ME340B – Stanford University – Winter 2004

Lecture Notes – Elasticity of Microscopic Structures

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Chapter 1

Introduction to Elasticity Equations

1.1 Index notation

In order to communicate properly the ideas and equations of elasticity, we need to establish a standard convention for writing them. The most common one used is the Einstein convention. This set of rules states that every index that is repeated once in a product implies a summation over that index from 1 to n for an n-dimensional problem. Any free index (i.e. not repeated in a product) implies a set of formulas, one formula for each of the degrees of freedom. Generally, an index does not appear three or more times in a product (otherwise something is wrong). If there is a need to deviate from this convention, then the meaning should be explicitly written. This enables us to write a vector, $\mathbf{v}$ as

$$\mathbf{v} = v_i\mathbf{e}_i = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \tag{1.1}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit (basis) vectors specifying the coordinate system.

Often we do not want to write out the basis of the vectors explicitly. Thus, we can denote the vector $\mathbf{v}$ by just its components v_i. For example, suppose that the v_i is the square of n_i, then we can write v_i as

$$v_i = n_i^2 \Rightarrow \; v_1 = n_1^2 \qquad v_2 = n_2^2 \qquad v_3 = n_3^2 \tag{1.2}$$

Also, if we want to write a scalar a as the sum of the square of the components of $\mathbf{v}$, we can write

$$a = v_iv_i = v_1v_1 + v_2v_2 + v_3v_3 = v_1^2 + v_2^2 + v_3^2 \tag{1.3}$$

Two special tensors worthy of introduction are the Kronecker delta δ_{ij} and the permutation tensor ϵ_{ijk}.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.4}$$

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations of } ijk \\ -1 & \text{for odd permutations of } ijk \\ 0 & \text{for repeated indices} \end{cases} \tag{1.5}$$

(a) (b) (c)

Figure 1.1: (a) Leopold Kronecker (1823-1891 Prussia, now Poland). (b) Johann Carl Friedrich Gauss (1777-1855, Brunswick, now Germany). (c) George Gabriel Stokes (1819-1903, Ireland).

The Kronecker delta and the permutation tensor are related by

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1.6}$$

The Kronecker delta is useful for expressing vector dot products without using vector notation. For example, the dot product of $\mathbf{a} \cdot \mathbf{b}$ can be written as $a_i b_j \delta_{ij} = a_i b_i$. In the same manner the permutation tensor allows us to to write the cross product as

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \mathbf{e_k} \tag{1.7}$$

Since most derivatives will be with respect to an implied cartesian coordinate system, the differentiation symbols need not be explicitly written. Instead, the notation $a_{i,j}$ will denote $\partial a_i / \partial x_j$. A useful identity that combines this notation and the Kronecker delta is $x_{i,j} = \delta_{ij}$.

Gauss's Theorem
If A and its first derivatives, $A_{,i}$, are continuous and single valued on a given volume V with surface S and outward normal n_i, then

$$\int_V A_{,i}\, \mathrm{d}V = \int_S A\, n_i\ \mathrm{d}S \tag{1.8}$$

Stoke's Theorem
If A and its first derivatives, $A_{,i}$, are continuous and single valued on a given surface S with boundary line L, then

$$\int_S \epsilon_{jik} A_{,j}\, n_k\ \mathrm{d}S = \int_L A\, v_i\ \mathrm{d}L \tag{1.9}$$

where n_k is the normal vector of surface S and v_i is the line direction unit vector of line L.

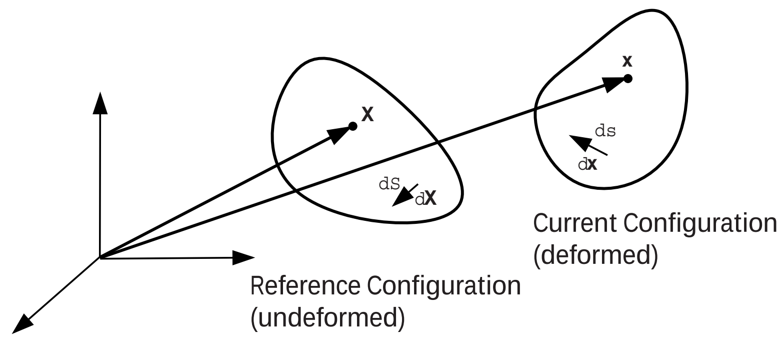


Figure 1.2: Configuration of a undeformed and deformed body

1.2 Deformation of an elastic body

Consider the body shown in Fig.1.2. In the reference configuration the body is undeformed and a point in the body can be denoted $\mathbf{X}$. After deformation, the point previously at $\mathbf{X}$ is now at a point $\mathbf{x}$. The displacement of a point $\mathbf{X}$, denoted $\mathbf{u}(\mathbf{X})$, is the difference between the point in the reference configuration and the current configuration. This is written as

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{X} \tag{1.10}$$

Thus, any point in the current configuration can be written as

$$\mathbf{x} = \mathbf{u} + \mathbf{X} \tag{1.11}$$

or in component form,

$$x_i = u_i + X_i \tag{1.12}$$

Consider a small vector $d\mathbf{X}$ in the undeformed body. The length of this vector is $dS = \sqrt{dX_i dX_i}$. After deformation, this vector becomes $d\mathbf{x}$. Its length now becomes $ds = \sqrt{dx_i dx_i}$. Later on we will use the relationship between ds and dS to define *strain*.

1.3 Stress and equilibrium

The stress tensor, σ_{ij}, is defined as the force per unit area on the i-face in the j-direction. From the stress tensor we can define a traction, T_j, as the force per unit area in the j-direction, on a surface with normal vector $\mathbf{n} = n_i \mathbf{e}_i$. The traction is related to the stress tensor by $\sigma_{ij} n_i = T_j$.

At equilibrium, every point in the elastic body is stationary. To derive the condition for σ_{ij} when the elastic body is at equilibrium, consider a body with a volume V, enclosed by a surface S with an outward normal $\mathbf{n}$ as shown in Fig.1.3. This body has two types of forces acting on it, tractions and body forces. The tractions act over the surface area, and are related to the stresses as described above. The body forces act per unit volume and

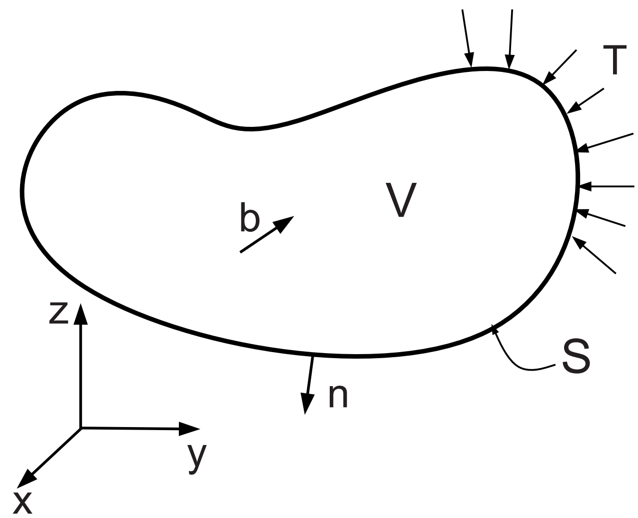


Figure 1.3: An elastic body V under applied loads. **T** is the traction force on the surface S, with normal vector **n** and **b** is body force.

represent external force fields such as gravity. Force equilibrium in the j-direction can be written as

$$\int_V b_j \, \mathrm{d}V + \int_S T_j \, \mathrm{d}S = 0 \tag{1.13}$$

where b_j is the body force and T_j are the tractions in the j-direction. Substituting in the definition of tractions,

$$\int_V b_j \, \mathrm{d}V + \int_S \sigma_{ij} n_i \, \mathrm{d}S = 0$$

and using Gauss's theorem we have,

$$\int_V (b_j + \sigma_{ij,i}) \, \mathrm{d}V = 0$$

The equilibrium equation above is valid for any arbitrary volume and thus must hold in the limit that the volume is vanishingly small. Thus, the above formula must hold point-wise, and the equation for equilibrium is

$$\sigma_{ij,i} + b_j = 0 \tag{1.14}$$

At equilibrium, the net moment around an arbitrary point should also be zero. Otherwise, the body will rotate around this point. For convenience, let this point be the origin. The moments caused by the tractions and the body forces can be written as the position crossed into the force. Using our expression for cross products, the moments in the k-direction can be written as

$$M_k = \int_V \epsilon_{ijk} x_i b_j \,\mathrm{d}V + \int_S \epsilon_{ijk} x_i T_j \,\mathrm{d}S \tag{1.15}$$

Substituting in the definition of tractions, and noting that M_k must be zero for equilibrium,

$$\int_V \epsilon_{ijk} x_i b_j \,\mathrm{d}V + \int_S \epsilon_{ijk} x_i \sigma_{mj} n_m \,\mathrm{d}S = 0 \tag{1.16}$$

Using Gauss's theorem,

$$\int_V \left[\epsilon_{ijk} x_i b_j + (\epsilon_{ijk} x_i \sigma_{mj})_{,m}\right] \mathrm{d}V = 0 \tag{1.17}$$

Distributing the differentiation and noting that the permutation tensor is a constant

$$\begin{aligned} &\int_V \epsilon_{ijk}\left[x_i b_j + x_{i,m}\sigma_{mj} + x_i \sigma_{mj,m}\right] \mathrm{d}V \\ &= \int_V \epsilon_{ijk}\left[x_i(b_j + \sigma_{mj,m}) + \delta_{im}\sigma_{mj}\right] \mathrm{d}V = 0 \end{aligned}$$

From force equilibrium $b_j + \sigma_{mj,m} = 0$, so that

$$\int_V \epsilon_{ijk}\sigma_{ij} \,\mathrm{d}V = 0$$

As before, this must hold for a vanishingly small volume resulting in

$$\epsilon_{ijk}\sigma_{ij} = 0 \tag{1.18}$$

Writing out one term of this formula gives

$$\epsilon_{123}\sigma_{12} + \epsilon_{213}\sigma_{21} = 0$$

$$\sigma_{12} = \sigma_{21}$$

Carrying this through for the other two equations, it is clear that

$$\sigma_{ij} = \sigma_{ji} \tag{1.19}$$

which says the stress tensor must be symmetric. We can also show that the stress tensor is also symmetric even if the body is not in equilibrium (see box below). Thus the symmetry of the stress tensor is independent of equilibrium conditions.

In summary, the equations of equilibrium are

$$\sigma_{ij,i} + b_j = 0$$

and

$$\sigma_{ij} = \sigma_{ji}$$

Noting that there are 9 components of the stress tensor and equilibrium specifies 6 equations (or 3 equations for the 6 unknowns of the symmetric stress tensor), at this moment we are unable to solve this set of partial differential equations.

Also, the reader should be aware that it is possible to define the stress tensor opposite to the definition used above. σ_{ij} could be defined as the force in the i-direction on the j-face. This would result in the force equation of equilibrium

$$\sigma_{ij,j} + b_i = 0$$

and the result from zero moment would be the same. However, since the stress tensor is symmetric, both equations are the same and it does not matter which definition of stress is used.

Symmetry of Stress Tensor in Dynamics

To derive the symmetry of the stress tensor in dynamics, we must first write the balance of forces using Newton's law. It is similar to the force equilibrium, except that the forces are now equal to the time rate of change of linear momentum. This conservation of linear momentum can be written as

$$\int_V \rho \dot{v}_j \, \mathrm{d}V = \int_S \sigma_{ij} n_i \, \mathrm{d}S + \int_V b_j \, \mathrm{d}V \tag{1.20}$$

where ρ is the point-wise density of the body. Following the equilibrium case, we can easily show that

$$\rho \dot{v}_j = \sigma_{ij,i} + b_j \tag{1.21}$$

Conservation of rotational momentum about the origin says

$$\int_V \rho \epsilon_{ijk} x_i \dot{v}_j \, \mathrm{d}V = \int_S \epsilon_{ijk} x_i \sigma_{mj} n_m \, \mathrm{d}S + \int_V \epsilon_{ijk} x_i b_j \, \mathrm{d}V \tag{1.22}$$

Using Guass's theorem, and rearranging terms we can write

$$\int_V \epsilon_{ijk} \big[x_i(-\rho \dot{v}_j + \sigma_{mj,m} + b_j) + \delta_{ij} \sigma_{mj} \big] \, \mathrm{d}V \;\; = 0 \tag{1.23}$$

Then, using the balance of linear momentum,

$$\int_V \epsilon_{ijk} \sigma_{ij} \, \mathrm{d}V = 0$$

The rest of the argument follows the static case, thus

$$\sigma_{ij} = \sigma_{ji}$$

and the stress tensor is symmetric, regardless of whether or not it is in equilibrium.

The physical interpretation of this result is the following. If $\sigma_{ij} \neq \sigma_{ji}$, say $\sigma_{12} \neq \sigma_{21}$, then for a small cubic volume $V = l^3$, there will be a net torque around the x_3-axis, which is on the order of $M = \mathcal{O}(l^3)$. Yet the moment of inertia for the cube is on the order of $I = \mathcal{O}(l^5)$. Thus the rotational acceleration of the cube is $\dot{\omega} = M/I = \mathcal{O}(l^{-2})$. For $l \to 0$, $\dot{\omega}$ diverges unless $\sigma_{ij} = \sigma_{ji}$.

1.4 Strain and compatibility

The strain tensor, which is a measure of the body's stretching, can be defined as

$$\mathrm{d}s^2 - \mathrm{d}S^2 = 2e_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j \tag{1.24}$$

where $\mathrm{d}s$, $\mathrm{d}S$, and $\mathrm{d}x_i$ are defined in Fig.1.2 of section 1.2. Why should strain be defined in this way? In fact, there are many different definitions of strain. Eq.(1.24) is a reasonable one because it describes how does the change of length of a differential segment in the elastic body depend on its orientation. If we re-write the left hand side of Eq.(1.24)

$$\mathrm{d}s^2 - \mathrm{d}S^2 = (\mathrm{d}s + \mathrm{d}S)(\mathrm{d}s - \mathrm{d}S)$$

and dividing by $\mathrm{d}s^2$

$$\frac{\mathrm{d}s^2 - \mathrm{d}S^2}{\mathrm{d}s^2} = \frac{(\mathrm{d}s + \mathrm{d}S)(\mathrm{d}s - \mathrm{d}S)}{\mathrm{d}s^2}$$

For small strains, $\mathrm{d}s + \mathrm{d}S \approx 2\,\mathrm{d}s$ and

$$\frac{\mathrm{d}s^2 - \mathrm{d}S^2}{\mathrm{d}s^2} \approx \frac{2\,\mathrm{d}s(\mathrm{d}s - \mathrm{d}S)}{\mathrm{d}s^2}$$

This simplifies to

$$\frac{\mathrm{d}s^2 - \mathrm{d}S^2}{\mathrm{d}s^2} \approx \frac{2(\mathrm{d}s - \mathrm{d}S)}{\mathrm{d}s}$$

This shows that in the small strain approximation, the above strain tensor is indeed a measure of a change in length per unit length, which is traditionally how engineering strain is defined. This simple example also shows the motivation for the factor of 2 in the definition for strain.

The relationship between strain and displacements is important to establish because it provides more equations that are needed to close the set of equations for the elastic fields of a deformed body. (More equations will be provided by Hooke's law in section 1.5.) Thus, we wish to write $\mathrm{d}s$ and $\mathrm{d}S$ in terms of displacements u_i.

$$\begin{aligned}
\mathrm{d}s^2 - \mathrm{d}S^2 &= \mathrm{d}x_i\,\mathrm{d}x_i - \mathrm{d}X_i\,\mathrm{d}X_i \\
&= \mathrm{d}x_i\,\mathrm{d}x_j\delta_{ij} - \left(\delta_{ij} - \frac{\partial u_i}{\partial x_j}\right)\mathrm{d}x_j\left(\delta_{ik} - \frac{\partial u_i}{\partial x_k}\right)\mathrm{d}x_k \\
&= \mathrm{d}x_i\,\mathrm{d}x_j\delta_{ij} - \left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} + \delta_{jk}\right) \\
&= (u_{j,k} + u_{k,j} - u_{i,j}u_{i,k})\,\mathrm{d}x_j\,\mathrm{d}x_k
\end{aligned} \tag{1.25}$$

Thus, the strain tensor is [2]

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) \tag{1.26}$$

Notice that strain tensor is symmetric, i.e. $e_{ij} = e_{ji}$. For strains much less than unity, higher order terms are negligible and the strain tensor becomes

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.27}$$

Often times this tensor is referred to the small strain tensor, or the linearized strain tensor. This form of the strain tensor is particularly useful since it allows for a linear relationship between strain and displacements. Because of this simplicity, the linearized strain tensor will be used in all further discussions.

The above definition of strain relates six components of the strain tensor to the three components of the displacement field. This implies that the six components of the strain tensor cannot be independent, and the equations that relate this interdependency are termed *compatability*. The equations of compatability can be obtained directly from the definition of the strain tensor, Eq.(1.27), which can be written out explicitly using x-y-z coordinates

$$\begin{aligned}
e_{xx} &= u_{x,x} \\
e_{yy} &= u_{y,y} \\
e_{zz} &= u_{z,z} \\
e_{xy} &= \frac{1}{2}(u_{x,y} + u_{y,x}) \\
e_{xz} &= \frac{1}{2}(u_{x,z} + u_{z,x}) \\
e_{yz} &= \frac{1}{2}(u_{y,z} + u_{z,y})
\end{aligned}$$

Now, the first equation of compatibility can be obtained by calculating $e_{xx,yy}$, $e_{yy,xx}$ and $e_{xy,xy}$

$$\begin{aligned}
e_{xx,yy} &= u_{x,xyy} \\
e_{yy,xx} &= u_{y,yxx} \\
e_{xy,xy} &= \frac{1}{2}(u_{x,xyy} + u_{y,yxx})
\end{aligned}$$

Thus $e_{xx,yy}$, $e_{yy,xx}$ and $e_{xy,xy}$ must satisfy the condition that

$$e_{xx,yy} + e_{yy,xx} - 2e_{xy,xy} = 0$$

Two more equations of compatibility are obtained by simply permuting the indices, giving a total of three equations. The fourth equation of compatibility can be found in a similar way to the first. Writing different second derivatives of the strain tensor

$$\begin{aligned}
e_{xx,yz} &= u_{x,xyz} \\
e_{xy,xz} &= \frac{1}{2}(u_{x,xyz} + u_{y,xxz}) \\
e_{xz,xy} &= \frac{1}{2}(u_{x,xyz} + u_{z,xxy}) \\
e_{yz,xx} &= \frac{1}{2}(u_{y,xxz} + u_{z,xxy})
\end{aligned}$$

Therefore these strain components must satisfy the condition

$$e_{xx,yz} = e_{xy,xz} + e_{xz,xy} - e_{yz,xx}$$

Two more equations can be obtained by permuting indices in the above equation, giving a total of six equations of compatibility. These six equations can be written in index notations as

$$\epsilon_{pmk}\epsilon_{qnj}e_{jk,nm} = 0 \tag{1.28}$$

The equations of compatibility are not very useful in solving three dimensional problems. However, in two dimensions only one of the equations is non-trivial and is often used to solve such problems [4]. To solve 3-dimensional problems, we usually use Eq.(1.27) to express the strain in terms of displacements and write the partial differential equations in terms of u_i, hence bypassing the need to invoke the compatibility Eq. (1.28) explicitly.

1.5 Hooke's law

In section 1.3 the equilibrium condition gave three equations for the six unknowns of the symmetric stress tensor. In section 1.4 strain was defined under the pretense that it would provide additional equations that would allow the equations of equilibrium to be solved. In order to get those additional equations, there must some way to relate stresses to strains. The most common way to relate stresses to strains is with a constant tensor (linear relationship) which is often termed Hooke's Law. Since stress and strain are both second order tensors, the most general relationship between stress and strain would involve a fourth order tensor. The tensor that relates strains to the stresses is called the elastic stiffness tensor (or elastic constant tensor) and is usually written as

$$\sigma_{ij} = C_{ijkl}e_{kl} \tag{1.29}$$

Thus, in the most general sense, the stiffness tensor C has $3 \times 3 \times 3 \times 3 = 81$ constants. However, both the stress and the strain tensor are symmetric so that the stiffness tensor must also have some symmetries, which are called *minor* symmetries, i.e.,

$$C_{ijkl} = C_{jikl} = C_{ijlk} \tag{1.30}$$

In elasticity, it is assumed that there exists a strain energy density function $W(e_{ij})$ which is related to the stress by

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}} \tag{1.31}$$

Using the definition of the elasticity tensor in Eq.(1.29), the stiffness tensor can be re-written as

$$C_{ijkl} = \frac{\partial^2 W}{\partial e_{ij}\partial e_{kl}} \tag{1.32}$$

Since the order of differentiation is irrelevant, the stiffness tensor must have the property that

$$C_{ijkl} = C_{klij} \tag{1.33}$$

which is often called the *major* symmetry of the stiffness tensor. This reduces the number of independent elastic constants to 21, the most for a completely anisotropic solid. Similarly, the strains can be related to the stresses by a fourth rank tensor S, called the compliance tensor.

$$e_{ij} = S_{ijkl}\ \sigma_{kl} \tag{1.34}$$

The compliance tensor is the inverse of the stiffness tensor and the two are related by

$$C_{ijkl}S_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{1.35}$$

In isotropic elasticity, there are only two independent elastic constants, in terms of which the stiffness tensor can be expressed as,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.36}$$

where λ and μ together are known as Lamé's constants. μ is commonly referred to as the shear modulus, and λ is related to Poisson's ratio, ν, by $\lambda = \frac{2\mu\nu}{1-2\nu}$. Substituting Eq.(1.36) into Eq.(1.29) gives

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \tag{1.37}$$

Now there are enough equations to solve for all of the unknowns in the equilibrium equations. Substituting Eq.(1.29) into Eq.(1.14)

$$C_{ijkl}e_{kl,i} + b_j = 0 \tag{1.38}$$

Substituting in the definition of the strain tensor

$$\frac{1}{2}C_{ijkl}(u_{k,li} + u_{l,ki}) + b_j = 0 \tag{1.39}$$

using the minor symmetry of C allows the formula to be re-written as

$$\frac{1}{2}(C_{ijkl}\ u_{k,li} + C_{ijlk}\ u_{l,ki}) + b_j = 0 \tag{1.40}$$

Since repeated indices are dummy indices, the above expression can be combined into

$$C_{ijkl}\ u_{k,li} + b_j = 0 \tag{1.41}$$

This is the final equilibrium equation written in terms of displacements. This set of linear partial differential equations has three equations for the three unknowns (displacements). Once the displacements are solved for, the strains can be determined from the definition of the strain tensor and the stresses can be determined from Hooke's law.

