ME340B – Elasticity of Microscopic Structures – Wei Cai – Stanford University – Winter 2004

Problem Set 4. Eshelby's Inclusion

Due date: Feb 2, 2005

Problem 4.1 (15') Spherical inclusion.

(a) Derive the expressions for the auxiliary tensor $\mathcal{D}_{ijkl}$ for a spherical inclusion in an isotropic medium with shear modulus μ and Poisson's ratio ν.
[Hint: many components of $\mathcal{D}_{ijkl}$ are zero, unless there are repeated indices.]

(b) Derive the corresponding expressions for Eshelby's tensor $\mathcal{S}_{ijkl}$.

Problem 4.2 (15') Dilation field.

The "constrained" dilation of a transformed inclusion (not necessarily ellipsoidal) is,

$$\begin{aligned} u^{\mathrm{c}}_{i,i} &= \int_{S_0} \sigma^*_{kj} n_k(\mathbf{x}') G_{ij,i}(\mathbf{x}-\mathbf{x}') dS(\mathbf{x}') \\ &= -\int_{V_0} \sigma^*_{kj} G_{ij,ik}(\mathbf{x}-\mathbf{x}') dV(\mathbf{x}') \end{aligned} \tag{1}$$

(a) Show that if $e^*_{ij} = \varepsilon\delta_{ij}$ (pure dilational eigenstrain), then in isotropic elasticity the constrained dilation is constant inside the inclusion and independent of inclusion shape.

(b) What is $u^{\mathrm{c}}_{i,i}$ inside the inclusion in terms of ε?

Hint: The Green's function $G_{ij}(\mathbf{x})$ can be expressed in terms of second derivatives of $R = |\mathbf{x}|$.

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi\mu}\left[\delta_{ij}\nabla^2 R - \frac{1}{2(1-\nu)}\partial_i\partial_j R\right] \tag{2}$$

Notice that

$$\nabla^2 R = \frac{2}{R} \tag{3}$$

$$\nabla^2 \frac{1}{R} = -4\pi\delta(\mathbf{x}) \tag{4}$$