Problem Set 2 Solution
Elasticity in one and two dimensions

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Problem 2.1 (10') Elastic constants.
The elastic stiffness tensor for the isotropic medium is $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Determine the compliance tensor, S_{ijkl}, which is the inverse of C_{ijkl}, i.e.,

$$C_{ijkl}S_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{1}$$

Solution:

$$\begin{aligned}
C_{ijkl}S_{klmn} &= [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\,[\alpha\delta_{ij}\delta_{kl} + \beta\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \\
&= (3\lambda\alpha + 2\mu\alpha + 2\beta\lambda)\delta_{ij}\delta_{mn} + 2\beta\mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \\
&= \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})
\end{aligned}$$

Thus

$$\begin{aligned}
2\beta\mu &= \frac{1}{2} \\
\beta &= \frac{1}{4\mu}
\end{aligned}$$

and

$$\begin{aligned}
3\lambda\alpha + 2\mu\alpha + 2\beta\lambda = 0 \\
3\lambda\alpha + 2\mu\alpha + \frac{\lambda}{2\mu} = 0 \\
(3\lambda + 2\mu)\alpha = -\frac{1}{2}\frac{\lambda}{\mu} \\
\alpha = -\frac{1}{2}\frac{\lambda}{\mu(3\lambda + 2\mu)} \\
S_{ijkl} = -\frac{1}{2}\frac{\lambda}{\mu(3\lambda + 2\mu)}\delta_{ij}\delta_{kl} + \frac{1}{4\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\end{aligned}$$

Explicit expressions of C_{ijkl} and S_{ijkl}

Let us write out the various terms of C_{ijkl} and S_{ijkl} explicitly in isotropic elasticity. There are only three different terms in C_{ijkl}: C_{1111}, C_{1122} and C_{1212}. Other terms can be obtained by symmetry, e.g. $C_{2233} = C_{1122}$. In contracted notation, these three terms are written as C_{11}, C_{12}, C_{44}.

$$C_{1111} = C_{11} = \lambda + 2\mu \tag{2}$$
$$C_{1122} = C_{12} = \lambda \tag{3}$$
$$C_{1212} = C_{44} = \mu \tag{4}$$

Therefore, an isotropic elastic medium has the property that

$$C_{11} = C_{12} + 2C_{44} \tag{5}$$

Hence the anisotropic factor,

$$A \equiv \frac{2C_{44}}{C_{11} - C_{12}} \tag{6}$$

equals to one for an isotropic medium. This is of course not the case for an anisotropic medium. For crystals with cubic symmetry, the elastic constants, C_{11}, C_{12} and C_{44} are independent of each other.

$$S_{1111} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} + \frac{1}{2\mu} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \equiv \frac{1}{E} \tag{7}$$
$$S_{1122} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \tag{8}$$
$$S_{1212} = \frac{1}{4\mu} \tag{9}$$

where E is called the Young's modulus. Notice that $\lambda = 2\mu\nu/(1 - 2\nu)$, thus

$$S_{1111} = \frac{1}{E} = \frac{1}{2\mu(1 + \nu)} \tag{10}$$
$$S_{1122} = -\frac{\nu}{E} \tag{11}$$
$$S_{1212} = \frac{1 + \nu}{2E} \tag{12}$$

These results are the bases of Problem 2.3(b).

Problem 2.2 (10') 1D elasticity.
Determine the displacement, strain and stress field of a long rod of length L standing vertically in a gravitational field g. Assume the rod is an isotropic elastic medium with shear modulus μ and Poisson's ratio ν.

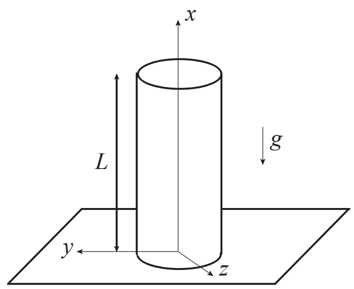

Figure 1: A rod of length L standing vertically in a gravitational field g.

Solution:

Choose the coordinate system such that x-axis goes along the axis of the rod pointing up with the origin at the bottom of the rod. The equation of equilibrium is,

$$\sigma_{xx,x} + b_x = 0 \tag{13}$$

while all the other stress components are zero, i.e.,

$$\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \tag{14}$$

The boundary condition for Eq. (13) is such that $\sigma_{xx} = 0$ at $x = L$. Because $b_x = -\rho g$ (ρ is the density of the rod), the solution is

$$\sigma_{xx} = \rho g(x - L) \tag{15}$$

From Problem 2.1, noting that for Hooke's Law we have

$$e_{xx} = S_{1111}\sigma_{xx} = \frac{\sigma_{xx}}{E} = \frac{\rho g}{E}(x - L) \tag{16}$$

where $E = 2\mu(1 + \nu)$ is the Young's modulus. Similarly,

$$\begin{aligned}
e_{yy} &= S_{1122}\sigma_{xx} = -\frac{\nu\rho g}{E}(x - L) && (17)\\
e_{zz} &= S_{1122}\sigma_{xx} = -\frac{\nu\rho g}{E}(x - L) && (18)\\
e_{xy} &= e_{yz} = e_{zx} = 0 && (19)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
u_{x,x} &= \frac{\rho g}{E}(x-L) && (20)\\
u_{y,y} &= -\frac{\nu\rho g}{E}(x-L) && (21)\\
u_{z,z} &= -\frac{\nu\rho g}{E}(x-L) && (22)\\
u_{x,y} &= -u_{y,x} && (23)\\
u_{x,z} &= -u_{z,x} && (24)\\
u_{y,z} &= -u_{z,y} && (25)
\end{aligned}
$$