1.6 Green's Function

The elastic Green's function, $G_{ij}(\mathbf{x}, \mathbf{x}')$, is defined as the displacement in the $i-$direction at $\mathbf{x}$ due to a point force in the $j-$direction at $\mathbf{x}'$. It is the solution $u_i(\mathbf{x})$ of Eq.(1.41) when the body force b_j is a delta function, i.e. $b_k(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')\delta_{jk}$. In the following, we give an alternative derivation of the equation satisfied by the Green's function $G_{ij}(\mathbf{x}, \mathbf{x}')$. (For an astonishing story on the life of George Green, see [5].)

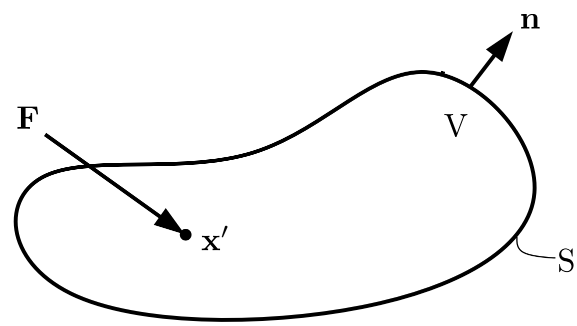


Figure 1.4: A point force is applied to point $\mathbf{x}'$ inside an infinite elastic body. V is a finite volume within the elastic body and S is its surface.

1.6.1 Equilibrium equation for an infinite body

In an infinite homogenous body the Green's function only depends on the relative displacement between the points and thus can be written as

$$G_{ij}(\mathbf{x}, \mathbf{x}') = G_{ij}(\mathbf{x} - \mathbf{x}') \tag{1.42}$$

We wish to construct the equations for the displacement field in response to a point force applied to an infinite body. Consider a constant point force $\mathbf{F}$ acting at $\mathbf{x}'$ as shown in Fig.1.4 within an infinite body. The volume V is any arbitrary volume enclosed by a surface S with an outward normal $\mathbf{n}$. The displacement field caused by this applied force is

$$u_i(\mathbf{x}) = G_{ij}(\mathbf{x} - \mathbf{x}')F_j \tag{1.43}$$

The displacement gradient is thus

$$u_{i,m}(\mathbf{x}) = G_{ij,m}(\mathbf{x} - \mathbf{x}')F_j \tag{1.44}$$

and the stress field can then be determined by Hooke's law

$$\sigma_{kp}(\mathbf{x}) = C_{kpim}G_{ij,m}(\mathbf{x} - \mathbf{x}')F_j \tag{1.45}$$

If the volume V encloses the point $\mathbf{x}'$, then the force $\mathbf{F}$ must be balanced by the tractions acting over the surface S. This can be written as

$$\begin{aligned} F_k + \int_S \sigma_{kp}(\mathbf{x}) n_p(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) &= 0 \\ F_k + \int_S C_{kpim} G_{ij,m}(\mathbf{x} - \mathbf{x}') n_p(\mathbf{x}) F_j\, \mathrm{d}S(\mathbf{x}) &= 0 \end{aligned}$$

Using Gauss's theorem on the surface integral

$$F_k + \int_V C_{kpim} G_{ij,mp}(\mathbf{x} - \mathbf{x}') F_j \, \mathrm{d}V(\mathbf{x}) = 0 \tag{1.46}$$

The definition of the three dimensional Dirac delta function is

$$\int_V \delta(\mathbf{x} - \mathbf{x}') \, \mathrm{d}V(\mathbf{x}) \quad = \quad \begin{cases} 1 & \text{if } \mathbf{x}' \in V \\ 0 & \text{if } \mathbf{x}' \notin V \end{cases} \tag{1.47}$$

This allows us to move the F_k term into the volume integral

$$\int_V \left[C_{kpim} G_{ij,mp}(\mathbf{x} - \mathbf{x}') F_j + F_k \delta(\mathbf{x} - \mathbf{x}') \right] \, \mathrm{d}V(\mathbf{x})$$

Replacing F_k with $F_j \delta_{kj}$ and factoring F_j gives

$$\int_V \left[C_{kpim} G_{ij,mp}(\mathbf{x} - \mathbf{x}') + \delta_{kj} \delta(\mathbf{x} - \mathbf{x}') \right] F_j \, \mathrm{d}V(\mathbf{x}) = 0$$

This must hold for any arbitrary volume V containing the point $\mathbf{x}'$ and any arbitrary constant force $\mathbf{F}$, thus it must hold pointwise resulting in the equilibrium condition

$$C_{kpim} G_{ij,mp}(\mathbf{x} - \mathbf{x}') + \delta_{jk} \delta(\mathbf{x} - \mathbf{x}') = 0 \tag{1.48}$$

This is the equilibrium equation satisfied by the Green's function in an infinite elastic body, which could be arbitrarily anisotropic. Eq.(1.48) is equivalent to Eq.(1.41) when the body force is a delta function, i.e., $b_k = \delta_{jk} \delta(\mathbf{x} - \mathbf{x}')$.

1.6.2 Green's function in Fourier space

Eq.(1.48) can be solved using Fourier transforms. Defining the Fourier transform of the elastic Green's function as $g_{km}(\mathbf{k})$, it is related to the Green's function as

$$g_{ij}(\mathbf{k}) = \int_{-\infty}^{\infty} \exp(i \, \mathbf{k} \cdot \mathbf{x}) G_{ij}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \tag{1.49}$$

$$G_{ij}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp(-i \, \mathbf{k} \cdot \mathbf{x}) g_{ij}(\mathbf{k}) \, \mathrm{d}\mathbf{k} \tag{1.50}$$

The three dimensional Dirac delta function is

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp(-i \, \mathbf{k} \cdot \mathbf{x}) \, \mathrm{d}\mathbf{k} \tag{1.51}$$

The equilibrium equation for the elastic Green's function can be solved in the Fourier space using the above definitions. Substituting in the definitions of $G_{ij}(\mathbf{x})$ and $\delta(\mathbf{x})$ (setting $\mathbf{x}' = 0$, which fixes the origin)

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \left[C_{kpim} \frac{\partial^2}{\partial x_m \partial x_p} g_{ij}(\mathbf{k}) + \delta_{kj} \right] \exp(-i \, \mathbf{k} \cdot \mathbf{x}) \, \mathrm{d}\mathbf{k} = 0 \tag{1.52}$$

Defining the vector $\mathbf{z}$ as

$$\mathbf{z} = \frac{\mathbf{k}}{|\mathbf{k}|} \tag{1.53}$$

Thus we can simplify Eq.(1.52) to

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \left[-C_{kpim} z_m z_p k^2 + g_{ij}(\mathbf{k}) + \delta_{kj} \right] \exp(-i\,\mathbf{k} \cdot \mathbf{x})\, \mathrm{d}\mathbf{k} = 0 \tag{1.54}$$

This leads to

$$C_{kpim}\, z_m\, z_p\, g_{ij}(\mathbf{k})\, k^2 = \delta_{jk} \tag{1.55}$$

Defining tensor $(zz)_{ki}$ as

$$(zz)_{ki} \equiv C_{pkim} z_p z_m \tag{1.56}$$

Substituting this definition into Eq.(1.55)

$$(zz)_{ki}\, g_{ij}\, k^2 = \delta_{kj} \tag{1.57}$$

The inverse of the $(zz)_{ij}$ tensor can be defined such that

$$(zz)^{-1}_{nk} (zz)_{ki} = \delta_{ni} \tag{1.58}$$

Thus the Green's function in Fourier space is

$$g_{ij}(\mathbf{k}) = \frac{(zz)^{-1}_{ij}}{k^2} \tag{1.59}$$

1.6.3 Green's function in real space

The Green's function in real space can be obtained by inverse Fourier transform of Eq.(1.59). However the analytical solution can only be obtained for isotropic and hexagonal medium. For general anisotropic materials, the Green's function only has a integral representation in real space. Substituting the solution for $g_{ij}(\mathbf{k})$ into Eq.(1.50)

$$G_{ij}(\mathbf{x}) = \left(\frac{1}{2\pi} \right)^3 \int_{-\infty}^{\infty} \exp(-i\,\mathbf{k} \cdot \mathbf{x}) \frac{(zz)^{-1}_{ij}}{k^2}\, \mathrm{d}\mathbf{k} \tag{1.60}$$

Using the spherical coordinate system as shown in Fig.1.5 the integral can be written as

$$G_{ij}(\mathbf{x}) = \left(\frac{1}{2\pi} \right)^3 \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \exp(-ikx\cos\phi) \frac{(zz)^{-1}_{ij}}{k^2} k^2 \sin\phi\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}k \tag{1.61}$$

Because $G_{ij}(\mathbf{x})$ must be real, the integral over k can be written from $-\infty$ to ∞ with a factor of 1/2 as

$$G_{ij}(\mathbf{x}) = \frac{1}{2(2\pi)^3} \int_{-\infty}^{\infty} \int_0^{\pi} \int_0^{2\pi} \exp(-ikx\cos\phi) (zz)^{-1}_{ij} \sin\phi\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}k \tag{1.62}$$

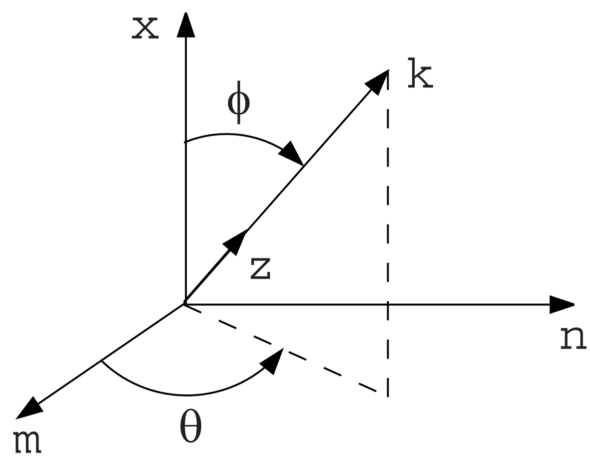


Figure 1.5: Spherical coordinate system. ϕ is the angle between $\mathbf{k}$ and $\mathbf{x}$. $\mathbf{z}$ is a unit vector along $\mathbf{k}$.

The k-integral is the one-dimensional inverse Fourier transform of the delta function

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi^2}\int_0^{\pi}\int_0^{2\pi}\delta(x\cos\phi)(zz)^{-1}_{ij}\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{1.63}$$

Using the property of delta functions that says $\delta(ax) = \frac{\delta(x)}{a}$ and using a transformation of variables such that $s = \cos\phi$

$$\begin{aligned} G_{ij}(\mathbf{x}) &= \frac{1}{8\pi^2 x}\int_1^{-1}\int_0^{2\pi} -\delta(s)(zz)^{-1}_{ij}\,\mathrm{d}\theta\,\mathrm{d}s && (1.64)\\ &= \frac{1}{8\pi^2 x}\int_{-1}^{1}\int_0^{2\pi}\delta(s)(zz)^{-1}_{ij}\,\mathrm{d}\theta\,\mathrm{d}s && (1.65)\end{aligned}$$

Using the definition of the delta function this reduces to

$$\begin{aligned} G_{ij}(\mathbf{x}) &= \frac{1}{8\pi^2 x}\int_0^{2\pi}(zz)^{-1}_{ij}\,\mathrm{d}\theta\Big|_{s=0} && (1.66)\\ G_{ij}(\mathbf{x}) &= \frac{1}{8\pi^2 x}\int_0^{2\pi}(zz)^{-1}_{ij}\,\mathrm{d}\theta\Big|_{\mathbf{x}\cdot\mathbf{z}=0} && (1.67)\end{aligned}$$

Eq.(1.67) represents the infinite medium Green's function for general anisotropic materials. This integral can be evaluated by integrating $(zz)^{-1}_{ij}$ over a unit circle normal to the point direction $\mathbf{x}$ as shown in Fig.1.6. The circle is normal to $\mathbf{x}$ and represents all possible values of the unit vector $\mathbf{z}$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two unit vectors perpendicular to each other and both in the plane normal to $\mathbf{x}$, so that $\mathbf{z} = \boldsymbol{\alpha}\cos\theta + \boldsymbol{\beta}\sin\theta$.

1.6.4 Green's function in isotropic medium

As previously mentioned, the integral in Eq.(1.67) can be evaluated analytically for isotropic materials. Substituting in the elastic constants for isotropic materials into the definition of

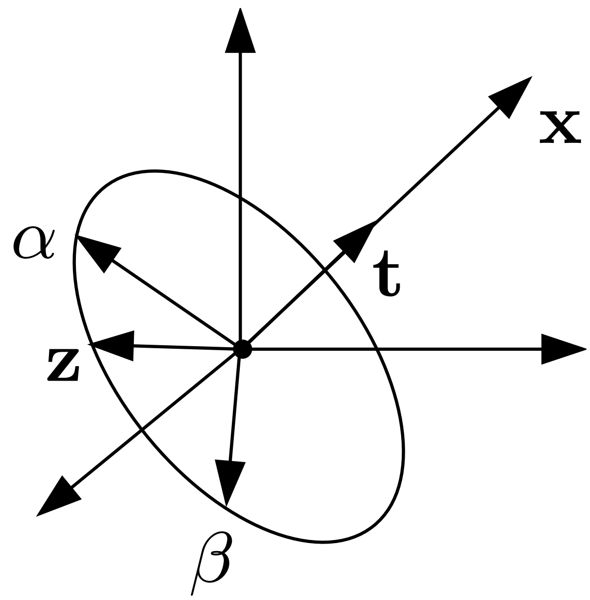


Figure 1.6: Coordinate system for evaluating the integral in Green's function expression.

$(zz)_{ij}$ gives

$$\begin{aligned}
(zz)_{ij} &= z_m z_n C_{imjn} \\
&= z_m z_n[\lambda\delta_{im}\delta_{jn} + \mu(\delta_{ij}\delta_{mn} + \delta_{in}\delta_{jm})] \\
&= \lambda z_i z_j + \mu(z_i z_j + \delta_{ij} z_n z_n) \\
&= (\lambda + \mu) z_i z_j + \mu\delta_{ij} \qquad (1.68)
\end{aligned}$$

Thus $(zz)_{ij}$ can be written as

$$(zz)_{ij} = \mu\left(\delta_{ij} + \frac{\lambda+\mu}{\mu} z_i z_j\right) \qquad (1.69)$$

and the inverse can be written as

$$(zz)^{-1}_{ij} = \frac{1}{\mu}\left(\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu} z_i z_j\right) \qquad (1.70)$$

This can be verified by showing $(zz)^{-1}_{ij}(zz)_{jk} = \delta_{ik}$. Substituting this into Eq.(1.67)

$$G_{ij} = \frac{1}{8\pi^2 x}\int_0^{2\pi} \frac{1}{\mu}\left(\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu} z_i z_j\right) \mathrm{d}\theta\Big|_{\mathbf{x}\cdot\mathbf{z}=0} \qquad (1.71)$$

In Fig.1.6 the unit vector $\mathbf{z}$ can be written as a function of two fixed perpendicular vectors α and β

$$z_i = \alpha_i\cos\theta + \beta_i\sin\theta \qquad (1.72)$$

$$z_i z_j = \alpha_i\alpha_j\cos^2\theta + (\alpha_i\beta_j + \alpha_j\beta_i)\cos\theta\sin\theta + \beta_i\beta_j\sin^2\theta \qquad (1.73)$$

Substituting into Eq.(1.71)

$$\begin{aligned} G_{ij} &= \frac{1}{8\pi^2 x\mu} \int_0^{2\pi} \left[\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu} \Big(\alpha_i\alpha_j \cos^2\theta + \right. \\ & \left. + (\alpha_i\beta_j + \alpha_j\beta_i)\cos\theta\sin\theta + \beta_i\beta_j \sin^2\theta \Big) \right] \mathrm{d}\theta \\ &= \frac{1}{8\pi^2 x\mu} \left[2\pi\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu} (\alpha_i\alpha_j\pi + \beta_i\beta_j\pi) \right] \\ &= \frac{1}{8\pi\mu x} \left[2\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu} (\alpha_i\alpha_j + \beta_i\beta_j) \right] \end{aligned} \tag{1.74}$$

By now we have evaluated the integral, but the Green's function is expressed by two (arbitrary) vectors in the plane perpendicular to $\mathbf{x}$. It would be much more convenient to express the Green's function in terms of the field point itself ($\mathbf{x}$) which can be done with a simple trick. The vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ form a basis with the vector $\mathbf{t}$ as shown in Fig.1.6 where $\mathbf{t} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Thus any vector $\mathbf{v}$ can be written in terms of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\mathbf{t}$.

$$\mathbf{v} = (\mathbf{v}\cdot\boldsymbol{\alpha})\boldsymbol{\alpha} + (\mathbf{v}\cdot\boldsymbol{\beta})\boldsymbol{\beta} + (\mathbf{v}\cdot\mathbf{t})\mathbf{t} \tag{1.75}$$

and in component form

$$v_i = v_j\alpha_j\alpha_i + v_j\beta_j\beta_i + v_j t_j t_i \tag{1.76}$$

This means that

$$\delta_{ij} = \alpha_i\alpha_j + \beta_i\beta_j + t_i t_j \tag{1.77}$$

Substituting 1.77 into 1.74

$$G_{ij} = \frac{1}{8\pi\mu x} \left[2\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu} (\delta_{ij} - t_i t_j) \right]$$

Simplifying results in

$$G_{ij} = \frac{1}{8\pi\mu x} \left[\frac{\lambda+3\mu}{\lambda+2\mu}\delta_{ij} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{x_i x_j}{x^2} \right] \tag{1.78}$$

or, in terms of μ and ν

$$G_{ij} = \frac{1}{16\pi\mu(1-\nu)x} \left[(3-4\nu)\delta_{ij} + \frac{x_i x_j}{x^2} \right] \tag{1.79}$$

or in terms of $R \equiv |\mathbf{x}|$

$$G_{ij} = \frac{1}{8\pi\mu} \left[\delta_{ij} R_{,kk} - \frac{1}{2(1-\nu)} R_{,ij} \right] \tag{1.80}$$

1.7 Betti's Theorem and reciprocity

Betti's Theorem

Consider a linear elastic body with two sets of equilibrating tractions and body forces applied to it. Let $\mathbf{u}^{(1)}$ be the displacement field in response to traction force $\mathbf{t}^{(1)}$ and body force $\mathbf{b}^{(1)}$. Let $\mathbf{u}^{(2)}$ be the displacement field in response to traction force $\mathbf{t}^{(2)}$ and body force $\mathbf{b}^{(2)}$. Under the assumptions of linear elasticity theory, the Betti's Theorem states,

$$\int_S \mathbf{t}^{(1)} \cdot \mathbf{u}^{(2)}\,\mathrm{d}S + \int_V \mathbf{b}^{(1)} \cdot \mathbf{u}^{(2)}\,\mathrm{d}V = \int_S \mathbf{t}^{(2)} \cdot \mathbf{u}^{(1)}\,\mathrm{d}S + \int_V \mathbf{b}^{(2)} \cdot \mathbf{u}^{(1)}\,\mathrm{d}V \tag{1.81}$$

and in component form

$$\int_S t_i^{(1)} u_i^{(2)}\,\mathrm{d}S + \int_V b_i^{(1)} u_i^{(2)}\,\mathrm{d}V = \int_S t_i^{(2)} u_i^{(1)}\,\mathrm{d}S + \int_V b_i^{(2)} u_i^{(1)}\,\mathrm{d}V \tag{1.82}$$

Proof

First, lets establish the fact that $\sigma_{ij}^{(1)} e_{ij}^{(2)} = \sigma_{ij}^{(2)} e_{ij}^{(1)}$. This is because,

$$\begin{aligned}
\sigma_{ij}^{(1)} e_{ij}^{(2)} &= C_{ijkl} e_{kl}^{(1)} e_{ij}^{(2)} \\
\sigma_{ij}^{(2)} e_{ij}^{(1)} &= C_{ijkl} e_{kl}^{(2)} e_{ij}^{(1)} \\
C_{ijkl} &= C_{klij}
\end{aligned}$$

Integrating this identity over the volume of the solid, we have

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)}\,\mathrm{d}V = \int_V \sigma_{ij}^{(2)} e_{ij}^{(1)}\,\mathrm{d}V \tag{1.83}$$

The left hand side can be re-written as,

$$\begin{aligned}
\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)}\,\mathrm{d}V &= \int_V \sigma_{ij}^{(1)} u_{j,i}^{(2)}\,\mathrm{d}V \\
&= \int_V [\sigma_{ij}^{(1)} u_j^{(2)}]_{,i} - \sigma_{ij,i}^{(1)} u_j^{(2)}\,\mathrm{d}V
\end{aligned}$$

From equilibrium condition,

$$\sigma_{ij,i}^{(1)} + b_j^{(1)} = 0 \tag{1.84}$$

we have,

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)}\,\mathrm{d}V = \int_V [\sigma_{ij}^{(1)} u_j^{(2)}]_{,i} + b_j^{(1)} u_j^{(2)}\,\mathrm{d}V$$

Applying Gauss's Theorem on the first term, we have,

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)}\,\mathrm{d}V = \int_S \sigma_{ij}^{(1)} u_j^{(2)} n_i\,\mathrm{d}S + \int_V b_j^{(1)} u_j^{(2)}\,\mathrm{d}V$$

Noticing the definition of traction force,

$$t_j^{(1)} = \sigma_{ij}^{(1)} n_i \tag{1.85}$$

we obtain,

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} \, \mathrm{d}V = \int_S t_j^{(1)} u_j^{(2)} \, \mathrm{d}S + \int_V b_j^{(1)} u_j^{(2)} \, \mathrm{d}V$$

Similarly, the right hand side of Eq. (1.83) can be written as,

$$\int_V \sigma_{ij}^{(2)} e_{ij}^{(1)} \, \mathrm{d}V = \int_S t_j^{(2)} u_j^{(1)} \, \mathrm{d}S + \int_V b_j^{(2)} u_j^{(1)} \, \mathrm{d}V$$

Therefore,

$$\int_S t_j^{(1)} u_j^{(2)} \, \mathrm{d}S + \int_V b_j^{(1)} u_j^{(2)} \, \mathrm{d}V = \int_S t_j^{(2)} u_j^{(1)} \, \mathrm{d}S + \int_V b_j^{(2)} u_j^{(1)} \, \mathrm{d}V$$

which is Betti's Theorem.

