Problem 3.1 (10’) Numerical calculation of Green’s function.
(a) Write a Matlab program that returns $C_{ijkl}$ given $C_{11}$, $C_{12}$, and $C_{44}$ of an anisotropic elastic medium with cubic symmetry.

(b) Write a Matlab program that computes $(zz)_{ij}$ and $(zz)^{-1}_{ij}$ given $C_{ijkl}$ and $z_i$. The elastic constants of Silicon are $C_{11} = 161.6$ GPa, $C_{12} = 81.6$ GPa, $C_{44} = 60.3$ GPa. What are the values for all components of $g_{ij}(k)$ for $k = [112]$ ($k$ in unit of $\mu m^{-1}$)?

(c) Write a Matlab program that computes $G_{ij}(x)$ given $C_{ijkl}$ and $x$. What are the values for all components of $G_{ij}(x)$ for $x = [112]$ ($x$ in unit of $\mu m$)? Plot $G_{33}(x, y)$ on plane $z = 1$.

Include a print out of your source code in your report. You may feel free to use other softwares (e.g. Mathematica) instead of Matlab if you prefer to do so.

Problem 3.2 (10’) Reciprocal Theorem.
Use Betti’s theorem (under zero body force),
\[ \int_S t^{(1)} \cdot u^{(2)} dS = \int_S t^{(2)} \cdot u^{(1)} dS \] (1)
to show that, the volume change of an isotropic medium with Young’s modulus $E$ and Possion’s ratio $\nu$ under surface traction $t^{(1)}$ is,
\[ \delta V_1 = \int_S \frac{1 - 2\nu}{E} x_i t_i^{(1)} dS \] (2)

Notice that the traction force satisfies,
\[ \int_S t_i^{(1)} dS = 0 \] (3)
\[ \int_S \epsilon_{ijk} x_j t_k^{(1)} dS = 0 \] (4)

[ Hint: use auxiliary solution $\sigma_{ij}^{(2)} = \delta_{ij}$, i.e. the medium under unit hydrostatic tension. ]
Problem 3.3 (10’) Contact problem.
Consider a semi-infinite isotropic elastic medium filling the half space \(x_3 \geq 0\). Let the shear modulus be \(\mu\) and Poisson’s ratio be \(\nu\). The Green’s function for the half space is \(G^h_{ij}(x, x')\). If the force is only applied to the surface, i.e. \(x'_3 = 0\), then the Green’s function can be written as,

\[
G^h_{ij}(x, x') = G^h_{ij}(x - x')
\]

Introduce function \(F(x) = x_3 \ln(x_3 + R)\) where \(R = |x|\). Then the surface Green’s function can be expressed as (when the surface force is applied at \(x' = 0\)),

\[
G^h_{ij}(x) = \frac{1}{4\pi \mu} \left[ \delta_{ij} \nabla^2 R - \partial_i \partial_j R - (-1)^{\delta_{i3}} (1 - 2\nu) \partial_i \partial_j F \right]
\]

(a) What is the explicit form of \(G^h_{33}(x)\), i.e. the normal displacement in response to a normal surface force? What is the normal displacement \(G^h_{33}(x, y)\) on the surface \((x_3 = 0)\)?

(b) Consider a spherical indentor with radius of curvature \(\rho\) punching on the surface along the \(x_3\) axis. Let \(a\) be the radius of the contact area. The indentor is much stiffer than the substrate so that we can assume the substrate conforms to the shape of the indentor in the contact area, i.e.,

\[
u(x, y) = d - \frac{x^2 + y^2}{2\rho}
\]

where \(d\) is the maximum displacement on the surface and \(r = \sqrt{x^2 + y^2}\). Let the total indenting force be \(F\). What is the pressure distribution on the surface \(p(x, y)\)? [Hint: try the form \(p(x, y) = B \sqrt{1 - (x/a)^2 - (y/a)^2}\) and determine \(B\) in terms of \(F\). Show that \(p(x, y)\) indeed gives rise to displacement according to Eq. (7).]

(c) What is the expression for the contact radius \(a\) in terms of indenting force \(F\) and indentor radius of curvature \(\rho\)?

(d) What is the expression for the maximum displacement \(d\) in terms of indenting force \(F\) and indentor radius of curvature \(\rho\)?

Note: you may find the following identity useful,

\[
\int_{x'^2 + y'^2 \leq 1} \frac{\sqrt{1 - x'^2 - y'^2}}{\sqrt{(x - x')^2 + (y - y')^2}} dx' dy' = \frac{\pi^2}{2} \left( 1 - \frac{x^2 + y^2}{2} \right)
\]