We wish to apply the boundary condition of $u_x = 0$ at $x = 0$. The following solution satisfies the boundary condition of $u_x = 0$ at $x = y = z = 0$ (i.e. boundary condition only imposed at a single point at the bottom of the rod),

$$
\begin{aligned}
u_x &= \frac{\rho g}{E}\left[\frac{1}{2}x^2 - Lx + \frac{\nu}{2}(y^2+z^2)\right] && (26)\\
u_y &= -\frac{\nu\rho g}{E}(x-L)y && (27)\\
u_z &= -\frac{\nu\rho g}{E}(x-L)z && (28)
\end{aligned}
$$

The above solution does not satisfy the boundary condition at the entire plane of $x = 0$. Therefore solution is not valid near the end of the rod. (The rod is now standing on a quadratic surface.) To fully account for the end effect of a flat surface, the stress will no longer be a simple one-dimensional function as given by Eq. (15).

Problem 2.3 (10') 2D elaticity.
Lets look at equilibrium in 2-D elasticity using x-y cartesian coordinates under zero body force. Assume the 2-d body is in a state of plane stress, i.e.,

$$\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$$

which corresponds to a free standing thin film. The equilibrium equations reduce to

$$
\begin{aligned}
&\sigma_{xx,x} + \sigma_{yx,y} = 0 && (29)\\
&\sigma_{yy,y} + \sigma_{xy,x} = 0 && (30)
\end{aligned}
$$

And the compatability equations reduce to

$$e_{xx,yy} - 2e_{xy,xy} + e_{yy,xx} = 0 \qquad (31)$$

One popular method to solve such problems is to introduce the Airy's stress function ϕ such that,

$$
\begin{aligned}
\sigma_{xx} &= \phi_{,yy} && (32)\\
\sigma_{yy} &= \phi_{,xx} && (33)\\
\sigma_{xy} &= -\phi_{,xy} && (34)
\end{aligned}
$$

(a) Show that this particular choice of stress function automatically satisfies equilibrium.

Solution:

$$\begin{aligned}\sigma_{xx,x} + \sigma_{yx,y} = \phi_{,yyx} + (-\phi_{,xyy}) = 0 \\ \sigma_{yy,y} + \sigma_{xy,x} = \phi_{,xxy} + (-\phi_{,yxy}) = 0\end{aligned}$$

(b) Assuming that Hooke's Law is of the form

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} \tag{35}$$

$$e_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} \tag{36}$$

$$e_{xy} = \frac{\sigma_{xy}(1+\nu)}{E} \tag{37}$$

show that the compatability equation reduces to

$$\phi_{,xxxx} + 2\phi_{,xxyy} + \phi_{,yyyy} = 0 \tag{38}$$

Solution:

Starting from the compatibility equation,

$$e_{xx,yy} - 2e_{xy,xy} + e_{yy,xx} \quad = \quad 0$$

plug in the Hooke's law,

$$\begin{aligned}\frac{1}{E}\left(\sigma_{xx,yy} - \nu\sigma_{yy,yy} + \sigma_{yy,xx} - \nu\sigma_{xx,xx}\right) - 2\frac{(1+\nu)}{E}\sigma_{xy,xy} &= 0 \\ \phi_{,yyyy} - \nu\phi_{,xxyy} + \phi_{,xxxx} - \nu\phi_{,yyxx} + 2(1+\nu)\phi_{,xyxy} &= 0 \\ \phi_{,xxxx} + 2\phi_{,xyxy} + \phi_{,yyyy} &= 0\end{aligned}$$

This is the biharmonic equation, which is often written as $\nabla^4\phi = 0$.

(c) What is the relation between E and the shear modulus μ and Poisson's ration ν?

Solution:

$$E = 2\mu(1+\nu)$$

see box in Problem 2.1.

(d) Note that the solution of Eq.(38) does not depend on elastic constants. Let's use this solution to solve a very simple stress problem. Consider a square of length a under hydrostatic pressure P. What are the stress components inside the box? (guess!) What is the stress function ϕ?

Solution:

The box is under uniform stress,

$$\begin{aligned}
\sigma_{xx} &= -P \\
\sigma_{yy} &= -P \\
\sigma_{xy} &= 0 \\
\phi &= -\frac{1}{2}Py^2 - \frac{1}{2}Px^2
\end{aligned}$$

The strain tensor is,

$$\begin{aligned}
e_{xx} &= e_{yy} = -P\frac{1-\nu}{E} \\
e_{xy} &= 0
\end{aligned}$$

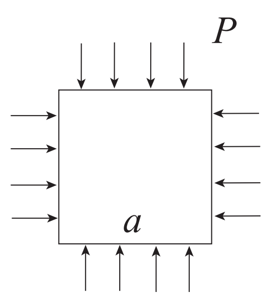

Figure 2: A square of length a under hydrostatic pressure P.