Reciprocity of Green's function

Betti's Theorem can be used to prove the reciprocity of Green's function,

$$G_{ij}(\mathbf{x}, \mathbf{x}') = G_{ji}(\mathbf{x}', \mathbf{x}) \tag{1.86}$$

Proof

Consider a specific situation onto which we will apply the Betti's Theorem. Let $\mathbf{b}^{(1)}$ be a concentrated body force $\mathbf{F}$ at point $\mathbf{x}^{(1)}$. Let $\mathbf{b}^{(2)}$ be a concentrated body force $\mathbf{H}$ at point $\mathbf{x}^{(2)}$. We would like to show that the contribution of the traction integral from Betti's theorem is zero, however they cannot be set to zero identically since the body must be in equilibrium. Let's consider a body that has a displacement restraints over part of the surface such that $u_i = 0$ on S^*, where S^* is a subsection of the total surface S. Let's also further assume that there are no other tractions on S. In this case,

$$\begin{aligned}
u_i^{(1)}(\mathbf{x}) &= G_{ij}(\mathbf{x}, \mathbf{x}^{(1)}) F_j \\
b_i^{(1)}(\mathbf{x}) &= F_i \delta(\mathbf{x} - \mathbf{x}^{(1)}) \\
u_i^{(2)}(\mathbf{x}) &= G_{ij}(\mathbf{x}, \mathbf{x}^{(2)}) H_j \\
b_i^{(2)}(\mathbf{x}) &= F_i \delta(\mathbf{x} - \mathbf{x}^{(2)}) \\
t_j^{(1)} &= t_j^{(2)} = 0 \qquad \text{on } S - S^* \\
u_j^{(1)} &= u_j^{(2)} = 0 \qquad \text{on } S^*
\end{aligned} \tag{1.87}$$

Applying Betti's Theorem and noting that the surface integrals are zero, we get,

$$\int_V F_i \delta(\mathbf{x} - \mathbf{x}^{(1)}) G_{ij}(\mathbf{x}, \mathbf{x}^{(2)}) H_j \, \mathrm{d}V(\mathbf{x}) = \int_V H_j \delta(\mathbf{x} - \mathbf{x}^{(2)}) G_{ji}(\mathbf{x}, \mathbf{x}^{(1)}) F_i \, \mathrm{d}V(\mathbf{x}) \tag{1.88}$$

Using the property of δ function, we have,

$$F_i H_j G_{ij}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = F_i H_j G_{ji}(\mathbf{x}^{(2)}, \mathbf{x}^{(1)}) \tag{1.89}$$

This condition must be true for arbitrary forces **F** and **H**. Therefore,

$$G_{ij}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = G_{ji}(\mathbf{x}^{(2)}, \mathbf{x}^{(1)}) \tag{1.90}$$

which is the reciprocity of Green's function.

Chapter 2

Eshelby's Inclusion I: Stress and Strain

2.1 Inclusion and eigenstrain

Consider a homogeneous linear elastic solid with volume V and surface area S, with elastic constant C_{ijkl}, as shown in Fig. 2.1. Let a sub-volume V_0 with surface area S_0 undergo a uniform permanent (inelastic) deformation, such as a martensitic phase transformation. The material inside V_0 is called an *inclusion* and the material outside is called the *matrix*. If we remove V_0 from its surrounding matrix, it should assume a uniform strain e^*_{ij} and will experience zero stress. e^*_{ij} is called the *eigenstrain*, meaning the strain under zero stress. Notice that both the inclusion and the matrix have the same elastic constants. The *eigenstress* is defined as $\sigma^*_{ij} \equiv C_{ijkl} e^*_{kl}$.

In reality, the inclusion is surrounded by the matrix. Therefore, it is not able to reach the state of eigenstrain and zero stress. Instead, both the inclusion and the matrix will deform and experience an elastic stress field. The Eshelby's transformed inclusion problem is to solve the stress, strain and displacement fields both in the inclusion and in the matrix.

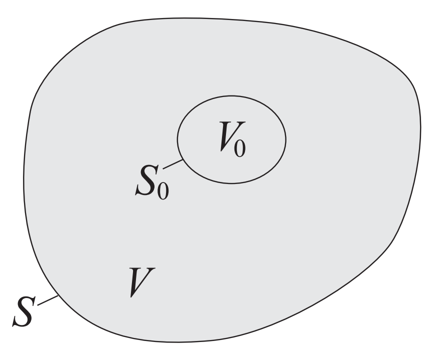


Figure 2.1: A linear elastic solid with volume V and surface S. A subvolume V_0 with surface S_0 undergoes a permanent (inelastic) deformation. The material inside V_0 is called an *inclusion* and the material outside is called the *matrix*.

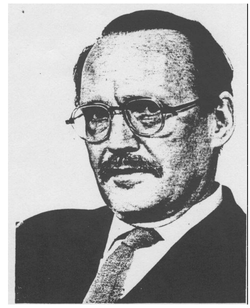
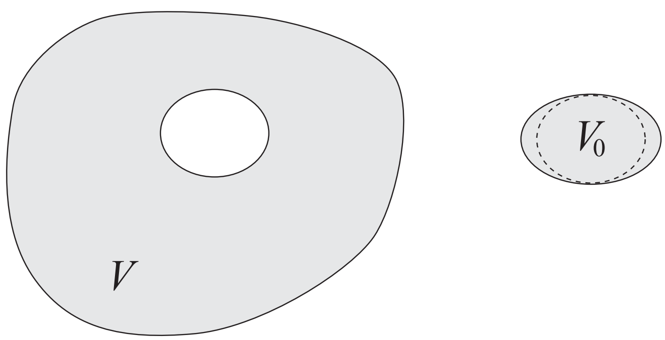


Figure 2.2: John Douglas Eshelby (1916-1981, United Kingdom).

2.2 Green's function and Eshelby's tensor $\mathcal{S}_{ijkl}$

Eshelby showed that the problem stated above can be solved elegantly by the superposition principle of linear elasticity and using the Green's function [6]. Eshelby used the following 4 steps of a "virtual" experiment to construct the desired solution.

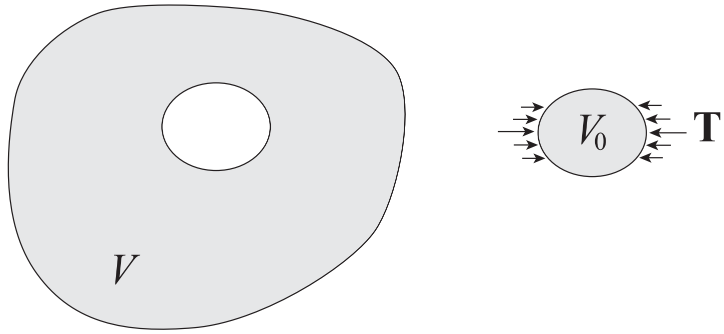
Step 1. Remove the inclusion from the matrix.



Apply no force to the inclusion, nor to the matrix. The strain, stress and displacement fields in the matrix and the inclusion are,

matrix	inclusion
$e_{ij} = 0$	$e_{ij} = e^*_{ij}$
$\sigma_{ij} = 0$	$\sigma_{ij} = 0$
$u_i = 0$	$u_i = e^*_{ij} x_j$

Step 2. Apply surface traction to S_0 in order to make the inclusion return to its original shape

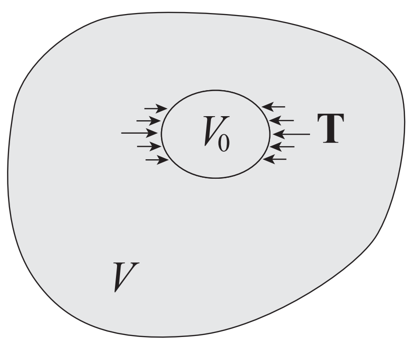


The elastic strain of the inclusion should exactly cancel the eigenstrain, i.e. $e^{\text{el}}_{ij} = -e^*_{ij}$. The strain, stress and displacement fields in the matrix and the inclusion are,

matrix	inclusion
$e_{ij} = 0$	$e_{ij} = e^{\text{el}}_{ij} + e^*_{ij} = 0$
$\sigma_{ij} = 0$	$\sigma_{ij} = C_{ijkl} e^{\text{el}}_{ij} = -C_{ijkl} e^*_{ij} = -\sigma^*_{ij}$
$u_i = 0$	$u_i = 0$

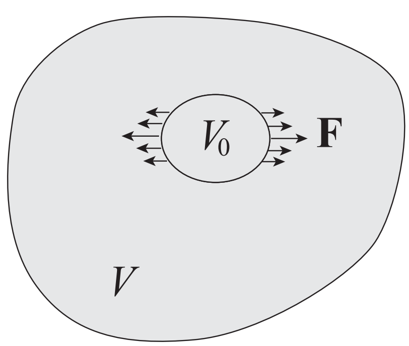
The traction force on S_0 is $T_j = \sigma_{ij} n_i = -\sigma^*_{ij} n_i$.

Step 3. Put the inclusion back to the matrix.



The same force $\mathbf{T}$ is applied to the internal surface S_0. There is no change in the deformation fields in either the inclusion or the matrix from step 2.

Step 4. Now remove the traction $\mathbf{T}$. This returns us to the original inclusion problem as shown in Fig. 2.1. The change from step 3 to step 4 is equivalent to applying a cancelling body force $\mathbf{F} = -\mathbf{T}$ to the internal surface S_0 of the elastic body.



Let $u_i^{\mathrm{c}}(\mathbf{x})$ be the displacement field in response to body force F_j on S_0. $u_i^{\mathrm{c}}(\mathbf{x})$ is called the *constrained* displacement field. It can be easily expressed in terms of the Green's function of the elastic body, (notice that $F_j = -T_j = \sigma_{jk}^* n_k$)

$$u_i^{\mathrm{c}}(\mathbf{x}) = \int_{S_0} F_j(\mathbf{x}')G_{ij}(\mathbf{x},\mathbf{x}')dS(\mathbf{x}') = \int_{S_0} \sigma_{jk}^* n_k(\mathbf{x}')G_{ij}(\mathbf{x},\mathbf{x}')dS(\mathbf{x}') \tag{2.1}$$

The displacement gradient, strain, and stress of the constrained field are

$$u_{i,j}^{\mathrm{c}}(\mathbf{x}) = \int_{S_0} \sigma_{lk}^* n_k(\mathbf{x}')G_{il,j}(\mathbf{x},\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \tag{2.2}$$

$$e_{ij}^{\mathrm{c}}(\mathbf{x}) = \frac{1}{2}(u_{i,j}^{\mathrm{c}} + u_{i,j}^{\mathrm{c}}) = \frac{1}{2}\int_{S_0} \sigma_{lk}^* n_k(\mathbf{x}')\left[G_{il,j}(\mathbf{x},\mathbf{x}') + G_{jl,i}(\mathbf{x},\mathbf{x}')\right]\,\mathrm{d}S(\mathbf{x}') \tag{2.3}$$

$$\sigma_{ij}^{\mathrm{c}}(\mathbf{x}) = C_{ijkl}e_{kl}^{\mathrm{c}}(\mathbf{x}) \tag{2.4}$$

In terms of the constrained field, the strain, stress and displacement fields in the matrix and the inclusion are,

matrix	inclusion
$e_{ij} = e_{ij}^{\mathrm{c}}$	$e_{ij} = e_{ij}^{\mathrm{c}}$
$\sigma_{ij} = \sigma_{ij}^{\mathrm{c}}$	$\sigma_{ij} = \sigma_{ij}^{\mathrm{c}} - \sigma_{ij}^* = C_{ijkl}(e_{kl}^{\mathrm{c}} - e_{kl}^*)$
$u_i = u_i^{\mathrm{c}}$	$u_i = u_i^{\mathrm{c}}$

To obtain explicit expressions for the stresses and strains everywhere, the constrained field must be determined both inside and outside the inclusion. We can define a fourth order tensor $\mathcal{S}_{ijkl}$ that relates the constrained strain inside the inclusion to its eigenstrain,

$$e_{ij}^{c} = \mathcal{S}_{ijkl}e_{kl}^* \tag{2.5}$$

$\mathcal{S}_{ijkl}$ is often referred to as Eshelby's tensor. Because it relates two symmetric strain tensors, the Eshelby's tensor satisfies minor symmetries,

$$\mathcal{S}_{ijkl} = \mathcal{S}_{jikl} = \mathcal{S}_{ijlk} \tag{2.6}$$

However, in general it does not satisfy the major symmetry, i.e. $\mathcal{S}_{ijkl} \neq \mathcal{S}_{klij}$. In the following sections, we derive the explicit expressions of Eshelby's tensor in an infinite elastic medium ($V \to \infty$). In principle, Eshelby's tensor is a function of space, i.e. $\mathcal{S}_{ijkl}(\mathbf{x})$. However, an amazing result obtained by Eshelby is that,

For an *ellipsoidal* inclusion in a homogeneous infinite matrix, the Eshelby tensor $\mathcal{S}_{ijkl}$ is a *constant* tensor. Hence the stress-strain fields inside the inclusion are *uniform*.

2.3 Auxiliary tensor $\mathcal{D}_{ijkl}$

For convenience, let us define another tensor $\mathcal{D}_{ijkl}$ that relates the constrained displacement gradients to the eigenstress inside the inclusion [7],

$$u_{i,l}^{c}(\mathbf{x}) = -\sigma_{kj}^* \mathcal{D}_{ijkl}(\mathbf{x}) \tag{2.7}$$

Obviously, tensor $\mathcal{D}_{ijkl}$ is related to Eshelby's tensor,

$$\begin{aligned}
\mathcal{S}_{ijmn}e^*_{mn} &= e^c_{ij} && (2.8)\\
&= \frac{1}{2}(u^c_{i,j} + u^c_{j,i}) && (2.9)\\
&= -\frac{1}{2}(\sigma^*_{lk}\mathcal{D}_{iklj} + \sigma^*_{lk}\mathcal{D}_{jkli}) && (2.10)\\
&= -\frac{1}{2}\sigma^*_{lk}(\mathcal{D}_{iklj} + \mathcal{D}_{jkli}) && (2.11)\\
&= -\frac{1}{2}C_{lkmn}e^*_{mn}(\mathcal{D}_{iklj} + \mathcal{D}_{jkli}) && (2.12)
\end{aligned}$$

Therefore,

$$\mathcal{S}_{ijmn}(\mathbf{x}) = -\frac{1}{2}C_{lkmn}(\mathcal{D}_{iklj}(\mathbf{x}) + \mathcal{D}_{jkli}(\mathbf{x})) \tag{2.13}$$

Rewrite Eq. (2.7) as $u^c_{i,j}(\mathbf{x}) = -\sigma^*_{kl}\mathcal{D}_{ilkj}(\mathbf{x})$ and compare it with From Eq. (2.2), we obtain,

$$\mathcal{D}_{ilkj}(\mathbf{x}) = -\int_{S_0} n_k(\mathbf{x}')G_{il,j}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \tag{2.14}$$

or equivalently,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\int_{S_0} G_{ij,l}(\mathbf{x}-\mathbf{x}')n_k(\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \tag{2.15}$$

Notice that we have used the fact that $G_{ij}(\mathbf{x},\mathbf{x}') = G_{ij}(\mathbf{x}-\mathbf{x}')$ for an infinite homogeneous medium. Applying Gauss's Theorem, we obtain

$$\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= -\int_{V_0} \frac{\partial}{\partial x'_k}G_{ij,l}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')\\
&= \int_{V_0} \frac{\partial}{\partial x_k}G_{ij,l}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')
\end{aligned}$$

Therefore,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = \int_{V_0} G_{ij,kl}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \tag{2.16}$$

Recall that the Green's function for an anisotropic medium is,

$$G_{ij}(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]\frac{(zz)^{-1}_{ij}}{k^2}\,\mathrm{d}\mathbf{k} \tag{2.17}$$

where $\mathbf{z} = \mathbf{k}/k$. Substituting this into Eq. (2.16), we get

$$\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= \int_{V_0} \frac{\partial^2}{\partial x_k \partial x_l}\left[\frac{1}{(2\pi)^3}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]\frac{(zz)^{-1}_{ij}}{k^2}\,\mathrm{d}\mathbf{k}\right]\mathrm{d}V(\mathbf{x}')\\
&= \frac{1}{(2\pi)^3}\int_{V_0}\int_{-\infty}^{\infty}\left[\mathrm{d}\mathbf{k}(-ik_k)(-ik_l)\exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]\frac{(zz)^{-1}_{ij}}{k^2}\right]\mathrm{d}V(\mathbf{x}')\\
&= -\frac{1}{(2\pi)^3}\int_{V_0}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right](zz)^{-1}_{ij}z_k\,z_l\,\mathrm{d}\mathbf{k}\,\mathrm{d}V(\mathbf{x}') && (2.18)
\end{aligned}$$

Because the integration over the inclusion volume V_0 only depends on $\mathbf{x}'$, but not on $\mathbf{x}$, it is helpful to rearrange integrals as,

$$\begin{aligned}\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\,\exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k\, z_l \int_{V_0} \exp(i\mathbf{k}\cdot\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\ &= -\frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\,\exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k\, z_l\, Q(\mathbf{k}) \qquad (2.19)\end{aligned}$$

where

$$Q(\mathbf{k}) \equiv \int_{V_0} \exp(i\mathbf{k}\cdot\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \qquad (2.20)$$

Therefore, for an infinite homogeneous medium, the auxiliary tensor $\mathcal{D}_{ijkl}$ also satisfies minor symmetries,

$$\mathcal{D}_{ijkl} = \mathcal{D}_{jikl} = \mathcal{D}_{ijlk} \qquad (2.21)$$

But in general it does not satisfy the major symmetry, i.e. $\mathcal{D}_{ijkl} \neq \mathcal{D}_{klij}$ (similar to Eshelby's tensor $\mathcal{S}_{ijkl}$).

2.4 Ellipsoidal inclusion

Now let us restrict our attention to inclusions that are ellipsoidal in shape. The goal is to prove that $\mathcal{D}_{ijkl}(\mathbf{x})$ is a constant inside an ellipsoidal inclusion. The volume V_0 occupied by the inclusion can be expressed as,

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 + \left(\frac{z'}{c}\right)^2 \le 1 \qquad (2.22)$$

where a, b, c specify the size of the ellipsoid. Define new variables,

$$\begin{aligned} X' &\equiv \frac{x'}{a} && (2.23)\\ Y' &\equiv \frac{y'}{b} && (2.24)\\ Z' &\equiv \frac{z'}{c} && (2.25)\\ \mathbf{R} &\equiv X'\mathbf{e}_1 + Y'\mathbf{e}_2 + Z'\mathbf{e}_3 && (2.26)\\ R &\equiv |\mathbf{R}| && (2.27)\end{aligned}$$

Then the integration over V_0 becomes an integration over a unit sphere in the space of $\mathbf{R}$,

$$\int_{V_0} \mathrm{d}V(\mathbf{x}') \;\Rightarrow\; abc\int_{|\mathbf{R}|\le 1} \mathrm{d}\mathbf{R} \qquad (2.28)$$

Also define new variables in Fourier space,

$$\begin{aligned}
\lambda_x &\equiv ak_x && (2.29)\\
\lambda_y &\equiv bk_y && (2.30)\\
\lambda_z &\equiv ck_z && (2.31)\\
\boldsymbol{\lambda} &\equiv \lambda_x \mathbf{e}_1 + \lambda_y \mathbf{e}_2 + \lambda_z \mathbf{e}_3 && (2.32)\\
\lambda &\equiv |\boldsymbol{\lambda}| = \sqrt{a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2} && (2.33)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{k}\cdot\mathbf{x}' &= \boldsymbol{\lambda}\cdot\mathbf{R} && (2.34)\\
Q(\mathbf{k}) &\equiv \int_{V_0} \exp(i\mathbf{k}\cdot\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')\\
&= abc \int_{|\mathbf{R}|\le 1} \exp(i\boldsymbol{\lambda}\cdot\mathbf{R})\,\mathrm{d}\mathbf{R} && (2.35)
\end{aligned}$$

In polar coordinates,

$$\begin{aligned}
Q(\mathbf{k}) &= abc \int_0^1 \int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi \exp(i\lambda R\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}R && (2.36)\\
&= 2\pi abc \int_0^1 \mathrm{d}R\,R^2 \int_{-1}^{1} \mathrm{d}s\ \exp(i\lambda Rs) && (2.37)\\
&= 2\pi abc \int_0^1 R^2 \left[\frac{2\sin(\lambda R)}{\lambda R}\right] \mathrm{d}R && (2.38)\\
&= 4\pi \frac{abc}{\lambda} \int_0^1 R\sin\lambda R\,\mathrm{d}R && (2.39)\\
&= 4\pi \frac{abc}{\lambda^3}\left(\sin\lambda - \lambda\cos\lambda\right) && (2.40)
\end{aligned}$$

Substituting this result into Eq. (2.19), we have

$$\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{1}{8\pi^3}\int_{-\infty}^{\infty} \mathrm{d}\mathbf{k}\ \exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k\, z_l\, \frac{4\pi}{\lambda^3} abc(\sin\lambda - \lambda\cos\lambda)\\
&= -\frac{abc}{2\pi^2}\int_{-\infty}^{\infty} (zz)^{-1}_{ij} z_k\, z_l\ \exp(-i\mathbf{k}\cdot\mathbf{x})\,\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\,\mathrm{d}\mathbf{k} && (2.41)
\end{aligned}$$

