ME340B – Elasticity of Microscopic Structures – Wei Cai – Stanford University – Winter 2004

Problem Set 2. Elasticity in one and two dimensions

Due date: Jan 19, 2005

Problem 2.1 (10') Elastic constants.
The elastic stiffness tensor for the isotropic medium is $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Determine the compliance tensor, S_{ijkl}, which is the inverse of C_{ijkl}, i.e.,

$$C_{ijkl}S_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{1}$$

[Hint: assume that S_{ijkl} has the form $\alpha\delta_{ij}\delta_{kl} + \beta\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.]

Problem 2.2 (10') 1D elasticity.
Determine the displacement, strain and stress field of a long rod of length L standing vertically in a gravitational field g. Assume the rod is an isotropic elastic medium with shear modulus μ and Poisson's ratio ν.

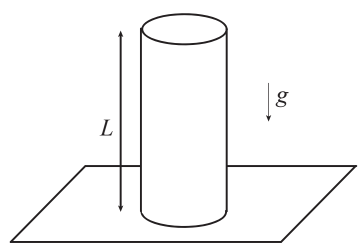

Figure 1: A rod of length L standing vertically in a gravitational field g.

Problem 2.3 (10') 2D elaticity.
Lets look at equilibrium in 2-D elasticity using x-y cartesian coordinates under zero body force. Assume the 2-d body is in a state of plane stress, i.e.,

$$\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$$

which corresponds to a free standing thin film. The equilibrium equations reduce to

$$\sigma_{xx,x} + \sigma_{yx,y} = 0 \tag{2}$$
$$\sigma_{yy,y} + \sigma_{xy,x} = 0 \tag{3}$$

And the compatability equations reduce to

$$e_{xx,yy} - 2e_{xy,xy} + e_{yy,xx} = 0 \tag{4}$$

One popular method to solve such problems is to introduce the Airy's stress function ϕ such that,

$$\sigma_{xx} = \phi_{,yy} \tag{5}$$
$$\sigma_{yy} = \phi_{,xx} \tag{6}$$
$$\sigma_{xy} = -\phi_{,xy} \tag{7}$$

(a) Show that this particular choice of stress function automatically satisfies equilibrium.

(b) Assuming that Hooke's Law is of the form

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} \tag{8}$$
$$e_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} \tag{9}$$
$$e_{xy} = \frac{\sigma_{xy}(1+\nu)}{E} \tag{10}$$

show that the compatability equation reduces to

$$\phi_{,xxxx} + 2\phi_{,xxyy} + \phi_{,yyyy} = 0 \tag{11}$$

This is the biharmonic equation, which is often written as $\nabla^4\phi = 0$.

(c) What is the relation between E and the shear modulus μ and Poisson's ration ν?

(d) Note that the solution of Eq.(11) does not depend on elastic constants. Let's use this solution to solve a very simple stress problem. Consider a square of length a under hydrostatic pressure P. What are the stress components inside the box? (guess!) What is the stress function ϕ?

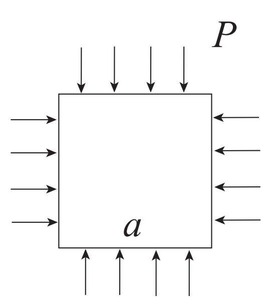

Figure 2: A square of length a under hydrostatic pressure P.