&= \frac{3}{4} \frac{F(1-\nu)}{\mu\pi^2 a} \int_{x'^2+y'^2<1} \frac{\sqrt{1-(x')^2-(y')^2} dx' dy'}{\sqrt{(x/a-x')^2+(y/a-y')^2}} \\
&= \frac{3}{4} \frac{F(1-\nu)}{\mu\pi^2 a} \frac{\pi^2}{2} \left(1 - \frac{(x/a)^2 + (y/a)^2}{2} \right) \\
&= \frac{3}{8} \frac{F(1-\nu)}{\mu a} \left(1 - \frac{r^2}{2a^2} \right) \\
&= d - \frac{r^2}{2\rho^2}
\end{aligned} \tag{19}$$

Hence,

$$d = \frac{3}{8} \frac{F(1-\nu)}{\mu a} \tag{20}$$

$$\frac{1}{2\rho^2} = \frac{3}{8} \frac{F(1-\nu)}{\mu a} \frac{1}{2a^2} = \frac{d}{2a^2} \tag{21}$$

(c) From Eq. (21),

$$a = \sqrt{d\rho} \tag{22}$$

$$d = a^2/\rho \tag{23}$$

Plug into Eq. (20), we have

$$\begin{aligned}
\frac{a^2}{\rho} &= \frac{3}{8} \frac{F(1-\nu)}{\mu a} \\
a &= \left[\frac{3F(1-\nu)\rho}{8\mu} \right]^{1/3}
\end{aligned} \tag{24}$$

(d)

$$d = a^2\rho = \left[\frac{3F(1-\nu)}{8\mu} \right]^{2/3} \cdot \rho^{-1/3} \tag{25}$$

Notice that the displacement at the edge of the contact area is

$$u_3(r=a) = d - \frac{a^2}{2\rho} = \frac{d}{2} \neq 0 \tag{26}$$

```

%green.m
%
% Chris Weinberger and Wei Cai
% ME340B Elasticity of Microscopic Structures, Stanford University, Winter 2004
% Homework 3.1 (Due: Jan 26, 2005)
%
% This program computes the Green's function (with singularity)
%
clear;

% cubic material model, aligned with global coordinates
c11=161.6; c12=81.6; c44=60.3; %(in GPa)

%Isotroic model (for debugging purposes)
%mu=130*10^9;
%v=0.309;
%c11=mu*(1+1/(1-2*v));
%c44=mu;
%c12=c11-2*c44;

c=zeros(3,3,3,3);
c(1,1,1,1) = c11; c(2,2,2,2) = c11; c(3,3,3,3) = c11;
c(1,1,2,2) = c12; c(1,1,3,3) = c12; c(2,2,1,1) = c12;
c(2,2,3,3) = c12; c(3,3,1,1) = c12; c(3,3,2,2) = c12;
c(1,2,1,2) = c44; c(2,1,2,1) = c44; c(1,3,1,3) = c44;
c(3,1,3,1) = c44; c(2,3,2,3) = c44; c(3,2,3,2) = c44;
c(1,2,2,1) = c44; c(2,1,1,2) = c44; c(1,3,3,1) = c44;
c(3,1,1,3) = c44; c(2,3,3,2) = c44; c(3,2,2,3) = c44;

% which part of problem 1 ?

part = 'c' ;

switch part

case 'a'

    c

case 'b'

    K = [1 1 2];
    z = K/norm(K);
    zz=zeros(3,3);

```

```

for i=1:3
    for j=1:3
        for k=1:3
            for l=1:3
                zz(j,k) = c(i,j,k,l)*z(i)*z(l) + zz(j,k);
            end
        end
    end
end

zz
zzinv = zz^(-1)
g = zzinv/norm(K)^2
disp('g in unit of 10^{-21} m^4.N^{-1}');

case 'c'
x = [1 1 2];

v1=(cross(x,[1 1 1]));
if v1==0
    v1=(cross(x,[0 1 0]));
end
v1=v1/norm(v1);
v2=(cross(x,v1));
v2=v2/norm(v2);

G_theta = zeros(3,3);
nint=40;
theta=[0:nint-1]/nint*(2*pi);
dtheta = (2*pi)/nint;

for m=1:nint
    z = v1*cos(theta(m)) + v2*sin(theta(m));
    zz=zeros(3,3);
    for i=1:3
        for j=1:3
            for k=1:3
                for l=1:3
                    zz(j,k) = c(i,j,k,l)*z(i)*z(l) + zz(j,k);
                end
            end
        end
    end
end

```

```

zzinv = inv(zz);

G_theta = zzinv + G_theta;
end
G = G_theta /8/pi^2/norm(x) * dtheta;

X=linspace(-10,10,51)';
Y=linspace(-10,10,51)';
nx=length(X);
ny=length(Y);

for p=1:nx
    for q=1:ny
        x = [X(p) Y(q) 1];
        v1=(cross(x,[1 1 1]));
        if v1==0
            v1=(cross(x,[0 1 0]));
        end
        v1=v1/norm(v1);
        v2=(cross(x,v1));
        v2=v2/norm(v2);
        zzinv = zeros(3,3);

        for m=1:nint
            z = v1*cos(theta(m)) + v2*sin(theta(m));
            zz=zeros(3,3);
            for i=1:3
                for j=1:3
                    for k=1:3
                        for l=1:3
                            zz(j,k) = c(i,j,k,l)*z(i)*z(l) + zz(j,k);
                        end
                    end
                end
            end
            zzinv = inv(zz) + zzinv;
        end
        G33(p,q) = zzinv(3,3) / 8/ pi^2/ norm(x) * dtheta;
    end
end

disp('G in unit of 10^{-3} m.N^{-1}');
%contourf(Y,X,G33,31);
mesh(Y,X,G33);

```

```
set(gca,'FontSize',17);
xlabel('y');
ylabel('x');
title('G_{33}');
end
```