Again we go to polar coordinates. Define new variables Φ, Θ, γ through,

$$\begin{aligned}
k_x &= k\sin\Phi\cos\Theta && (2.42)\\
k_y &= k\sin\Phi\sin\Theta && (2.43)\\
k_z &= k\cos\Phi && (2.44)\\
\gamma &\equiv (\mathbf{k}\cdot\mathbf{x})/k = x\sin\Phi\cos\Theta + y\sin\Phi\sin\Theta + z\cos\Phi && (2.45)\\
\beta &\equiv \lambda/k = \sqrt{(a^2\cos^2\Theta + b^2\sin^2\Theta)\sin^2\Phi + c^2\cos^2\Phi} && (2.46)
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{abc}{2\pi^2}\int_0^\infty\int_0^\pi\int_0^{2\pi} k^2 (zz)^{-1}_{ij} z_k\, z_l\, \exp(-ik\gamma)\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\sin\Phi\,\mathrm{d}\Theta\,\mathrm{d}\Phi\,\mathrm{d}k \\
&= -\frac{abc}{2\pi^2}\int_0^\pi\int_0^{2\pi} (zz)^{-1}_{ij} z_k\, z_l\, \kappa(\gamma)\sin\Phi\,\mathrm{d}\Theta\,\mathrm{d}\Phi \qquad (2.47)
\end{aligned}$$

where

$$\begin{aligned}
\kappa(\gamma) &\equiv \int_0^\infty \mathrm{d}k\, k^2\exp(-ik\gamma)\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3} \\
&= \int_0^\infty \mathrm{d}k\, k^2\exp(-ik\gamma)\frac{\sin k\beta-k\beta\cos k\beta}{k^3\beta^3} \\
&= \frac{1}{\beta^3}\int_0^\infty \mathrm{d}k\,\exp(-ik\gamma)\left[\frac{\sin k\beta}{k}-\beta\cos k\beta\right] \qquad (2.48)
\end{aligned}$$

Notice that the dependence of $\mathcal{D}_{ijkl}$ on $\mathbf{x}$ is through $\gamma = (\mathbf{k}\cdot\mathbf{x})/k$ in $\kappa(\gamma)$. To evaluate $\kappa(\gamma)$, notice that the term in the square bracket is an even function of k. Because $\mathcal{D}_{ijkl}$ is real, $\kappa(\gamma)$ must be real as well. Therefore, we can rewrite the integral as,

$$\kappa(\gamma) = \frac{1}{2\beta^3}\int_{-\infty}^\infty \mathrm{d}k\,\exp(-ik\gamma)\left[\frac{\sin k\beta}{k}-\beta\cos k\beta\right] \qquad (2.49)$$

Notice that

$$\begin{aligned}
\int_{-\infty}^\infty \mathrm{d}k\,\exp(-ik\gamma)\cos k\beta &= \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}k\,\mathrm{e}^{-ik\gamma}(\mathrm{e}^{ik\beta}+\mathrm{e}^{-ik\beta}) \\
&= \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}k\,\left[\mathrm{e}^{-ik(\gamma-\beta)}+\mathrm{e}^{-ik(\gamma+\beta)}\right] \\
&= \pi\left[\delta(\beta-\gamma)+\delta(\beta+\gamma)\right] \qquad (2.50) \\
\frac{d}{d\beta}\int_{-\infty}^\infty \mathrm{d}k\,\exp(-ik\gamma)\frac{\sin k\beta}{k} &= \int_{-\infty}^\infty \mathrm{d}k\,\exp(-ik\gamma)\cos k\beta \qquad (2.51) \\
\int_{-\infty}^\infty \mathrm{d}k\,\exp(-ik\gamma)\frac{\sin k\beta}{k} &= \pi\left[h(\beta-\gamma)+h(\beta+\gamma)\right] \qquad (2.52)
\end{aligned}$$

where

$$h(\alpha) = \begin{cases} -\frac{1}{2} & \text{if } \alpha<0 \\ 0 & \text{if } \alpha=0 \\ \frac{1}{2} & \text{if } \alpha>0 \end{cases} \qquad (2.53)$$

Therefore, if $\beta\pm\gamma>0$ then the $\kappa(\gamma)$ reduces to

$$\begin{aligned}
\kappa(\gamma) &= \frac{\pi}{2\beta^3}\left[h(\beta-\gamma)+h(\beta+\gamma)-\beta\delta(\beta-\gamma)-\beta\delta(\beta+\gamma)\right] \\
&= \frac{\pi}{2\beta^3} \qquad (2.54)
\end{aligned}$$

In other words, $\kappa(\gamma)$ becomes a constant if $\beta \pm \gamma > 0$. In this case, $\mathcal{D}_{ijkl}(\mathbf{x})$ reduces to a surface integral that is independent of $\mathbf{x}$,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{abc}{2\pi^2} \int_0^{\pi} \int_0^{2\pi} (zz)^{-1}_{ij} z_k\, z_l\, \frac{\pi}{2\beta^3}\, \sin\Phi\, \mathrm{d}\Theta\, \mathrm{d}\Phi \tag{2.55}$$

We will now show that if $\mathbf{x}$ is within the ellipsoid, then $\beta \pm \gamma > 0$. This will then prove that $\mathcal{D}_{ijkl}$ and $\mathcal{S}_{ijkl}$ are constants within the ellipsoidal inclusion. To see why this is the case, consider vector $\boldsymbol{\rho}$ such that,

$$\boldsymbol{\rho} = \frac{x}{a}\mathbf{e_1} + \frac{y}{b}\mathbf{e_2} + \frac{z}{c}\mathbf{e_3} \tag{2.56}$$

If $\mathbf{x}$ lies within the ellipsoid, then

$$\rho \equiv |\boldsymbol{\rho}| = \sqrt{(x/a)^2 + (y/b)^2 + (z/c)^2} < 1 \tag{2.57}$$

At the same time,

$$\gamma \equiv (\mathbf{k} \cdot \mathbf{x})/k = (\boldsymbol{\lambda} \cdot \boldsymbol{\rho})/k \tag{2.58}$$

$$\beta \equiv \lambda/k \tag{2.59}$$

Therefore,

$$\begin{aligned} |\gamma| &= |\boldsymbol{\lambda} \cdot \boldsymbol{\rho}|/k \le \lambda\rho/k < \lambda/k = \beta \\ \beta \pm \gamma &> 0 \end{aligned} \tag{2.60}$$

Therefore, when $\mathbf{x}$ lies within the ellipsoid, the $\mathcal{D}_{ijkl}$ tensor can be calculated by simply performing a surface integral over a unit sphere,

$$\mathcal{D}_{ijkl} = -\frac{abc}{4\pi} \int_0^{\pi} \int_0^{2\pi} (zz)^{-1}_{ij} z_k\, z_l\, \frac{\sin\Phi}{\beta^3}\, \mathrm{d}\Theta\, \mathrm{d}\Phi \tag{2.61}$$

When $\mathbf{x}$ lies outside the ellipsoid, $\beta \pm \gamma$ is positive for some values of θ and ϕ but is negative elsewhere, hence $\mathcal{D}_{ijkl}$ will depend on $\mathbf{x}$, and can be calculated directly from the Green's function,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\int_S G_{ij,l}(\mathbf{x} - \mathbf{x}')\, n_k(\mathbf{x}')\, \mathrm{d}S(\mathbf{x}') \tag{2.62}$$

Once $\mathcal{D}_{ijkl}$ is obtained, Eshelby's tensor $\mathcal{S}_{ijkl}$ can be found by Eq. (2.13).

2.5 Discontinuities across inclusion interface

We now consider the possible discontinuity of elastic fields across the interface S_0 of the inclusion. Let us define $[[f]]$ as the jump of field f from the inside of the inclusion to the outside, i.e.,

$$[[f]] \equiv f^M - f^I \tag{2.63}$$

where f can be the displacement u_j, stress σ_{ij} or strain e_{ij} and ^{M} indicates the matrix side of the surface S_0 and ^{I} indicates the inclusion side of S_0. First we notice that the displacement field must be continuous everywhere, i.e.,

$$[[u_i]] = 0$$

Since the total displacements are equal to the constrained displacements, the jump in the constrained displacements are zero as well, i.e.,

$$[[u_i^c]] = 0$$

Because the traction forces are continuous across the interface,

$$[[\sigma_{ij} n_i]] = 0$$

Since

$$\begin{aligned} [[\sigma_{ij}]] &= \sigma_{ij}^M - \sigma_{ij}^I \\ \sigma_{ij}^M &= \sigma_{ij}^c \\ \sigma_{ij}^I &= \sigma_{ij}^c - \sigma_{ij}^* \end{aligned}$$

the jump in the total stress is related to the jump in constrained stress field through,

$$[[\sigma_{ij}]] = [[\sigma_{ij}^c]] - \sigma_{ij}^*$$

Therefore, the jump in the constrained tractions must be

$$[[\sigma_{ij}^c n_i]] = -\sigma_{ij}^* n_i \tag{2.64}$$

Even though the constrained displacements u_i^c are continuous across S_0, its gradients $u_{k,l}^c$ are not necessarily continuous. Yet, the continuity of u^c along the entire S_0 surface requires that the derivative of u_i^c along the direction within the local tangent plane of S_0 must be continuous across S_0. Let τ_l be a vector contained in the local tangent plane of S_0, then,

$$[[u_{k,l}^c \tau_l]] = 0 \tag{2.65}$$

Thus we can write

$$[[u_{k,l}^c]] = \mu_k n_l \tag{2.66}$$

where μ_k is a (yet unknown) vector field and n_l is the normal unit vector of the local tangent plane of S_0. From Eq. (2.64) and (2.66), we can establish an equation based on the jump of the constrained traction field,

$$\begin{aligned} [[\sigma_{ij}^c n_i]] &= [[C_{ijkl} u_{k,l}^c n_i]] \\ &= C_{ijkl} \mu_k n_i n_l \\ &= -\sigma_{ij}^* n_i \end{aligned}$$

Recall that we have defined $(nn)_{ij}$ as

$$(nn)_{ij} \equiv C_{iklj} n_k n_l$$

Then we have

$$\begin{aligned} (nn)_{jk}\mu_k &= -\sigma^*_{ij} n_i \\ \mu_k &= -(nn)^{-1}_{kj} \sigma^*_{ij} n_i \end{aligned}$$

The jump in constrained displacement gradients is then

$$[[u^c_{k,l}]] = -(nn)^{-1}_{kj} \sigma^*_{ij} n_i n_l \tag{2.67}$$

And the jump in constrained stress is

$$[[\sigma^c_{ij}]] = -C_{ijkl}(nn)^{-1}_{kp} \sigma^*_{pm} n_m n_l \tag{2.68}$$

Finally the total jump in stress is

$$[[\sigma_{ij}]] = \sigma^*_{ij} - C_{ijkl}(nn)^{-1}_{kp} \sigma^*_{pm} n_m n_l \tag{2.69}$$

2.6 Eshelby's tensor in isotropic medium

The derivation of the Eshelby tensor in isotropic materials can be found in [6] and [3]. For isotropic medium, the Eshelby's tensor for an ellipsoidal inclusion with semi-axes a, b, c can be expressed in terms of elliptic integrals.

For a **spherical inclusion** ($a = b = c$), Eshelby's tensor has the following compact expression,

$$\mathcal{S}_{ijkl} = \frac{5\nu - 1}{15(1-\nu)} \delta_{ij}\delta_{kl} + \frac{4 - 5\nu}{15(1-\nu)} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.70}$$

Notice that the tensor itself does not depend on the radius of the sphere.

In the most **general case** where $a > b > c$ and the semi axis a aligns with the coordinate x (and similarly b with y and c with z), the Eshelby's tensor is,

$$\begin{aligned} \mathcal{S}_{1111} &= \frac{3}{8\pi(1-\nu)} a^2 I_{11} + \frac{1-2\nu}{8\pi(1-\nu)} I_1 \\ \mathcal{S}_{1122} &= \frac{1}{8\pi(1-\nu)} b^2 I_{12} + \frac{1-2\nu}{8\pi(1-\nu)} I_1 \\ \mathcal{S}_{1133} &= \frac{1}{8\pi(1-\nu)} c^2 I_{13} + \frac{1-2\nu}{8\pi(1-\nu)} I_1 \\ \mathcal{S}_{1212} &= \frac{a^2+b^2}{16\pi(1-\nu)} I_{12} + \frac{1-2\nu}{16\pi(1-\nu)} (I_1 + I_2) \\ \mathcal{S}_{1112} &= \mathcal{S}_{1223} = \mathcal{S}_{1232} = 0 \end{aligned}$$

The rest of the nonzero terms can be found by cyclic permutation of the above formulas. Notice that we should also let $a \to b \to c$ together with $1 \to 2 \to 3$. The I terms are defined in terms of standard elliptic integrals,

$$\begin{aligned} I_1 &= \frac{4\pi abc}{(a^2-b^2)(a^2-c^2)^{1/2}}\left[F(\theta,k)-E(\theta,k)\right] \\ I_3 &= \frac{4\pi abc}{(b^2-c^2)(a^2-c^2)^{1/2}}\left[\frac{b(a^2-c^2)^{1/2}}{ac}-E(\theta,k)\right] \end{aligned}$$

where

$$\begin{aligned} \theta = \arcsin\sqrt{\frac{a^2-c^2}{a^2}} \\ k = \sqrt{\frac{a^2-b^2}{a^2-c^2}} \end{aligned}$$

and

$$\begin{aligned} I_1+I_2+I_3 = 4\pi \\ 3I_{11}+I_{12}+I_{13} = \frac{4\pi}{a^2} \\ 3a^2I_{11}+b^2I_{12}+c^2I_{13} = 3I_1 \\ I_{12} = \frac{I_2-I_1}{a^2-b^2} \end{aligned}$$

and the standard elliptic integrals are defined as

$$F(\theta,k) = \int_0^\theta \frac{\mathrm{d}w}{(1-k^2\sin^2 w)^{1/2}} \tag{2.71}$$

$$E(\theta,k) = \int_0^\theta (1-k^2\sin^2 w)^{1/2}\,\mathrm{d}w \tag{2.72}$$

For an **elliptic cylinder** ($c \to \infty$)

$$
\begin{aligned}
\mathcal{S}_{1111} &= \frac{1}{2(1-\nu)}\left[\frac{b^2+2ab}{(a+b)^2}+(1-2\nu)\frac{b}{a+b}\right] \\
\mathcal{S}_{2222} &= \frac{1}{2(1-\nu)}\left[\frac{a^2+2ab}{(a+b)^2}+(1-2\nu)\frac{a}{a+b}\right] \\
\mathcal{S}_{3333} &= 0 \\
\mathcal{S}_{1122} &= \frac{1}{2(1-\nu)}\left[\frac{b^2}{(a+b)^2}-(1-2\nu)\frac{b}{a+b}\right] \\
\mathcal{S}_{2233} &= \frac{1}{2(1-\nu)}\frac{2\nu a}{a+b} \\
\mathcal{S}_{2211} &= \frac{1}{2(1-\nu)}\left[\frac{a^2}{(a+b)^2}-(1-2\nu)\frac{a}{a+b}\right] \\
\mathcal{S}_{3311} &= \mathcal{S}_{3322} = 0 \\
\mathcal{S}_{1212} &= \frac{1}{2(1-\nu)}\left[\frac{a^2+b^2}{2(a+b)^2}+\frac{(1-2\nu)}{2}\right] \\
\mathcal{S}_{1133} &= \frac{1}{2(1-\nu)}\frac{2\nu b}{a+b} \\
\mathcal{S}_{2323} &= \frac{a}{2(a+b)} \qquad \mathcal{S}_{3131} = \frac{b}{2(a+b)}
\end{aligned}
$$

For a **flat ellipsoid** ($a > b \gg c$). The I integrals in this limiting case reduce to

$$
\begin{aligned}
I_1 &= 4\pi(F(k)-E(k))\frac{bc}{a^2-b^2} \\
I_2 &= 4\pi\left(E(k)\frac{c}{b}-(F(k)-E(k))\,\frac{bc}{a^2-b^2}\right) \\
I_3 &= 4\pi\left(1-E(k)\frac{c}{b}\right) \\
I_{12} &= 4\pi\left[E(k)\frac{c}{b}-2(F(k)-E(k))\frac{bc}{a^2-b^2}\right]/(a^2-b^2) \\
I_{23} &= 4\pi\left[1-2E(k)\frac{c}{b}+(F(k)-E(k))\frac{bc}{a^2-b^2}\right]/b^2 \\
I_{31} &= 4\pi\left[1-E(k)\frac{c}{b}-(F(k)-E(k))\frac{bc}{a^2-b^2}\right]/a^2 \\
I_{33} &= \frac{4\pi}{3c^2}
\end{aligned}
$$

where $E(k)$ and $F(k)$ are complete elliptic integrals defined as

$$
F(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}w}{(1-k^2\sin^2 w)^{1/2}} \tag{2.73}
$$

$$
E(k) = \int_0^{\frac{\pi}{2}} (1-k^2\sin^2 w)^{1/2}\,\mathrm{d}w \tag{2.74}
$$

We have a **penny shaped** inclusion if we let $a = b$ in the flat ellipsoid. The Eshelby's tensor further reduces to

$$
\begin{aligned}
\mathcal{S}_{1111} &= \mathcal{S}_{2222} = \frac{\pi(13-8\nu)}{32(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{3333} &= 1 - \frac{\pi(1-2\nu)}{4(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{1122} &= \mathcal{S}_{2211} = \frac{\pi(8\nu-1)}{32(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{1133} &= \mathcal{S}_{2233} = \frac{\pi(2\nu-1)}{8(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{3311} &= \mathcal{S}_{3322} = \frac{v}{1-v}\left(1 - \frac{\pi(4\nu+1)}{8\nu}\frac{c}{a}\right) \\
\mathcal{S}_{1212} &= \frac{\pi(7-8\nu)}{32(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{3131} &= \mathcal{S}_{2323} = \frac{1}{2}\left(1 + \frac{\pi(\nu-2)}{4(1-\nu)}\frac{c}{a}\right)
\end{aligned}
$$

Eshelby's tensor for various other shapes can be found in [3] and [8].

2.7 Eshelby's inclusion in 2-dimensions

The derivations on ellipsoidal inclusions in 3D space given above can be repeated for elliptic inclusions in 2D space (corresponding to elliptic cylinder in 3D). As an illustration, in this section we show that the Eshelby's tensor $\mathcal{S}$ is a constant within the ellipse and we derive the explicit expression of $\mathcal{S}$ for a circular (i.e. cylindrical) inclusion.

Constant $\mathcal{D}_{ijkl}$

Consider an elliptic inclusion in the 2D medium that can occupies the area,

$$
\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 \leq 1 \tag{2.75}
$$

For consistency of notation, we will still use V_0 to represent the area (or volume) occupied by the inclusion and S_0 as its boundary. Let its eigenstrain be e^*_{ij} $(i, j = 1, 2)$. Define Eshelby's tensor $\mathcal{S}_{ijkl}$ and auxiliary tensor $\mathcal{D}_{ijkl}$ similarly as before, but with $i, j, k, l = 1, 2$. We will consider the plane strain condition, so that the elastic constants tensor in 2D c_{ijkl} simply equals to the elastic constants tensor in 3D C_{ijkl} for $i, j, k, l = 1, 2$, i.e.,

$$
\begin{aligned}
c_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&= \frac{2\mu\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&= \mu\left(\frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)
\end{aligned}
$$

Similar to the 3D case, the Fourier space expression for the Green's function in 2D is

$$g_{ij}(\mathbf{k}) = \frac{(zz)^{-1}_{ij}}{k^2}$$

The real space expression is then,

$$G_{ij}(\mathbf{x}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \exp(-i\mathbf{k}\cdot\mathbf{x})\frac{(zz)^{-1}_{ij}}{k^2}\,\mathrm{d}\mathbf{k} \tag{2.76}$$

Similar to Eq. (2.16), the auxiliary tensor for an elliptic inclusion in 2D is,

$$\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= \int_{V_0} G_{ij,kl}(\mathbf{x}-\mathbf{x}')dV(\mathbf{x}') \\
&= \int_{V_0} \frac{\partial^2}{\partial x_k \partial x_l}\left[\frac{1}{(2\pi)^2}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]\frac{(zz)^{-1}_{ij}}{k^2}\,\mathrm{d}\mathbf{k}\right]\mathrm{d}V(\mathbf{x}') \\
&= -\frac{1}{(2\pi)^2}\int_{V_0}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right](zz)^{-1}_{ij} z_k z_l\,\mathrm{d}\mathbf{k}\,\mathrm{d}V(\mathbf{x}') \\
&= -\frac{1}{(2\pi)^2}\int \exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k z_l Q(\mathbf{k})\,\mathrm{d}\mathbf{k}
\end{aligned} \tag{2.77}$$

where

$$Q(\mathbf{k}) \equiv \int_{V_0} \exp(i\mathbf{k}\cdot\mathbf{x}')dV(\mathbf{x}') \tag{2.78}$$

Define

$$\begin{aligned}
\boldsymbol{\lambda} &\equiv (\lambda_1,\lambda_2) = (k_1 a, k_2 b)\ , \quad \lambda = |\boldsymbol{\lambda}| \\
\mathbf{R} &\equiv (R_1,R_2) = (x_1/a, x_2/b)\ , \quad R = |\mathbf{R}| \\
\gamma &= (\mathbf{k}\cdot\mathbf{x})/k = (\boldsymbol{\lambda}\cdot\mathbf{R})/k \\
\beta &= \lambda/k
\end{aligned} \tag{2.79}$$

Then

$$\begin{aligned}
Q(\mathbf{k}) &\equiv \int_{V_0} \exp(i\mathbf{k}\cdot\mathbf{x}')dV(\mathbf{x}') \\
&= ab\int_{|\mathbf{R}|\le 1} \exp(i\boldsymbol{\lambda}\cdot\mathbf{R})\,\mathrm{d}\mathbf{R} \\
&= ab\int_0^1\int_0^{2\pi} R\exp(i\lambda R\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}R \\
&= 2\pi ab\int_0^1 R\,J_0(\lambda R)\,\mathrm{d}R \\
&= 2\pi ab\frac{J_1(\lambda)}{\lambda}
\end{aligned} \tag{2.80}$$

Therefore,

$$\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{ab}{2\pi}\int \exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k z_l \frac{J_1(\lambda)}{\lambda}\,\mathrm{d}\mathbf{k} \\
&= -\frac{ab}{2\pi}\int_0^{2\pi}\int_0^{\infty} \exp(-ik\gamma)(zz)^{-1}_{ij} z_k z_l \frac{J_1(k\beta)}{k\beta} k\,\mathrm{d}k\,\mathrm{d}\theta \\
&= -\frac{ab}{2\pi}\int_0^{2\pi} (zz)^{-1}_{ij} z_k z_l \kappa(\gamma)\,\mathrm{d}\theta \qquad (2.81)
\end{aligned}$$

where

$$\begin{aligned}
\kappa(\gamma) &= \frac{1}{\beta}\int_0^{\infty}\exp(-ik\gamma)J_1(k\beta)\,\mathrm{d}k \\
&= \frac{1}{\beta^2}\left[1-\frac{i|\gamma|}{\sqrt{\beta^2-\gamma^2}}\right] \qquad (2.82)
\end{aligned}$$

Notice that $D_{ijkl}(\mathbf{x})$ is real. Since $(zz)^{-1}_{ij} z_k z_l$ is also real, the imaginary part of $\kappa(\gamma)$ can be neglected. Therefore, as long as $\beta > |\gamma|$, we can write

$$\kappa(\gamma) = \frac{1}{\beta^2} \qquad (2.83)$$

which is independent of γ. Therefore $\mathcal{D}_{ijkl}(\mathbf{x})$ is independent of $\mathbf{x}$. $\beta > |\gamma|$ is satisfied if $\mathbf{x}$ is within the inclusion. This can be shown by the following. If $\mathbf{x}$ is inside the ellipse, then

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = R_1^2 + R_2^2 < 1 \qquad (2.84)$$

which means $R < 1$. Therefore,

$$|\gamma| = |\boldsymbol{\lambda}\cdot\mathbf{R}|/k \le |\boldsymbol{\lambda}|\cdot|\mathbf{R}|/k = \lambda R/k < \lambda/k = \beta \qquad (2.85)$$

$\mathcal{S}_{ijkl}$ for circular inclusion

We have shown that inside an elliptic inclusion of an isotropic medium

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{ab}{2\pi}\int_0^{2\pi}(zz)^{-1}_{ij} z_k z_l \frac{1}{\beta^2}\,\mathrm{d}\theta$$

For a circular inclusion, $a = b$, then $\beta = a$ and $\mathcal{D}_{ijkl}$ becomes

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{1}{2\pi}\int_0^{2\pi}(zz)^{-1}_{ij} z_k z_l\,\mathrm{d}\theta$$

Notice that

$$\begin{aligned}
c_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
(zz)_{ij} &= \mu\delta_{ij} + (\lambda+\mu) z_i z_j \\
(zz)^{-1}_{ij} &= \frac{1}{\mu}\left(\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu} z_i z_j\right) = \frac{1}{\mu}\left(\delta_{ij} - \frac{1}{2(1-\nu)} z_i z_j\right)
\end{aligned}$$

Therefore,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\mu}\left(\delta_{ij} - \frac{1}{2(1-\nu)}z_i z_j\right) z_k z_l \,\mathrm{d}\theta$$

Notice that $z_1 = \cos\theta$ and $z_2 = \sin\theta$, $\mathcal{D}_{ijkl}$ can be evaluated explicitly. Let us define

$$H_{kl} \equiv \int_0^{2\pi} z_k z_l \,\mathrm{d}\theta$$

and

$$J_{ijkl} \equiv \int_0^{2\pi} z_i z_j z_k z_l \,\mathrm{d}\theta \tag{2.86}$$

The only non-zero elements of H_{kl} are H_{11} and H_{22}, i.e.,

$$H_{kl} = \delta_{kl}\int_0^{2\pi}\cos^2\theta \,\mathrm{d}\theta = \pi\delta_{kl}$$

Similarly J_{ijkl} is non-zero only when all four indices are the same or they come in pairs.

$$J_{1111} = J_{2222} = \int_0^{2\pi}\cos^4\theta \,\mathrm{d}\theta = \frac{3\pi}{4}$$

$$J_{1122} = J_{2211} = J_{1212} = J_{2121} = \int_0^{2\pi}\cos^2\theta\sin^2\theta = \frac{\pi}{4}$$

therefore

$$J_{ijkl} = \frac{\pi}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Thus

$$\begin{aligned}
\mathcal{D}_{ijkl} &= -\frac{1}{2\pi\mu}\left(\delta_{ij}H_{kl} - \frac{1}{2(1-\nu)}J_{ijkl}\right) \\
&= -\frac{1}{2\pi\mu}\left(\delta_{ij}\delta_{kl}\pi - \frac{1}{2(1-\nu)}\frac{\pi}{4}(\delta_{ij}\delta_{kl}\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right) \\
&= -\frac{1}{16\mu(1-\nu)}\left((8-8\nu)\delta_{ij}\delta_{kl} - \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) \\
&= -\frac{1}{16\mu(1-\nu)}\left((7-8\nu)\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{S}_{ijmn} &= -\frac{1}{2}c_{lkmn}(\mathcal{D}_{iklj} + \mathcal{D}_{jkli}) \\
&= -\lambda\mathcal{D}_{ikkj}\delta_{mn} - \mu(\mathcal{D}_{inmj} + \mathcal{D}_{jnmi})
\end{aligned}$$

$\mathcal{D}_{ikkj}$ can be evaluated by

$$\mathcal{D}_{ikkj} = -\frac{1}{16\mu(1-\nu)}\left((7-8\nu)\delta_{ik}\delta_{kj} - \delta_{ik}\delta_{kj} - \delta_{ij}\delta_{kk}\right)$$

Note, that now in two dimensions, $\delta_{kk} = 2$

$$\begin{aligned}
\mathcal{D}_{ikkj} &= -\frac{1}{16\mu(1-\nu)}\left((7-8\nu)\delta_{ij}\delta_{kj} - \delta_{ij} - 2\delta_{ij}\right) \\
&= -\frac{(4-8\nu)}{16\mu(1-\nu)}\delta_{ij} \\
\lambda &= \frac{2\mu\nu}{1-2\nu} \\
\lambda\mathcal{D}_{ikkj} &= -\frac{\nu}{2(1-\nu)}\delta_{ij}
\end{aligned} \tag{2.87}$$

Thus

$$\begin{aligned}
\mathcal{S}_{ijmn} &= \frac{\nu}{2(1-\nu)}\delta_{ij}\delta_{mn} + \frac{1}{16(1-\nu)}\left((6-8\nu)(\delta_{in}\delta_{jm} + \delta_{jn}\delta_{im}) - 2\delta_{ij}\delta_{mn}\right) \\
&= \frac{4\nu-1}{8(1-\nu)}\delta_{ij}\delta_{mn} + \frac{3-4\nu}{8(1-\nu)}\left(\delta_{in}\delta_{jm} + \delta_{jn}\delta_{im}\right)
\end{aligned}$$

This is the Eshelby's tensor for a circular inclusion in 2D, which is the same as a cylindrical inclusion in 3D under plane strain.

Chapter 3

Eshelby's Inclusion II: Energy

3.1 Inclusion energy in an infinite solid

So far we have obtained the expressions for the stress, strain and displacement field both inside and outside the inclusion. An important question is: "what is the total elastic energy E of the solid containing an inclusion?" In this and subsequent sections, we derive the expressions for E, which we refer to as the *inclusion energy* for brevity. However, we emphasize that E is the total elastic energy of the solid containing an inclusion. E includes the elastic energy stored both inside *and* outside the inclusion. For example, if we obtain E as a function of the inclusion size, then the derivative of E provides the driving force for the expansion (or shrinkage) of the inclusion. Notice that this is the case only if E is the total elastic energy, not just the energy stored inside the inclusion.

There are two ways to obtain the expression for the total energy E. First, we can integrate the elastic energy density both inside and outside the inclusion, using the field expressions we have already obtained. Second, we can obtain the elastic energy E by measuring the work done in a virtual experiment that transforms a solid system with zero elastic energy to the solid containing an inclusion. In this section, we take the first approach. The work method is discussed in the next section, which leads to identical results but may provide more physical insight.

For clarity, let us introduce some symbols to describe the elastic fields inside and outside the inclusion. Let the elastic (stress, strain, displacement) fields inside the inclusion be denoted by a superscript I, and the elastic fields outside the inclusion (i.e. in the matrix) be denoted by a superscript M. Notice that whenever the superscript I or M is used, the fields only include the elastic component. For a homogeneous infinite solid, the elastic fields in the matrix and the inclusion are,

matrix	inclusion
$e^M_{ij} = e^c_{ij}$	$e^I_{ij} = e^c_{ij} - e^*_{ij}$
$\sigma^M_{ij} = \sigma^c_{ij}$	$\sigma^I_{ij} = \sigma^c_{ij} - \sigma^*_{ij}$
$u^M_i = u^c_i$	$u^I_i = u^c_i - e^*_{ij} x_j$

Therefore, the total elastic energy is,

$$E = \frac{1}{2}\int_{V_0} \sigma^I_{ij} e^I_{ij}\,\mathrm{d}V + \frac{1}{2}\int_{V_\infty - V_0} \sigma^M_{ij} e^M_{ij}\,\mathrm{d}V \tag{3.1}$$

Rewriting E in terms of displacements, we have

$$E = \frac{1}{4}\int_{V_0} \sigma^I_{ij}(u^I_{i,j} + u^I_{j,i})\,\mathrm{d}V + \frac{1}{4}\int_{V_\infty - V_0} \sigma^M_{ij}(u^M_{i,j} + u^M_{j,i})\,\mathrm{d}V \tag{3.2}$$

and noting the symmetry of the stress tensor

$$E = \frac{1}{2}\int_{V_0} \sigma^I_{ij}u^I_{j,i}\,\mathrm{d}V + \frac{1}{2}\int_{V_\infty - V_0} \sigma^M_{ij}u^M_{j,i}\,\mathrm{d}V \tag{3.3}$$

Now, the derivative can be factored out using the following rule

$$\sigma_{ij}u_{i,j} = (\sigma_{ij}u_j)_{,i} - \sigma_{ij,i}u_j \tag{3.4}$$

$$E = \frac{1}{2}\int_{V_0} (\sigma^I_{ij}u^I_j)_{,i} - \sigma^I_{ij,i}u^I_j\,\mathrm{d}V + \frac{1}{2}\int_{V_\infty - V_0} (\sigma^M_{ij}u^M_j)_{,i} - \sigma^M_{ij,i}u^M_j\,\mathrm{d}V \tag{3.5}$$

The body is assumed not to have any body forces acting on it, thus the divergence of the stress tensor, $\sigma_{ij,i}$, is zero. Thus

$$E = \frac{1}{2}\int_{V_0} (\sigma^I_{ij}u^I_j)_{,i}\,\mathrm{d}V + \frac{1}{2}\int_{V_\infty - V_0} (\sigma^M_{ij}u^M_j)_{,i}\,\mathrm{d}V \tag{3.6}$$

We wish to now use Gauss's theorem on this equation. We need to be careful about the sign of the unit normal vector that points outside the integration volume. Let the normal vector pointing out of the inclusion volume V_0 be n_i^{out}. Let the unit normal vector pointing out of the outer surface of the matrix V_∞ (at infinity) be n_i^∞. Applying Gauss's theorem,

$$E = \frac{1}{2}\int_{S_0} \sigma^I_{ij}u^I_j n_i^{\mathrm{out}}\,\mathrm{d}S - \frac{1}{2}\int_{S_0} \sigma^M_{ij}u^M_j n_i^{\mathrm{out}}\,\mathrm{d}S + \frac{1}{2}\int_{S_\infty} \sigma^M_{ij}u^M_j n_i^{\infty}\,\mathrm{d}S \tag{3.7}$$

We expect that the surface integral over S_∞ should vanish as it approaches infinity. To show this, let S_∞ be a spherical surface whose radius R approaches infinity. Notice that

$$\mathcal{D}_{ijkl}(\mathbf{x}) = \int_{V_0} G_{ij,kl}(\mathbf{x}-\mathbf{x}')dV(\mathbf{x}') \tag{3.8}$$

Because $G_{ijkl}(\mathbf{x}-\mathbf{x}') \to R^{-3}$ where $R = |\mathbf{x}|$, for large R, then $\mathcal{D}_{ijkl}(\mathbf{x}) \to R^{-3}$. Therefore,

$$e^M_{ij} = \mathcal{O}\left(\frac{1}{R^3}\right) \tag{3.9}$$

$$\sigma^M_{ij} = \mathcal{O}\left(\frac{1}{R^3}\right) \tag{3.10}$$

$$\mathrm{d}S = \mathcal{O}\left(R^2\right) \tag{3.11}$$

Thus

$$\int_{S_\infty} \sigma^M_{ij}u^M_j n_i^{\mathrm{out}}\,\mathrm{d}V \to 0 \qquad \text{as} \qquad R \to \infty \tag{3.12}$$

Combining the two integrals over S_0,

$$E = \frac{1}{2}\int_{S_0} \left(\sigma^I_{ij} u^I_j - \sigma^M_{ij} u^M_j\right) n^{\rm out}_i \, {\rm d}S \tag{3.13}$$

Although the stress across the inclusion interface S_0 does not have to be continuous, the traction force across the interface must be continuous, i.e.,

$$\sigma^I_{ij} n^{\rm out}_i = \sigma^M_{ij} n^{\rm out}_i \tag{3.14}$$

which leads to

$$E = \frac{1}{2}\int_{S_0} \sigma^I_{ij}\left(u^I_j - u^M_j\right) n^{\rm out}_i \, {\rm d}S \tag{3.15}$$

From the definition of (elastic displacement fields) u^I_j and u^M_j, we have

$$u^I_j - u^M_j \;\;=\;\; (u^c_j - e^*_{jk} x_k) - u^c_j = -e^*_{jk} x_k \tag{3.16}$$

Thus

$$E = -\frac{1}{2}\int_{S_0} \sigma^I_{ij} n^{\rm out}_i e^*_{jk} x_k \, {\rm d}S \tag{3.17}$$

Therefore, we have expressed the total elastic energy E in terms of a surface integral over S_0, the inclusion interface. We can further simplify this expression by transforming the integral back into a volume integral (over the inclusion volume V_0).

$$\begin{aligned}
E &= -\frac{1}{2}\int_{V_0} \left(\sigma^I_{ij} e^*_{jk} x_k\right)_{,i} \, {\rm d}V \\
&= -\frac{1}{2}\int_{V_0} e^*_{jk}\left(\sigma^I_{ij,i} x_k + \sigma^I_{ij} x_{k,i}\right) \, {\rm d}V \\
&= -\frac{1}{2}\int_{V_0} e^*_{jk} \sigma^I_{kj} \, {\rm d}V \\
&= -\frac{1}{2} e^*_{ij} \int_{V_0} \sigma^I_{ij} \, {\rm d}V \\
&= -\frac{1}{2} e^*_{ij} \int_{V_0} \left(\sigma^c_{ij} - \sigma^*_{ij}\right) \, {\rm d}V
\end{aligned} \tag{3.18}$$

For an ellipsoidal inclusion, the stress inside is a constant, thus

$$E = -\frac{1}{2}\left(\sigma^c_{ij} - \sigma^*_{ij}\right) e^*_{ij} \, V_0 = -\frac{1}{2}\sigma^I_{ij} e^*_{ij} \, V_0 \tag{3.19}$$

If the volume is not an ellipsoid, we can still write the energy in terms of the average stress in the inclusion

$$E = -\frac{1}{2}\overline{\sigma^I_{ij}} e^*_{ij} \; V_0 \tag{3.20}$$

where

$$\overline{\sigma_{ij}^{I}} \equiv \frac{1}{V_0} \int_{V_0} \sigma_{ij}^{c}(\mathbf{x})\, \mathrm{d}V(\mathbf{x}) - \sigma_{ij}^{*} \tag{3.21}$$

Suppose that we wish to account for how much of the energy is stored inside the inclusion and how much is stored in the matrix. The energy store inside the inclusion is

$$E^{I} = \frac{1}{2} \int_{V_0} \sigma_{ij}^{I} e_{ij}^{I}\, \mathrm{d}V$$

For ellipsoidal inclusion, the stress and strain are constant inside, hence

$$E^{I} = \frac{1}{2} \sigma_{ij}^{I} e_{ij}^{I} V_0 = \frac{1}{2} \sigma_{ij}^{I} \left(e_{ij}^{\mathrm{c}} - e_{ij}^{*} \right) V_0$$

Since the total elastic energy is

$$E = -\frac{1}{2} \sigma_{ij}^{I} e_{ij}^{*} V_0$$

the elastic energy stored inside the matrix must be,

$$E^{M} = E - E^{I} = -\frac{1}{2} \sigma_{ij}^{I} e_{ij}^{c} V_0$$

3.2 Inclusion energy by the work method

In this section, we re-derive the expressions in the previous section concerning the inclusion energy using a different approach. Rather than integrating the strain energy density over the entire volume, we make use of the fact that the stored elastic (potential) energy in the solid must equal the work done to it in a reversible process. By considering a virtual reversible experiment that transforms a stress-free solid into a solid containing an inclusion, and accounting for the work done along the way, we can derive the total elastic energy (or the elastic energy stored within the inclusion or the matrix) using considerably less math than before.

To better illustrate this method, let us consider a simple example. Consider a mass M attached to a linear spring with stiffness k. Let E_0 be the equilibrium state of the system under no applied force. Obviously $E_0 = 0$. Define the origin as the position of the mass at this state. Suppose we gradually apply a force to the mass until the force reaches F_1. At this point the mass must have moved by a distance $x_1 = F_1/k$. Let the energy of this state be E_1. The work done in moving the mass from 0 to x_1 equals the average force $\overline{F}$ applied to the mass times the distance travelled (x_1). Because the initial force is 0 and the final force is F_1, the average force is $\overline{F} = F_1/2$. Therefore, the work done in moving the mass from 0 to x_1 is,

$$W_{01} = \overline{F} x_1 = \frac{1}{2} F_1 x_1 = \frac{1}{2} k x_1^2 \tag{3.22}$$

Hence

$$E_1 = E_0 + W_{01} = \frac{1}{2} k x_1^2 \tag{3.23}$$

Suppose we further increase the force to F_2 and the system reaches a new state at $x_2 = F_2/k$ with energy E_2. Since the initial force during this transformation is F_1 and the final force is F_2, the average force is $\overline{F} = (F_1 + F_2)/2$. The mass moves by a distance of $x_2 - x_1$ under this force. Therefore the work done is

$$W_{12} = \frac{1}{2}(F_1 + F_2)(x_2 - x_1) = \frac{1}{2} k (x_2^2 - x_1^2) \tag{3.24}$$

Hence

$$E_2 = E_1 + W_{12} = \frac{1}{2} k x_2^2 \tag{3.25}$$

Now, let's apply this method to Eshelby's inclusion problem. Let us consider the four steps in Eshelby's construction of a solid containing an inclusion. Recall that after step 1, the inclusion is outside the matrix. The inclusion has undergone a deformation due to its eigenstrain. No forces are applied to either the inclusion or the matrix. Obviously, the total elastic energy at this state is $E_1 = 0$.

In step 2, we apply a set of traction forces on the inclusion surface S_0. At the end of step 2, the traction forces are $T_j = -\sigma^*_{ij} n_j$ and the displacements on the surface are $u_j = -e^*_{kj} x_k$. Therefore, the work done in step 2 is

$$\begin{aligned} W_{12} &= \frac{1}{2} \int_{S_0} T_j(\mathbf{x}) u_j(\mathbf{x}) \, \mathrm{d}S(\mathbf{x}) \\ &= \frac{1}{2} \int_{S_0} \sigma^*_{ij} n_i e^*_{kj} x_k \, \mathrm{d}S(\mathbf{x}) \\ &= \frac{1}{2} \sigma^*_{ij} e^*_{kj} \int_{S_0} x_k n_i \, \mathrm{d}S(\mathbf{x}) \end{aligned} \tag{3.26}$$

and using Gauss's theorem

$$
\begin{aligned}
W_{12} &= \frac{1}{2}\sigma^*_{ij}e^*_{kj}\int_{V_0} x_{k,i}\,\mathrm{d}V(\mathbf{x}) \\
&= \frac{1}{2}\sigma^*_{ij}e^*_{kj}\int_{V_0} \delta_{ki}\,\mathrm{d}V(\mathbf{x}) \\
&= \frac{1}{2}\sigma^*_{ij}e^*_{ij}V_0 \qquad (3.27)
\end{aligned}
$$

In step 3, the inclusion is put inside the matrix with the traction force unchanged. No work is done in this step, i.e. $W_{23} = 0$. In step 4, the traction force T_j is gradually reduced to zero. Both the inclusion and the matrix displace over a distance of u^c_j. Since the traction force is T_j at the beginning of step 4 and 0 at the end of step 4, the average traction force is, again, $T_j/2$. The work done to the entire system (inclusion + matrix) is

$$
\begin{aligned}
W_{34} &= \frac{1}{2}\int_{S_0} T_j(\mathbf{x})u^c_j(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \\
&= -\frac{1}{2}\int_{S_0} \sigma^*_{ij}n_iu^c_j(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \\
&= -\frac{1}{2}\sigma^*_{ij}\int_{V_0} u^c_{j,i}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \\
&= -\frac{1}{2}\sigma^*_{ij}\int_{V_0} e^c_{ij}\,\mathrm{d}S(\mathbf{x}) \\
&= -\frac{1}{2}\sigma^*_{ij}e^c_{ij}V_0 \qquad (3.28)
\end{aligned}
$$

The total elastic energy at the end of step 4 is,

$$
\begin{aligned}
E &= E_1 + W_{12} + W_{23} + W_{34} \\
&= 0 + \frac{1}{2}\sigma^*_{ij}e^*_{ij}V_0 + 0 - \frac{1}{2}\sigma^c_{ij}e^*_{ij}V_0 \\
&= -\frac{1}{2}(\sigma^c_{ij} - \sigma^*_{ij})e^*_{ij}V_0 \\
&= -\frac{1}{2}\sigma^I_{ij}e^*_{ij}V_0 \qquad (3.29)
\end{aligned}
$$

which is exactly the same as Eq. (3.19).

The same approach can also be applied to obtain the elastic energy stored inside the inclusion (E^I) or inside the matrix (E^M). In step 4, the matrix also exerts force on the inclusion, which does work as the interface S_0 moves. This leads to a transfer of elastic energy from the inclusion to the matrix.

3.3 Inclusion energy in a finite solid

Let us now consider a problem with an inclusion in a finite solid. Again, the stress-strain fields in this case can be solved by superpositions. Suppose the finite solid assumes the stress-strain fields of an infinite solid containing an inclusion, as solved previously. Then we must apply a set of traction forces $\tilde{T}_j$ to the outer surface $S_{\rm ext}$ of the solid to maintain equilibrium. To obtain the solution of a finite solid with zero traction on its outer surface, we need to remove $\tilde{T}_j$ on $S_{\rm ext}$. This is equivalent to applying a cancelling traction force $\tilde{F}_j = -\tilde{T}_j$ on the outer surface $S_{\rm ext}$ of the finite solid. Let $e_{ij}^{\rm im}$, $\sigma_{ij}^{\rm im}$ and $u_i^{\rm im}$ be the strain, stress and displacement fields in response to the surface traction $\tilde{F}_j$. They are called *image* fields. In this case, the elastic fields inside the matrix and the inclusion are,

matrix	inclusion
$e_{ij}^M = e_{ij}^{\rm c} + e_{ij}^{\rm im}$	$e_{ij}^I = e_{ij}^{\rm c} - e_{ij}^* + e_{ij}^{\rm im}$
$\sigma_{ij}^M = \sigma_{ij}^{\rm c} + \sigma_{ij}^{\rm im}$	$\sigma_{ij}^I = \sigma_{ij}^{\rm c} - \sigma_{ij}^* + \sigma_{ij}^{\rm im}$
$u_i^M = u_i^{\rm c} + u_i^{\rm im}$	$u_i^I = u_i^{\rm c} - e_{ij}^* x_j + u_i^{\rm im}$

Obviously, the image fields satisfy the condition,

$$e_{ij}^{\rm im}(\mathbf{x}) = \frac{1}{2}(u_{i,j}^{\rm im}(\mathbf{x}) + u_{j,i}^{\rm im}(\mathbf{x})) \tag{3.30}$$

$$\sigma_{ij}^{\rm im}(\mathbf{x}) = C_{ijkl}\, e_{kl}^{\rm im}(\mathbf{x}) \tag{3.31}$$

Notice that the image fields are generally not uniform within the inclusion. The free traction boundary condition on the outer surface $S_{\rm ext}$ can be expressed as,

$$\sigma_{ij}^M n_i^{\rm ext} = 0 \quad (\text{on } S_{\rm ext}) \tag{3.32}$$

Similar to Eq. (3.7), the total elastic energy in the solid can be expressed in terms of surface integrals,

$$E = \frac{1}{2}\int_{S_0} (\sigma_{ij}^I u_j^I - \sigma_{ij}^M u_j^M) n_i^{\rm out}\, \mathrm{d}S + \int_{S_{\rm ext}} \sigma_{ij}^M u_j^M n_i^{\rm ext}\, \mathrm{d}S \tag{3.33}$$

Because of Eq. (3.32), the second integral does not contribute. Using the traction continuity argument ($\sigma_{ij}^I n_i^{\rm out} = \sigma_{ij}^M n_i^{\rm out}$) as before, we get

$$E = \frac{1}{2}\int_{S_0} \sigma_{ij}^I (u_j^I - u_j^M) n_i^{\rm out}\, \mathrm{d}S \tag{3.34}$$

Again, using $u_j^I - u_j^M = -e_{ik}^* x_k$, we get

$$E = -\frac{1}{2}\int_{S_0} \sigma_{ij}^I e_{jk}^* x_k n_i^{\rm out}\, \mathrm{d}S \tag{3.35}$$

This is the same as Eq. (3.17) except that the stress field inside the inclusion now contains the image component. Define

$$\sigma_{ij}^{I,\infty} \equiv \sigma_{ij}^{\rm c} - \sigma_{ij}^* \tag{3.36}$$

as the stress field inside the inclusion in an infinite medium. Then

$$\sigma_{ij}^I(\mathbf{x}) = \sigma_{ij}^{I,\infty} + \sigma_{ij}^{\rm im}(\mathbf{x}) \tag{3.37}$$

Similarly, define

$$E_\infty \equiv -\frac{1}{2}\int_{S_0} \sigma_{ij}^{I,\infty} e_{jk}^* x_k n_i^{\text{out}}\, \mathrm{d}S = -\frac{1}{2}\sigma_{ij}^{I,\infty} e_{ij}^* V_0 \tag{3.38}$$

as the inclusion energy in an infinite solid. Then the inclusion energy in a finite solid is,

$$E = E_\infty - \frac{1}{2}\int_{S_0} \sigma_{ij}^{\text{im}} e_{jk}^* x_k n_i^{\text{out}}\, \mathrm{d}S \tag{3.39}$$

Converting the second integral into volume integral, we have

$$\begin{aligned}
E &= E_\infty - \frac{1}{2}\int_{V_0} (\sigma_{ij}^{\text{im}} e_{jk}^* x_k)_{,i}\, \mathrm{d}V \\
&= E_\infty - \frac{1}{2}\int_{V_0} \sigma_{ij}^{\text{im}} e_{ij}^*\, \mathrm{d}V \\
&= E_\infty - \frac{1}{2} e_{ij}^* \int_{V_0} \sigma_{ij}^{\text{im}}\, \mathrm{d}V \\
&= E_\infty - \frac{1}{2}\overline{\sigma_{ij}^{\text{im}}} e_{ij}^* V_0 \\
&= E_\infty + E_{\text{im}}
\end{aligned}$$

where

$$\overline{\sigma_{ij}^{\text{im}}} \equiv \frac{1}{V_0}\int_{V_0} \sigma_{ij}^{\text{im}}(\mathbf{x})\, \mathrm{d}V(\mathbf{x}) \tag{3.40}$$

is the averaged image stress inside the inclusion.

$$E_{\text{im}} \equiv -\frac{1}{2}\overline{\sigma_{ij}^{\text{im}}} e_{ij}^* V_0 \tag{3.41}$$

is the "image" contribution to the total inclusion energy. The average stress inside the inclusion is,

$$\overline{\sigma_{ij}^{I}} \equiv \sigma_{ij}^{I,\infty} + \overline{\sigma_{ij}^{\text{im}}} \tag{3.42}$$

Thus the total inclusion energy is still related to the averaged stress inside the inclusion as

$$E = -\frac{1}{2}\overline{\sigma_{ij}^{I}} e_{ij}^* V_0 \tag{3.43}$$

The results of the total inclusion energy for ellipsoidal inclusion under various boundary conditions are summarized below.

	total elastic energy	
infinite solid	$E = -\frac{1}{2}\sigma_{ij}^I e_{ij}^* V_0$	
finite solid with zero traction	$E = -\frac{1}{2}\overline{\sigma_{ij}^I} e_{ij}^* V_0$	$\overline{\sigma_{ij}^I} = \sigma_{ij}^{I,\infty} + \overline{\sigma_{ij}^{\text{im}}}$

3.4 Colonetti's theorem

We now wish to study the energy of a solid containing an inclusion subjected to applied forces at its outer surface. Before we do that, let us first prove Colonetti's theorem, which is very useful when studying such problems. Colonetti's theorem [9] states that

> *There is no cross term in the total elastic energy of a solid, between the internal stress field and the applied stress field.*

However, there is an interaction energy term between the internal and applied fields when the energy of the applied loads is included. Colonetti's theorem can greatly simplify the energy expressions when we apply stress to a finite solid containing an inclusion. To appreciate Colonetti's theorem, we need to be specific about the meaning of *internal* and *applied* stress fields. Let us start with a stress-free homogenous solid with outer surface $S_{\rm ext}$. Define *internal* stress fields as the response to a heterogeneous field of eigenstrain inside the solid with zero traction on $S_{\rm ext}$. Define *applied* stress fields as the response to a set of tractions on $S_{\rm ext}$ when there is no eigenstrain inside the solid.

Let us consider two states of stress. State 1 is purely internal, and state 2 is "applied". The total elastic energy inside the solid for these two states are,

$$\begin{aligned} E^{(1)} &= \frac{1}{2}\int_V \sigma_{ij}^{(1)} e_{ij}^{(1)}\,\mathrm{d}V \\ E^{(2)} &= \frac{1}{2}\int_V \sigma_{ij}^{(2)} e_{ij}^{(2)}\,\mathrm{d}V \end{aligned}$$

Now consider a state 1+2 which is the superposition of state 1 and 2. Its total elastic energy should be,

$$\begin{aligned} E^{(1+2)} &= \frac{1}{2}\int_V (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)})(e_{ij}^{(1)} + e_{ij}^{(2)})\,\mathrm{d}V \\ &= \frac{1}{2}\int_V (\sigma_{ij}^{(1)} e_{ij}^{(1)} + \sigma_{ij}^{(1)} e_{ij}^{(2)} + \sigma_{ij}^{(2)} e_{ij}^{(1)} + \sigma_{ij}^{(2)} e_{ij}^{(2)})\,\mathrm{d}V \\ &= E^{(1)} + E^{(2)} + E^{(1-2)} \end{aligned}$$

where

$$E^{(1-2)} \equiv \frac{1}{2}\int_V (\sigma_{ij}^{(1)} e_{ij}^{(2)} + \sigma_{ij}^{(2)} e_{ij}^{(1)})\,\mathrm{d}V \tag{3.44}$$

is the "interaction term" between state 1 and state 2. Colonetti's theorem states that $E^{(1-2)}$ must be zero, which we will prove below. First, we note that

$$\begin{aligned} \sigma_{ij}^{(2)} e_{ij}^{(1)} &= C_{ijkl} e_{kl}^{(2)} e_{ij}^{(1)} \\ \sigma_{ij}^{(1)} e_{ij}^{(2)} &= C_{ijkl} e_{kl}^{(1)} e_{ij}^{(2)} \end{aligned}$$

so that

$$\sigma_{ij}^{(2)} e_{ij}^{(1)} = \sigma_{ij}^{(1)} e_{ij}^{(2)} \tag{3.45}$$

$$E^{(1-2)} = \int_V \sigma_{ij}^{(1)} e_{ij}^{(2)}\, \mathrm{d}V = \int_V \sigma_{ij}^{(2)} e_{ij}^{(1)}\, \mathrm{d}V \tag{3.46}$$

Since there is no body force, $\sigma_{ij,i}^{(1)} = 0$. Therefore,

$$E^{(1-2)} = \int_V (\sigma_{ij}^{(1)} u_j^{(2)})_{,i}\, \mathrm{d}V \tag{3.47}$$

Now, we wish to apply Gauss's theorem to convert the volume integral into a surface integral. However, to use Gauss's theorem, the integrand must be continuous inside the entire volume V. However, this is not necessarily the case if the eigenstrain field $e_{ij}^*(\mathbf{x})$ is not sufficiently smooth. For example, in Eshelby's transformed inclusion problem, $e_{ij}^*(\mathbf{x})$ is not continuous at the inclusion surface. As a result, the internal stress field $\sigma_{ij}^{(1)}(\mathbf{x})$ is not continuous at the inclusion surface either.

However, for clarity, let us assume for the moment that the eigenstrain field $e_{ij}^*(\mathbf{x})$ and the internal stress field $\sigma_{ij}^{(1)}(\mathbf{x})$ are sufficiently smooth for the Gauss's theorem to apply. This corresponds to the case of thermal strain induced by a smooth variation of temperature inside the solid. In this case,

$$\begin{aligned} E^{(1-2)} &= \int_V (\sigma_{ij}^{(1)} u_j^{(2)})_{,i}\, \mathrm{d}V \\ &= \int_{S_{\mathrm{ext}}} n_i^{\mathrm{ext}} \sigma_{ij}^{(1)} u_j^{(2)}\, \mathrm{d}S \end{aligned} \tag{3.48}$$

Since $\sigma_{ij}^{(1)}$ is a purely internal stress state,

$$\sigma_{ij}^{(1)} n_i^{\mathrm{ext}} = 0 \quad (\text{on } S_{\mathrm{ext}}) \tag{3.49}$$

Hence

$$E^{(1-2)} = 0 \tag{3.50}$$

which is Colonetti's theorem.

Let us now consider the case where the eigenstrain field $e_{ij}^*(\mathbf{x})$ and the internal stress field $\sigma_{ij}^{(1)}(\mathbf{x})$ are piecewise smooth inside various inclusion volumes V_K as well as in the matrix $V - \sum_K V_K$. Let $n_i^{\mathrm{out},K}$ be the outward normal vector of inclusion volume V_K. We can apply Gauss's theorem in each inclusion and the matrix separately, which gives,

$$E^{(1-2)} = \int_{S_{\mathrm{ext}}} \sigma_{ij}^{(1)} u_j^{(2)} n_i^{\mathrm{ext}}\, \mathrm{d}S + \sum_K \int_{S_K} (\sigma_{ij}^{(1),K} - \sigma_{ij}^{(1)}) u_j^{(2)} n_i^{\mathrm{out},K}\, \mathrm{d}S \tag{3.51}$$

where $\sigma_{ij}^{(1),K}$ is the stress inside the volume V_k and $\sigma_{ij}^{(1)}$ is the stress in matrix. The traction force across the inclusion interface must be continuous, i.e.,

$$(\sigma_{ij}^K - \sigma_{ij}^{(1)}) n_i^{\mathrm{out},K} = 0 \quad \text{for any } K\ , \tag{3.52}$$

where the summation is not implied over K. Therefore, again we have,

$$E^{(1-2)} = \int_{S_{\rm ext}} \sigma_{ij}^{(1)} u_j^{(2)} n_i^{\rm ext} \, {\rm d}S = 0 \tag{3.53}$$

which is Colonetti's theorem.

Colonetti's theorem only deals with the elastic strain energy that is stored inside the solid (internal energy). When the system is under applied load, its evolution proceeds towards minimizing its *enthalpy*, which is the internal energy subtracting the work done by the external force. For example, the enthalpy of a system under external pressure p is $H = E + pV$. The enthalpy for the solid under study is,

$$H = E^{(1+2)} - \Delta W_{\rm LM} \tag{3.54}$$

$\Delta W_{\rm LM}$ is the work done by the loading mechanism,

$$\Delta W_{\rm LM} = \int_{S_{\rm ext}} \sigma_{ij}^{(2)} (u_j^{(1)} + u_j^{(2)}) n_i^{\rm ext} \, {\rm d}S \tag{3.55}$$

which can be written as

$$\Delta W_{\rm LM} = \Delta W_{\rm LM}^{(1-2)} + \Delta W_{\rm LM}^{(2)} \tag{3.56}$$

where

$$\begin{aligned} \Delta W_{\rm LM}^{(1-2)} &= \int_S \sigma_{ij}^{(2)} u_j^{(1)} n_i^{\rm ext} \, {\rm d}S \\ \Delta W_{\rm LM}^{(2)} &= \int_S \sigma_{ij}^{(2)} u_j^{(2)} n_i^{\rm ext} \, {\rm d}S \end{aligned}$$

$\Delta W_{\rm LM}^{(1-2)}$ can be regarded as the cross term between the two stress states in the total enthalpy.

3.5 Finite solid with applied tractions

We now apply Colonetti's theorem to our problem of an inclusion in a finite solid under a set of applied tractions. We will use superscript A to denote the fields in response to the applied tractions when the eigenstrain vanishes (no inclusion). Let superscript F denote the fields of an inclusion in a finite solid under zero external tractions (as in section 3). From Colonetti's theorem,

$$E = E^A + E^F \tag{3.57}$$

where

$$E^A = \frac{1}{2} \int_V \sigma_{ij}^A e_{ij}^A \, {\rm d}V \tag{3.58}$$

$$E^F = -\frac{1}{2} (\sigma_{ij}^{I,\infty} + \overline{\sigma_{ij}^{\rm im}}) e_{ij}^* V_0 \tag{3.59}$$

The enthalpy of the system is

$$H = E - \Delta W_{\rm LM} \tag{3.60}$$

where the A and F fields do have interaction terms in the work term $\Delta W_{\rm LM}$, i.e.,

$$\begin{aligned}
\Delta W_{\rm LM} &= \Delta W^{A}_{\rm LM} + \Delta W^{A-F}_{\rm LM} & (3.61)\\
\Delta W^{A}_{\rm LM} &= \int_{S_{\rm ext}} \sigma^A_{ij} u^A_j n^{\rm ext}_i \,{\rm d}S = \int_V \sigma^A_{ij} e^A_{ij}\,{\rm d}V = 2E^A & (3.62)\\
\Delta W^{A-F}_{\rm LM} &= \int_{S_{\rm ext}} \sigma^A_{ij} u^F_j n^{\rm ext}_i \,{\rm d}S & (3.63)
\end{aligned}$$

We would like to express $\Delta W^{A-F}_{\rm LM}$ in terms of an integral over the inclusion volume V_0. The result is,

$$\Delta W^{A-F}_{LM} = e^*_{ij}\overline{\sigma^A_{ij}}\; V_0 \tag{3.64}$$

where

$$\overline{\sigma^A_{ij}} \equiv \frac{1}{V_0}\int_{V_0} \sigma^A_{ij}\,{\rm d}V \tag{3.65}$$

To show that this is the case, first note that $\sigma_{ij}^{F} n_i^{\text{ext}} = 0$ on the surface S_{ext}. Thus

$$\Delta W_{LM}^{A-F} = \int_{S_{\text{ext}}} (\sigma_{ij}^{A} u_j^{F,M} - \sigma_{ij}^{F,M} u_j^{A})\, n_i^{\text{ext}} \, \mathrm{d}S$$

where the superscript M denotes the fields in the matrix. Consider a volume integral of the same integrand over the matrix volume $V_M = V - V_0$,

$$\begin{aligned} 0 &= \int_{V_M} (\sigma_{ij}^{A} e_{ij}^{F,M} - \sigma_{ij}^{F,M} e_{ij}^{A})\, \mathrm{d}V \\ &= \int_{S_{\text{ext}}} (\sigma_{ij}^{A} u_j^{F,M} - \sigma_{ij}^{F,M} u_j^{A}) n_i^{\text{ext}} \, \mathrm{d}S - \int_{S_0} (\sigma_{ij}^{A} u_j^{F,M} - \sigma_{ij}^{F,M} u_j^{A}) n_i^{\text{out}} \, \mathrm{d}S \end{aligned}$$

which means

$$\int_{S_{\text{ext}}} (\sigma_{ij}^{A} u_j^{F,M} - \sigma_{ij}^{F,M} u_j^{A}) n_i^{\text{ext}} \, \mathrm{d}S = \int_{S_0} (\sigma_{ij}^{A} u_j^{F,M} - \sigma_{ij}^{F,M} u_j^{A}) n_i^{\text{out}} \, \mathrm{d}S \quad (3.66)$$

Hence

$$\Delta W_{LM}^{A-F} = \int_{S_0} (\sigma_{ij}^{A} u_j^{F,M} - \sigma_{ij}^{F,M} u_j^{A}) n_i^{\text{out}} \, \mathrm{d}S \quad (3.67)$$

Notice that the integral is on the matrix side of the inclusion interface. We can similarly write out the volume integral inside the inclusion

$$\begin{aligned} 0 &= \int_{V_0} (\sigma_{ij}^{A} e_{ij}^{F,I} - \sigma_{ij}^{F,I} e_{ij}^{A})\, \mathrm{d}V \\ &= \int_{S_0} (\sigma_{ij}^{A} u_j^{F,I} - \sigma_{ij}^{F,I} u_j^{A}) n_i^{\text{out}} \, \mathrm{d}S \end{aligned}$$

which means that

$$\int_{S_0} \sigma_{ij}^{A} u_j^{F,I} n_i^{\text{out}} \, \mathrm{d}S = \int_{S_0} \sigma_{ij}^{F,I} u_j^{A} n_i^{\text{out}} \, \mathrm{d}S \quad (3.68)$$

Substituting this into Eq. (3.67) and noting the traction continuity condition $\sigma_{ij}^{F,I} n_i^{\text{out}} = \sigma_{ij}^{F,M} n_i^{\text{out}}$, we have,

$$\begin{aligned} \Delta W_{LM}^{A-F} &= \int_{S_0} \sigma_{ij}^{A} (u_j^{F,M} - u_j^{F,I}) n_i^{\text{out}} \, \mathrm{d}S \\ &= \int_{S_0} \sigma_{ij}^{A} e_{jk}^{*} x_k n_i^{\text{out}} \, \mathrm{d}S \end{aligned} \quad (3.69)$$

Using Gauss's theorem, we get,

$$\begin{aligned}\Delta W_{LM}^{A-F} &= \int_{V_0} \sigma_{ij}^A e_{ij}^* \, \mathrm{d}V \\ &= -e_{ij}^* \int_{V_0} \sigma_{ij}^A \, \mathrm{d}V \\ &= -e_{ij}^* \overline{\sigma_{ij}^A} \; V_0\end{aligned} \tag{3.70}$$

The major results of this chapter are summarized below.

	total elastic energy		total enthalpy
infinite solid	$E = -\frac{1}{2}\sigma_{ij}^I e_{ij}^* V_0$		
finite solid with zero traction	$E = -\frac{1}{2}\overline{\sigma_{ij}^I} e_{ij}^* V_0$	$\overline{\sigma_{ij}^I} = \sigma_{ij}^{I,\infty} + \overline{\sigma_{ij}^{\mathrm{im}}}$	
finite solid with traction	$E = E^A + E^F$ $E^A = \frac{1}{2}\int_V \sigma_{ij}^A e_{ij}^A \, \mathrm{d}V$ $E^F = -\frac{1}{2}\overline{\sigma_{ij}^I} e_{ij}^* V_0$	$\overline{\sigma_{ij}^I} = \sigma_{ij}^{I,\infty} + \overline{\sigma_{ij}^{\mathrm{im}}}$	$H = E - \Delta W_{\mathrm{LM}}$ $= E^A + E^F - \Delta W_{\mathrm{LM}}^A - \Delta W_{\mathrm{LM}}^{A-F}$ $= E^F - E^A - \overline{\sigma_{ij}^A} e_{ij}^* V_0$

Chapter 4

Eshelby's Inhomogeneity

4.1 Introduction

We can apply Eshelby's solution of inclusions to other problems such as inhomogeneities, cracks, and dislocations to name a few. These solutions are modeled using a technique called the equivalent inclusion method, where the eigenstrain is chosen to model the specific problem. This is possible for ellipsoidal inhomogeneities because the stress and strain inside ellipsoidal inclusions are constant.

To start, let us consider a simple example. Suppose we cut a volume V_0 out of an infinite solid and fill it with a liquid to a pressure p_0. What are the stress, strain, and displacement fields inside the matrix? In principle, we could use the Green's function as a direct method to solve this problem. Because the liquid exerts a force $T_j = p_0 \delta_{kj} n_k$ on the surface of the void, the displacement field inside the matrix should be,

$$u_i(\mathbf{x}) = \int p_0 \delta_{kj} n_k \tilde{G}_{ij}(\mathbf{x}, \mathbf{x}') \, \mathrm{d}\mathbf{x}' \tag{4.1}$$

where $\tilde{G}_{ij}(\mathbf{x}, \mathbf{x}')$ is the Green's function for an infinite body with a cavity. However, we do not know the expression for $\tilde{G}_{ij}(\mathbf{x}, \mathbf{x}')$, which is different from the Green's function of an infinite body (without the cavity). Therefore, the formal solution in Eq. (4.1) is not very helpful in practice. Thus the remaining question is, can we express the displacement field in terms of infinite medium Green's function, $G_{ij}(\mathbf{x}, \mathbf{x}') = G_{ij}(\mathbf{x} - \mathbf{x}')$? This turns out to be possible if the shape of the cavity is an ellipsoid.

The solution can be constructed using Eshelby's equivalent inclusion method. The idea is to replace the liquid with an inclusion whose eigenstrain is chosen such that the stress field inside exactly matches that in the liquid, i.e., $\sigma^I_{ij} = -p_0 \delta_{ij}$. This is possible because we know the stress and strain in both the inclusion and liquid are constant. The required eigenstrain e^*_{ij} of the equivalent inclusion can be obtained from Eshelby's tensor $\mathcal{S}_{ijkl}$. Because

$$\sigma^I_{ij} = \sigma^c_{ij} - \sigma^*_{ij} = C_{ijkl}(e^c_{kl} - e^*_{kl}) = C_{ijkl}(\mathcal{S}_{klmn} e^*_{mn} - e^*_{kl}) \tag{4.2}$$

Therefore

$$C_{ijkl}(\mathcal{S}_{klmn} - \delta_{km}\delta_{ln}) e^*_{mn} \quad = \quad -p_0 \delta_{ij} \tag{4.3}$$

From this set of six equations we can solve for the six unkown equivalent eigenstrains e^*_{ij}. Once the eigenstrain is known, the displacements on the void surface S_0 can be calculated from

$$u_i = u_i^c = \mathcal{S}_{ijkl} e^*_{kl} x_j \tag{4.4}$$

What is the elastic energy inside the matrix? It must be the same as the elastic energy inside the matrix when in contains the equivalent inclusion, instead of the liquid. The total elastic energy inside the matrix and the inclusion is,

$$E = E^I + E^M = -\frac{1}{2}\sigma^I_{ij} e^*_{ij} V_0 \tag{4.5}$$

and the energy in the inclusion is

$$E^I = \frac{1}{2}\sigma^I_{ij} e^I_{ij} V_0 = \frac{1}{2}\sigma^I_{ij}(e^c_{ij} - e^*_{ij}) V_0 \tag{4.6}$$

Therefore, the energy in the matrix is

$$\begin{aligned} E^M &= E - E^I = -\frac{1}{2}\sigma^I_{ij} e^c_{ij} V_0 \\ &= -\frac{1}{2}(-p_0 \delta_{ij}) \mathcal{S}_{ijkl} e^*_{kl} V_0 \\ &= \frac{1}{2} p_0 \mathcal{S}_{iikl} e^*_{kl} V_0 \end{aligned} \tag{4.7}$$

4.2 Transformed inhomogeneity

Let us now apply the same idea to solve the transformed inhomogeneity problem. A transformed inhomogeneity is otherwise the same as a transformed inclusion, except that it has a different elastic constant C'_{ijkl} than the matrix. Let us assume that the inhomogeneity is ellipsoidal in shape and has a volume V_0 bounded by a surface S_0. Suppose it undergoes a permanent transformation described by eigenstrain $e^{*'}_{ij}$. Our problem is to determine the stresses and strains distribution in the solid as well as its total elastic energy. Notice that we use superscript $'$ to express all properties related to the inhomogeneity.

This problem is more complicated than the liquid-in-void problem in the previous section. This is because the inhomogeneity is a solid. To replace it with an equivalent inclusion, both the traction force and the displacement field on the interface S_0 should be matched. A sufficient condition is to match both the elastic stress and the total strain field inside the transformed in homogeneity and inside the equivalent inclusion.

The stress inside the inhomogeneity is,

$$\sigma^{I'}_{ij} = \sigma^{c'}_{ij} - \sigma^{*'}_{ij} = C'_{ijkl}(e^{c'}_{kl} - e^{*'}_{kl}) \tag{4.8}$$

This should match the stress inside the equivalent inclusion,

$$\sigma^I_{ij} = \sigma^c_{ij} - \sigma^*_{ij} = C_{ijkl}(e^c_{kl} - e^*_{kl}) \tag{4.9}$$

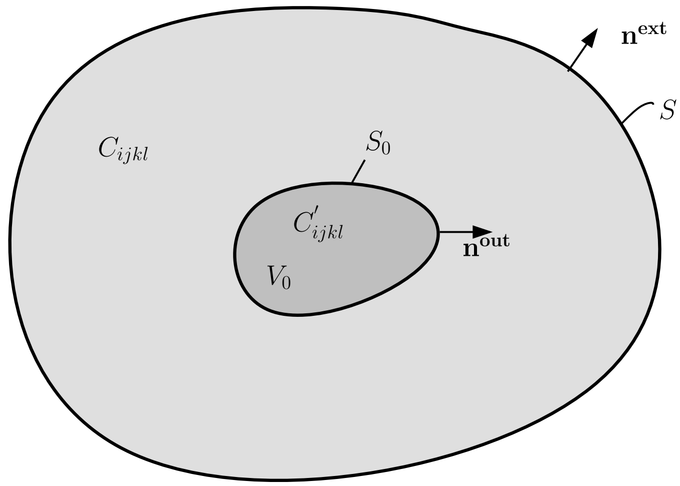


Figure 4.1: A linear elastic solid with volume V and a transformed inhomgeneity V_0, described by elastic constant C'_{ijkl} and eigenstrain $e^{*'}_{ij}$. While the problem can be defined when V_0 has a general shape, it can only be solved (elegantly) by Eshelby's equivalent inclusion method when V_0 is an ellipsoid.

The total strain inside the inhomogeneity is $e^{c'}_{ij}$, which must match the total strain side the equivalent inclusion e^c_{ij}. Therefore,

$$C'_{ijkl}(e^c_{kl} - e^{*'}_{kl}) = C_{ijkl}(e^c_{kl} - e^*_{kl}) \tag{4.10}$$

Because $e^c_{kl} = \mathcal{S}_{klmn}e^*_{mn}$, we have,

$$\left[(C'_{ijkl} - C_{ijkl})\mathcal{S}_{klmn} + C_{ijmn}\right] e^*_{mn} = C'_{ijkl}e^{*'}_{kl} \tag{4.11}$$

from which we can solve for the equivalent e^*_{mn} for the inclusion in terms of the eigenstrain $e^{*'}_{kl}$ of the transformed inhomogeneity.

The total strain inside the inhomogeneity is the same as the total strain inside the equivalent inclusion, i.e.,

$$e^{c'}_{ij} = e^c_{ij} = \mathcal{S}_{ijkl}e^*_{kl} \tag{4.12}$$

The stress inside the inhomogeneity is also the same as the stress inside the equivalent inclusion, i.e.,

$$\sigma^{I'}_{ij} = \sigma^I_{ij} = C_{ijkl}(e^c_{kl} - e^*_{kl}) = (C_{ijkl}\mathcal{S}_{klmn} - C_{ijmn})e^*_{mn} \tag{4.13}$$

The elastic energy inside the matrix is the same in both the transformed inhomogeneity problem and the equivalent inclusion problem, i.e.,

$$E^M = -\frac{1}{2}\sigma^I_{ij}e^c_{ij}V_0 \tag{4.14}$$

However, the elastic energy inside the transformed inhomogenity ($E^{I'}$) and that inside the equivalent inclusion (E^I) are not the same. Specifically,

$$E^{I'} = \frac{1}{2}\sigma^{I'}_{ij} e^{I'}_{ij} V_0 = \frac{1}{2}\sigma^{I'}_{ij}(e^{c'}_{ij} - e^{*'}_{ij})V_0 = \frac{1}{2}\sigma^{I}_{ij}(e^{c}_{ij} - e^{*'}_{ij})V_0 \tag{4.15}$$

whereas,

$$E^{I} = \frac{1}{2}\sigma^{I}_{ij} e^{I}_{ij} V_0 = \frac{1}{2}\sigma^{I}_{ij}(e^{c}_{ij} - e^{*}_{ij})V_0 \tag{4.16}$$

Thus, the total energy for the solid with a transformed inhomogeneity is,

$$E = E^{I'} + E^M = -\frac{1}{2}\sigma^{I}_{ij} e^{*'}_{ij} V_0 \tag{4.17}$$

whereas the total energy of the equivalent inclusion problem is,

$$E^{\text{eq.inc.}} = E^I + E^M = -\frac{1}{2}\sigma^{I}_{ij} e^{*}_{ij} V_0 \tag{4.18}$$

4.3 Inhomogeneity under uniform applied loads

Let us consider another important inhomogeneity problem where the inhomogeneity has no eigenstrain by itself. Instead, the solid containing the inhomogeneity is subjected to external loads. The load is uniform meaning that if the solid were homogeneous (with no inhomogeneity) the stress strain fields should be uniform throughout the solid. The question now is, What are the stress and strain fields when the solid does contain the inhomogeneity? We can solve this problem when the inhomogeneity is an ellipsoid.

Let us construct the stress strain fields inside the solid by superimposing two sets of fields. First, imagine that the solid containing the inhomogeneity is subjected to a uniform strain e^A_{ij}, which is the strain throughout the solid under the applied load if the entire solid has elastic constant C_{ijkl}. The stress field inside the matrix is $\sigma^A_{ij} = C_{ijkl}e^A_{kl}$ while the stress field inside the inhomogeneity is $\sigma^{A'}_{ij} = C'_{ijkl}e^A_{kl}$. The equilibrium condition would not be satisfied, unless a body force $T_j = (\sigma^{A'}_{ij} - \sigma^A_{ij})n_i$ is applied to the surface S_0 of the inhomogeneity.

To obtain the solution of the original problem, this body force must be removed. Thus, for the second set of elastic fields, imagine that we apply a body force $F_j = -T_j$ on the surface S_0 of the inhomogeneity. The solid is not subjected to external loads in this case. Let the stress and strain field due to F_j be $\sigma^{c'}_{ij}$ and $e^{c'}_{ij}$. Superimposing these two sets of fields, the elastic stress field inside the inhomogeneity is,

$$\sigma^{I'}_{ij} = \sigma^{A'}_{ij} + \sigma^{c'}_{ij} = C'_{ijkl}(e^A_{kl} + e^{c'}_{kl}) \tag{4.19}$$

The total strain field inside the inhomogeneity is the same as its elastic strain (since $e^{*'}_{ij} = 0$),

$$e^{I'}_{ij} = e^A_{ij} + e^{c'}_{ij} \tag{4.20}$$

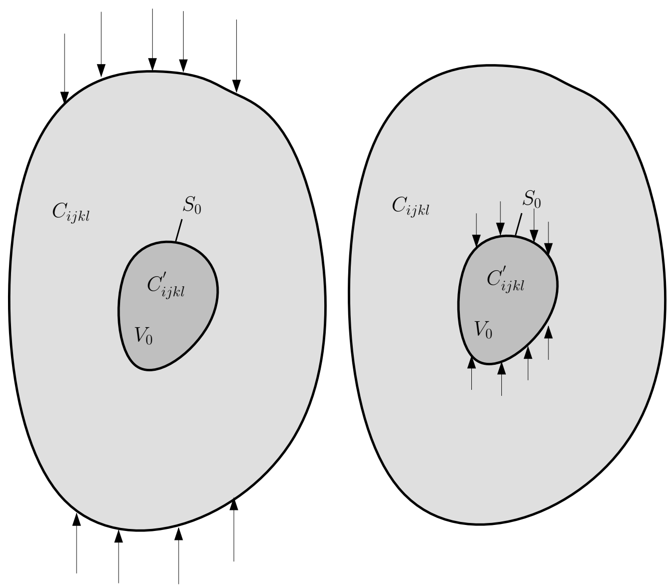


Figure 4.2: A solid containing an inhomogeneity under uniform loads. The total stress strain fields can be constructed as a superposition of two sets of fields. (a) Let the entire body have a uniform strain field e^A_{ij}. We need to apply a body force $T_j = (\sigma^{A'}_{ij} - \sigma^A_{ij})n_i$ on interior surface S_0 to maintain equilibrium. (b) Apply body force $F_j = -T_j$ on S_0 to cancel the extra body force. The resulting stress strain fields are called $\sigma^{c'}_{ij}$ and $e^{c'}_{ij}$. Notice that this problem has a simple solution only when the inhomogeneity is an ellipsoid.

At the same time, we can construct the stress strain fields of an equivalent inclusion with eigenstrain e^*_{ij} in a solid under a uniform applied load. The elastic stress field inside the inclusion is,

$$\sigma^I_{ij} = \sigma^A_{ij} + \sigma^c_{ij} - \sigma^*_{ij} = C_{ijkl}(e^A_{kl} + e^c_{kl} - e^*_{kl}) \tag{4.21}$$

The total strain field inside the inclusion is,

$$e^A_{ij} + e^c_{ij} \tag{4.22}$$

Similar to the problem in the previous section, both the elastic stress and the total strain have to match between the inhomogeneity and the inclusion problems. Therefore,

$$\begin{aligned} C'_{ijkl}(e^A_{kl} + e^{c'}_{kl}) &= C_{ijkl}(e^A_{kl} + e^c_{kl} - e^*_{kl}) && (4.23)\\ e^A_{ij} + e^{c'}_{ij} &= e^A_{ij} + e^c_{ij} && (4.24) \end{aligned}$$

Eq. (4.24) simply leads to $e^{c'}_{ij} = e^c_{ij}$. Plug it into Eq. (4.23), we get,

$$\begin{aligned} C'_{ijkl}(e^A_{kl} + e^c_{kl}) &= C_{ijkl}(e^A_{kl} + e^c_{kl} - e^*_{kl}) && (4.25)\\ \left[\left(C'_{ijkl} - C_{ijkl}\right)\mathcal{S}_{klmn} + C_{ijmn}\right] e^*_{mn} &= (C_{ijkl} - C'_{ijkl})e^A_{kl} && (4.26) \end{aligned}$$

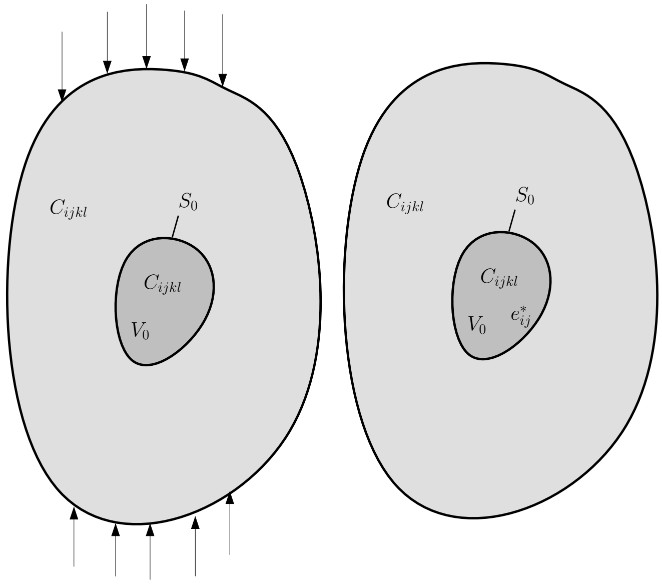


Figure 4.3: An equivalent inclusion problem that gives the same stress and total strain fields as the inhomogeneity problem in Fig.4.2. The stress strain fields can be constructed as superpositions of two sets of fields: (a) A homogeneous solid (zero eigenstrain) under uniform strain e^A_{ij}. (b) A solid containing an inclusion with eigenstrain e^*_{ij} and zero applied load.

From this we can solve for the equivalent eigenstrain e^*_{mn}. Notice that e^*_{mn} is proportional to the difference in the elastic constants $C_{ijkl} - C'_{ijkl}$ and the applied field e^A_{kl}, as it should. Once the equivalent eigenstrain is known, the stress and strain fields can be easily obtained.

Now, let us determine the total elastic energy and enthalpy of the inhomogeneity problem. To compute total elastic energy, we measure the work done during a reversible path that creates the final configuration. Let system 1 be the solid with inhomogeneity under uniform strain e^A_{ij}, as shown in Fig. 4.2(a). The elastic energy of this state is,

$$E_1 = \frac{1}{2}\sigma^A_{ij}e^A_{ij}V_M + \frac{1}{2}\sigma^{A'}_{ij}e^A_{ij}V_0 = \frac{1}{2}\sigma^A_{ij}e^A_{ij}V + \frac{1}{2}(\sigma^{A'}_{ij} - \sigma^A_{ij})e^A_{ij}V_0 \tag{4.27}$$

where V_M is the volume of the matrix, V_0 is the volume of the inhomogeneity, and V is the total volume of the solid. In system 1, a body force $T_j = (\sigma^{A'}_{ij} - \sigma^A_{ij})n_i$ is applied on S_0 to maintain equilibrium. We then gradually remove this body force and go to system 2, whose energy E_2 is the desired solution. Let ΔW_{12} be the work done to the solid during this transformation, then $E = E_2 = E_1 + \Delta W_{12}$. Notice that during this transformation, both the internal force on S_0 and the external force on S_{ext} do work. Let these two work contributions be $\Delta W^{\text{int}}_{12}$ and $\Delta W^{\text{ext}}_{12}$ respectively.

Let us first compute $\Delta W^{\text{int}}_{12}$. During the transformation from E_1 to E_2, the body force on S_0 decreases from T_j to 0, so that the average body force is $T_j/2$. The additional displacement on S_0 is $u_j^{c'}$. Thus,

$$\begin{aligned}
\Delta W_{12}^{\text{int}} &= \frac{1}{2}\int_{S_0} T_j u_j^{c'}\,\mathrm{d}S \\
&= \frac{1}{2}\int_{S_0} (\sigma_{ij}^{A'} - \sigma_{ij}^{A}) n_i u_j^{c'}\,\mathrm{d}S \\
&= \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^{A}) \int_{S_0} n_i u_j^{c'}\,\mathrm{d}S \\
&= \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^{A}) \int_{V_0} e_{ij}^{c'}\,\mathrm{d}V \\
&= \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^{A}) e_{ij}^{c} V_0 \qquad (4.28)
\end{aligned}$$

Because the applied load does not change, the factor of $\frac{1}{2}$ does not appear in $\Delta W_{12}^{\text{ext}}$. Let $T_j^A = \sigma_{ij}^A n_i$ be the traction force on the outer surface S_{ext}, then

$$\Delta W_{12}^{\text{ext}} = \int_{S_{\text{ext}}} T_j^A u_j^{c'}\,\mathrm{d}S \qquad (4.29)$$

Notice that $u_j^{c'}$ is the displacement field due to body force $F_j = -T_j$ on S_0 (see Fig. 4.2(b)). By Betti's theorem,

$$\Delta W_{12}^{\text{ext}} = \int_{S_0} F_j (u_j^A + u_j^{c'})\,\mathrm{d}S \qquad (4.30)$$

where $u_j^A + u_j^{c'}$ is the displacement field due to applied force T_j^A. Thus,

$$\begin{aligned}
\Delta W_{12}^{\text{ext}} &= -\int_{S_0} (\sigma_{ij}^{A'} - \sigma_{ij}^{A}) n_i (u_j^A + u_j^{c'})\,\mathrm{d}S \\
&= -(\sigma_{ij}^{A'} - \sigma_{ij}^{A})(e_{ij}^A + e_{ij}^c) V_0 \qquad (4.31)
\end{aligned}$$

Therefore,

$$\begin{aligned}
E &= E_1 + \Delta W_{12}^{\text{int}} + \Delta W_{12}^{\text{ext}} \\
&= \frac{1}{2}\sigma_{ij}^A e_{ij}^A V + \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^{A}) e_{ij}^A V_0 + \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^{A}) e_{ij}^c V_0 - (\sigma_{ij}^{A'} - \sigma_{ij}^{A})(e_{ij}^A + e_{ij}^c) V_0 \\
&= \frac{1}{2}\sigma_{ij}^A e_{ij}^A V - \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^{A})(e_{ij}^A + e_{ij}^c) V_0 \\
&= \frac{1}{2}\sigma_{ij}^A e_{ij}^A V - \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^{A}) e_{ij}^{I'} V_0 \qquad (4.32)
\end{aligned}$$

The enthalpy of the system is obtained by subtracting off the work done by the loading mechanism from internal energy E, i.e.,

$$H = E - \Delta W_{\text{LM}} \qquad (4.33)$$

where

$$
\begin{aligned}
\Delta W_{\mathrm{LM}} &= \int_{S_{\mathrm{ext}}} T_j^A (u_j^A + u_j^{c'})\,\mathrm{d}S \\
&= \int_{S_{\mathrm{ext}}} \sigma_{ij}^A n_i^{\mathrm{ext}} (u_j^A + u_j^{c'})\,\mathrm{d}S \\
&= \sigma_{ij}^A e_{ij}^A V + \Delta W_{12}^{\mathrm{ext}} \\
&= \sigma_{ij}^A e_{ij}^A V - (\sigma_{ij}^{A'} - \sigma_{ij}^A) e_{ij}^{I'} V_0 \\
&= 2E
\end{aligned}
\qquad (4.34)
$$

Therefore,

$$
\begin{aligned}
H &= -E \\
&= -\frac{1}{2}\sigma_{ij}^A e_{ij}^A V + \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^A) e_{ij}^{I'} V_0
\end{aligned}
\qquad (4.35)
$$

The fact that for an elastic medium under applied load, $\Delta W_{\mathrm{LM}} = 2E$ and $H = -E$ is a general result that holds for any solid if it is at a state of zero stress everywhere when zero external stress is applied. For example, this result is used in [10].

We can define

$$
H_0 \equiv -\frac{1}{2}\sigma_{ij}^A e_{ij}^A V \qquad (4.36)
$$

as the enthalpy of the solid without the inhomogeneity under applied load. Then

$$
\begin{aligned}
\Delta H &= H - H_0 \\
&= \frac{1}{2}(\sigma_{ij}^{A'} - \sigma_{ij}^A) e_{ij}^{I'} V_0 \\
&= \frac{1}{2}(C'_{ijkl} - C_{ijkl}) e_{ij}^A e_{ij}^{I'} V_0
\end{aligned}
\qquad (4.37)
$$

In the limit of $\delta C_{ijkl} \equiv C'_{ijkl} - C_{ijkl}$ very small, then

$$
e_{ij}^{I'} = e_{ij}^A + \mathcal{O}(\delta C_{ijkl}) \qquad (4.38)
$$

$$
\Delta H = \frac{1}{2}\delta C_{ijkl} e_{ij}^A e_{kl}^A V_0 + \mathcal{O}(\delta C_{ijkl})^2 \qquad (4.39)
$$

Eshelby calls the expression $\Delta H = \frac{1}{2}\delta C_{ijkl} e_{ij}^A e_{kl}^A V_0$ the *Feynman-Hellman theorem*.

In the above derivation, the volume V of the solid is assumed to be large enough so that the image effects at S_{ext} are ignored. When the image effects are accounted for, the above results can be rewritten as,

$$
\Delta H = \frac{1}{2}(C'_{ijkl} - C_{ijkl}) \int_{V_0} e_{ij}^A e_{ij}^{I'}\,\mathrm{d}V \qquad (4.40)
$$

where $e_{ij}^{I'} = e_{ij}^A + e_{ij}^c + e_{ij}^{\mathrm{im}}$, and e_{ij}^{im} accounts for the image contribution. Note that the identity $H = -E$ and the *Feynman-Hellman theorem* holds independent of the boundary condition on S_{ext}.

Chapter 5

Cracks I: Energy

5.1 Ellipsoidal void

When the elastic stiffness tensor C'_{ijkl} of the inhomogeneity goes to zero, we have a void. The solution for an ellipsoidal void under uniform load is no different from that of an inhomogeneity under uniform load, except that $C'_{ijkl} = 0$ further simplifies some of the expressions. For example, the total stress inside the void (inhomogeneity) has to be zero. Therefore, the match between the stress field inside the void (inhomogeneity) and the equivalent inclusion becomes,

$$0 = C'_{ijkl}(e^A_{kl} + e^{c'}_{kl}) = C_{ijkl}(e^A_{kl} + e^c_{kl} - e^*_{kl}) \tag{5.1}$$

Hence,

$$e^A_{kl} + e^c_{kl} - e^*_{kl} = 0 \tag{5.2}$$

or

$$\begin{aligned} -e^A_{kl} &= e^I_{kl} = e^c_{kl} - e^*_{kl} && (5.3) \\ -\sigma^A_{ij} &= \sigma^I_{ij} = C_{ijkl}(e^c_{kl} - e^*_{kl}) = C_{ijkl}(\mathcal{S}_{klmn} - \delta_{km}\delta_{ln})e^*_{mn} && (5.4) \end{aligned}$$

This means that the total stress inside the equivalent inclusion (when no stress is applied) must exactly cancel the applied stress. Eq. (5.4) provides a simple relationship between the applied stress and the equivalent eigenstrain.

The total strain inside the void (inhomogeneity) is

$$e^{I'}_{ij} = e^A_{ij} + e^{c'}_{ij} = e^A_{ij} + e^c_{ij} = e^*_{ij} \tag{5.5}$$

which is simply the eigenstrain of the equivalent inclusion. This should not be surprising, because the equivalent inclusion must be under zero stress, so that its total strain must be equal to its eigenstrain. The (extra) enthalpy of the void is,

$$\Delta H = \frac{1}{2}(C'_{ijkl} - C_{ijkl})e^A_{ij}e^{I'}_{kl}V_0 = -\frac{1}{2}\sigma^A_{ij}e^*_{ij}V_0 \tag{5.6}$$

Notice that from Eq. (5.4), e^*_{ij} can be solved from applied stress σ^A_{ij}.

5.2 Penny-shaped crack

We obtain a crack when one dimension of the ellipsoidal void (a, b, or c) goes to zero. Let us consider a simple case of penny-shaped crack, which corresponds to the condition: $a = b$, $c \to 0$. The Eshelby's tensor for such geometry in isotropic elasticity has been derived [3].

$$
\begin{aligned}
\mathcal{S}_{1111} &= \mathcal{S}_{2222} = \frac{\pi(13-8\nu)}{32(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{3333} &= 1 - \frac{\pi(1-2\nu)}{4(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{1122} &= \mathcal{S}_{2211} = \frac{\pi(8\nu-1)}{32(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{1133} &= \mathcal{S}_{2233} = \frac{\pi(2\nu-1)}{8(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{3311} &= \mathcal{S}_{3322} = \frac{v}{1-v}\left(1 - \frac{\pi(4\nu+1)}{8\nu}\frac{c}{a}\right) \\
\mathcal{S}_{1212} &= \frac{\pi(7-8\nu)}{32(1-\nu)}\frac{c}{a} \\
\mathcal{S}_{3131} &= \mathcal{S}_{2323} = \frac{1}{2}\left(1 + \frac{\pi(\nu-2)}{4(1-\nu)}\frac{c}{a}\right)
\end{aligned}
$$

These expressions are valid in the limit of $c \ll a$. Let us now apply a tensile load in the direction normal to the crack surface, i.e. the only non-zero component of the applied stress is σ^A_{33}. As a first step we need to obtain the equivalent eigenstrain e^*_{ij}.

5.2.1 Equivalent eigenstrain

In isotropic elasticity, the elastic stiffness tensor does not mix shear and normal strain components. Neither does the Eshelby's tensor in this case. Therefore, even though we need to solve 6 equations given by Eq. (5.4), and we already know that all shear eigenstrain components must be zero, i.e. $e^*_{12} = e^*_{23} = e^*_{31} = 0$. We only need to solve the normal eigenstrain components e^*_{11}, e^*_{22}, e^*_{33}. Plug in the Eshelby's tensor into Eq. (5.4), we obtain the following explicit equations.

$$
\begin{aligned}
-\sigma^A_{11} &= \left[-\frac{2\mu}{1-\nu} + \frac{13\mu\pi c}{16(1-\nu)a}\right] e^*_{11} + \left[-\frac{2\mu\nu}{1-\nu} + \frac{(16\nu-1)\mu\pi c}{16(1-\nu)a}\right] e^*_{22} - \frac{(2\nu+1)\mu\pi c}{4(1-\nu)a} e^*_{33} \\
-\sigma^A_{22} &= \left[-\frac{2\mu\nu}{1-\nu} + \frac{(16\nu-1)\mu\pi c}{16(1-\nu)a}\right] e^*_{11} + \left[-\frac{2\mu}{1-\nu} + \frac{13\mu\pi c}{16(1-\nu)a}\right] e^*_{22} - \frac{(2\nu+1)\mu\pi c}{4(1-\nu)a} e^*_{33} \\
-\sigma^A_{33} &= -\frac{(1+2\nu)\mu\pi c}{4(1-\nu)a} e^*_{11} - \frac{(1+2\nu)\mu\pi c}{4(1-\nu)a} e^*_{22} - \frac{\mu\pi c}{2(1-\nu)a} e^*_{33}
\end{aligned}
$$

Notice that $\sigma^A_{11} = \sigma^A_{22} = 0$. To construct a solution that leads to finite σ^A_{33} at $c \to 0$, we need to let $e^*_{33} \to \infty$ but let $e^*_{33}c$ remain finite. Let e^*_{11} and e^*_{22} remain finite. Define

$$
e^* \equiv \lim_{c\to 0} e^*_{33} c \tag{5.7}
$$

Then

$$
\begin{aligned}
0 &= -\frac{2\mu}{1-\nu}e^*_{11} - \frac{2\mu\nu}{1-\nu}e^*_{22} - \frac{(2\nu+1)\mu\pi}{4(1-\nu)a}e^* \\
0 &= -\frac{2\mu\nu}{1-\nu}e^*_{11} - \frac{2\mu}{1-\nu}e^*_{22} - \frac{(2\nu+1)\mu\pi}{4(1-\nu)a}e^* \\
-\sigma^A_{33} &= -\frac{\mu\pi}{2(1-\nu)a}e^*
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
e^* &= \frac{2(1-\nu)a}{\mu\pi}\sigma^A_{33} && (5.8) \\
e^*_{11} = e^*_{22} &= -\frac{(1+2\nu)\pi}{8(1+\nu)a}e^* = -\frac{(1+2\nu)(1-\nu)}{4(1+\nu)\mu}\sigma^A_{33} && (5.9)
\end{aligned}
$$

Notice that $e^*_{33} = e^*/c \gg e^*_{11} = e^*_{22}$.

5.2.2 Griffith criteria

The (extra) enthalpy of the crack is

$$
\begin{aligned}
\Delta H &= -\frac{1}{2}\sigma^A_{ij}e^*_{ij}V_0 \\
&= -\frac{1}{2}\sigma^A_{33}e^*_{33}\frac{4\pi}{3}a^2c \\
&= -\frac{2\pi}{3}\sigma^A_{33}e^*a^2 \\
&= -\frac{4(1-\nu)}{3\mu}(\sigma^A_{33})^2a^3 && (5.10)
\end{aligned}
$$

The driving force for crack growth from elastic interaction is,

$$
f^{\mathrm{el}}_a = -\frac{\partial \Delta H}{\partial a} = \frac{4(1-\nu)}{\mu}(\sigma^A_{33})^2a^2 \tag{5.11}
$$

Therefore, a larger crack has a larger driving force to grow. The elastic driving force for crack growth is always positive. On the other hand, there are situations where a crack is stable (stationary) when a finite load is applied. This means there must be other driving forces that inhibit crack growth. Griffith [10] noticed that when a crack grows, new surfaces must be created, which increases the total energy. Let the surface energy (per unit area) of the solid be γ and let the surface area of the penny shaped crack be A, $A = 2\pi a^2$. Then the Gibbs free energy of the system is,

$$
\Delta G = \Delta H + A\gamma = -\frac{4(1-\nu)}{3\mu}(\sigma^A_{33})^2a^3 + 2\pi\gamma a^2 \tag{5.12}
$$

The total driving force for crack growth is,

$$
f^{\mathrm{tot}}_a = -\frac{\partial \Delta G}{\partial a} = \frac{4(1-\nu)}{\mu}(\sigma^A_{33})^2a^2 - 4\pi\gamma a \tag{5.13}
$$

At the critical condition $f_a^{\rm tot} = 0$,

$$\sigma_{33}^A = \sqrt{\frac{\pi\mu\gamma}{(1-\nu)a}} \tag{5.14}$$

This is the Griffith criteria [10] for crack growth. For a penny shaped crack with a radius a, it will grow if the applied stress exceeds the value given by Eq. (5.14). The critical condition can also be written as,

$$a = \frac{\pi\gamma\mu}{(1-\nu)(\sigma_{33}^A)^2} \tag{5.15}$$

This means that under the applied stress σ_{33}^A, cracks with radii smaller than Eq. (5.15) are stable while those with larger radius will grow even larger (eventually propagate through the solid). The critical value a is usually called the "Griffith crack length".

Similarly, if we apply a constant shear stress σ_{13}^A, at infinity, the critical stress can be found in the same way as above. The result is

$$\sigma_{13}^A = \sqrt{\frac{\pi\mu\gamma(2-\nu)}{2(1-\nu)a}} \tag{5.16}$$

5.3 Slit-like crack

Many of the theoretical and experimental works on cracks deal with the 2-dimensional (plane-strain or plane-stress) problem. A 2-dimensional crack problem in plane-strain can be solved using Eshelby's approach by letting one of the dimensions of the ellipsoid go to infinity. In the following, we will take the limit: $c \to \infty$, $b \to 0$. The result is a slit-like crack with length $2a$, as shown in Fig. 5.1.

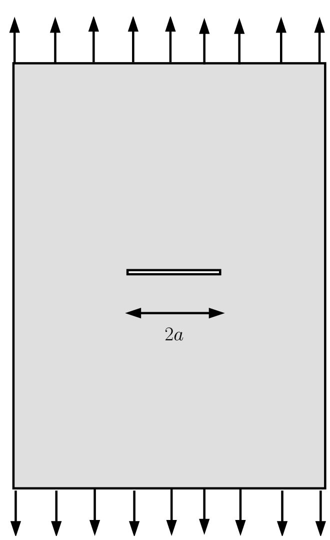


Figure 5.1: Slit like crack under uniform tension stress σ_{22}^A.

5.3.1 Equivalent eigenstrain

To solve this problem, we first need to obtain the Eshelby's tensor in this limit. Let us first take the limit of $c \to \infty$. The resulting Eshelby's tensor in isotropic elasticity is,

$$\begin{aligned}
\mathcal{S}_{1111} &= \frac{1}{2(1-\nu)}\left[\frac{b^2+2ab}{(a+b)^2}+(1-2\nu)\frac{b}{a+b}\right] \\
\mathcal{S}_{2222} &= \frac{1}{2(1-\nu)}\left[\frac{a^2+2ab}{(a+b)^2}+(1-2\nu)\frac{a}{a+b}\right] \\
\mathcal{S}_{1122} &= \frac{1}{2(1-\nu)}\left[\frac{b^2}{(a+b)^2}-(1-2\nu)\frac{b}{a+b}\right] \\
\mathcal{S}_{2211} &= \frac{1}{2(1-\nu)}\left[\frac{a^2}{(a+b)^2}-(1-2\nu)\frac{a}{a+b}\right]
\end{aligned}$$

Here we only list the "relevant" components of the Eshelby's tensor. Since we only apply σ_{22}^A, and shear and normal strain components do not couple to each other (either in the elastic stiffness tensor or in Eshelby's tensor), the shear components of eigenstrain must be zero. Since we are considering a plain strain problem, e_{33}^* is also zero. Thus, we only need to solve for e_{11}^* and e_{22}^* in terms of σ_{22}^A. Similar to the previous section, Eq. (5.4) becomes,

$$\begin{aligned}
-\sigma_{11}^A &= -\frac{(2a^2+ab)\mu}{(1-\nu)(a+b)^2}e_{11}^* - \frac{ab\mu}{(1-\nu)(a+b)^2}e_{22}^* \\
-\sigma_{22}^A &= -\frac{ab\mu}{(1-\nu)(a+b)^2}e_{11}^* - \frac{(ab+2b^2)\mu}{(1-\nu)(a+b)^2}e_{22}^*
\end{aligned}$$

Take the limit of $b \to 0$ and notice that $\sigma_{11}^A = 0$, we have,

$$\begin{aligned}
0 &= -\frac{2\mu}{1-\nu}e_{11}^* - \frac{b\mu}{(1-\nu)a}e_{22}^* \\
-\sigma_{22}^A &= -\frac{b\mu}{(1-\nu)a}e_{11}^* - \frac{b\mu}{(1-\nu)a}e_{22}^*
\end{aligned}$$

Define $e^* = \lim_{b\to 0} e_{22}^* b$, and let e_{11}^* remain finite as $b \to 0$, we have

$$\begin{aligned}
0 &= -\frac{2\mu}{1-\nu}e_{11}^* - \frac{\mu}{(1-\nu)a}e^* \\
-\sigma_{22}^A &= -\frac{\mu}{(1-\nu)a}e^*
\end{aligned}$$

Therefore,

$$\begin{aligned}
e^* &= \frac{(1-\nu)a}{\mu}\sigma_{22}^A \\
e_{11}^* &= -\frac{e^*}{2a} = -\frac{(1-\nu)}{2\mu}\sigma_{22}^A
\end{aligned}$$

Notice that $e_{22}^* = e^*/b \gg e_{11}^*$.

5.3.2 Griffith criteria

The volume of a elliptic cylinder of length c is

$$V_0 = \pi abc \tag{5.17}$$

Hence the (extra) enthalpy of the crack is

$$\begin{aligned}\Delta H &= -\frac{1}{2}\sigma^A_{ij}e^*_{ij}V_0 \\ &= -\frac{1}{2}\sigma^A_{22}e^*_{22}\pi abc \\ &= -\frac{2\pi}{3}\sigma^A_{22}e^* ac \\ &= -\frac{(1-\nu)\pi}{2\mu}(\sigma^A_{22})^2 a^2 c\end{aligned} \tag{5.18}$$

The enthalpy per unit length of crack is

$$\Delta H/c = -\frac{(1-\nu)\pi}{2\mu}(\sigma^A_{22})^2 a^2 \tag{5.19}$$

The driving force (per unit length) for the crack growth from elastic interaction is,

$$f^{\mathrm{el}}_a = -\frac{\partial \Delta H/c}{\partial a} = \frac{(1-\nu)\pi}{\mu}(\sigma^A_{22})^2 a \tag{5.20}$$

The surface area (per unit length) of a slit-like crack is $A/c = 4a$. Then the Gibbs free energy per unit length (along the crack) is,

$$\Delta G/c = \Delta H/c + A\gamma/c = -\frac{(1-\nu)\pi}{2\mu}(\sigma^A_{22})^2 a^2 + 4\gamma a \tag{5.21}$$

The total driving force (per unit length) for crack growth is,

$$f^{\mathrm{tot}}_a = -\frac{\partial \Delta G/c}{\partial a} = \frac{(1-\nu)\pi}{\mu}(\sigma^A_{22})^2 a - 4\gamma \tag{5.22}$$

At the critical condition $f^{\mathrm{tot}}_a = 0$,

$$\sigma^A_{22} = \sqrt{\frac{4\mu\gamma}{(1-\nu)\pi a}} \tag{5.23}$$

This is the Griffith criteria [10] for crack growth in plane strain.[1] This result can be easily converted to plain stress condition, which reads,

$$\sigma^A_{22} = \sqrt{\frac{4\mu(1+\nu)\gamma}{\pi a}} = \sqrt{\frac{2E\gamma}{\pi a}} \tag{5.24}$$

[1] The original Griffith paper contains a typo making it not in perfect agreement with Eq. (5.23).

where $E = 2\mu(1+\nu)$ is the Young's modulus. The conversion can be done by expressing the result in terms of the Kolosov's constant,

$$\kappa = \begin{cases} 3 - 4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases} \tag{5.25}$$

The elasticity solutions of plane strain and plane stress are the same if the result is expressed in terms of κ. For example, the critical stress expressed in terms of κ is,

$$\sigma_{22}^A = \sqrt{\frac{16\gamma\mu}{a(1+\kappa)\pi}} \tag{5.26}$$

If we apply a constant shear stress σ_{12}^A, we can show that the Griffith criteria for critical stress is (the same as in tension)

$$\sigma_{12}^A = \sqrt{\frac{16\gamma\mu}{a(1+\kappa)\pi}} \tag{5.27}$$

In plane stress, this means,

$$\sigma_{12}^A = \sqrt{\frac{2\gamma E}{a\pi}} \tag{5.28}$$

5.4 Flat ellipsoidal crack

A flat ellipsoidal crack ($a > b$, $c \to 0$) is a general situation between the two extreme cases considered above — penny-shaped and slit-shaped cracks. Studying the flat ellipsoidal crack would help us answer an important question: Would the crack tend to become more elongated (become slit-like) or less elongated (close to penny-shaped)?

Let us consider the case of simple tension: σ_{33}^A, with all other components of applied stress zero. It turns out that, similar to the penny-shaped crack case, as $c \to 0$, we need to keep

$$\lim_{c\to 0} e_{33}^* c = e^* \tag{5.29}$$

constant. The solution is (Mura 1987, p. 244)

$$e^* = \frac{(1-\nu)b}{\mu E(k)} \sigma_{33}^A \tag{5.30}$$

where $E(k)$ is the elliptic integral

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 w}\, dw \tag{5.31}$$

$$F(k) = \int_0^{\pi/2} \frac{dw}{\sqrt{1 - k^2 \sin^2 w}} \tag{5.32}$$

$$k = \sqrt{1 - b^2/a^2} \tag{5.33}$$

The extra enthalpy is,

$$\begin{aligned}\Delta H &= -\frac{1}{2}\sigma_{33}^{A} e_{33}^{*} \frac{4\pi}{3} abc \\ &= -\frac{2\pi}{3}\sigma_{33}^{A} e^{*} ab \\ &= -\frac{2\pi(1-\nu)}{3\mu}\frac{ab^2}{E(k)}(\sigma_{33}^{A})^2 \end{aligned} \tag{5.34}$$

The Gibbs free energy is,

$$\Delta G = -\frac{2\pi(1-\nu)}{3\mu}\frac{ab^2}{E(k)}(\sigma_{33}^{A})^2 + 2\pi\gamma ab \tag{5.35}$$

The Griffith critical conditions are,

$$\frac{\partial \Delta G}{\partial a} = 0 \tag{5.36}$$

$$\frac{\partial \Delta G}{\partial b} = 0 \tag{5.37}$$

If condition Eq. (5.36) is met before (at a lower σ_{33}^{A}) Eq. (5.37) is reached, the crack would extend along x_1 direction and become more elongated. Otherwise, the crack would extend along x_2 direction and become more penny-like.

Using the identities,

$$\frac{\mathrm{d}E(k)}{\mathrm{d}k} = \frac{E(k)-F(k)}{k} \tag{5.38}$$

$$\frac{\mathrm{d}F(k)}{\mathrm{d}k} = \frac{E(k)/(1-k^2)-F(k)}{k} \tag{5.39}$$

$$\frac{\mathrm{d}k}{\mathrm{d}a} = \frac{b^2}{a^3 k} \tag{5.40}$$

$$\frac{\mathrm{d}k}{\mathrm{d}b} = -\frac{b}{a^2 k} \tag{5.41}$$

we obtain,

$$\frac{\partial \Delta G}{\partial a} = -\frac{2}{3}\frac{b^2(1-\nu)\pi(\sigma_{33}^{A})^2}{\mu E^2(k)}\left[1-\frac{b^2}{a^2-b^2}\left(1-\frac{F(k)}{E(k)}\right)\right] + 2\pi\gamma b \tag{5.42}$$

$$\frac{\partial \Delta G}{\partial b} = -\frac{2}{3}\frac{ab(1-\nu)\pi(\sigma_{33}^{A})^2}{\mu E(k)}\left[2+\frac{b^2}{a^2-b^2}\left(1-\frac{F(k)}{E(k)}\right)\right] + 2\pi\gamma a \tag{5.43}$$

The two conditions gives rise to the following critical stress expressions.

$$\begin{aligned}\sigma_{33}^{A,a} &= \sqrt{\frac{3\mu\gamma k^2 E^2(k)}{b(1-\nu)\left[(-1+2k^2)E(k)+(1-k^2)F(k)\right]}} \\ \sigma_{33}^{A,b} &= \sqrt{\frac{3\mu\gamma k^2 E^2(k)}{b(1-\nu)\left[(1+k^2)E(k)-(1-k^2)F(k)\right]}} \end{aligned} \tag{5.44}$$

The crack would grow if the applied stress reaches the lower one of the two. It can be shown that for $a > b$ ($k > 0$), $\sigma_{33}^{A,b} < \sigma_{33}^{A,a}$ (Mura 1987, p.245). This means that the crack would always grow in the x_2 direction until it becomes penny shaped. Applying more stress components, e.g. σ_{31}^{A} together with σ_{33}^{A} could change the situation.

Chapter 6

Cracks II: Driving force

6.1 Crack Opening Displacement

We now consider the elastic fields – displacement, strain and stress – of a slit like crack. Under a tensile loading stress σ_{22}^A, the slit like crack will open up. Let $d(x)$ be defined as the distance between the crack faces as a function of x. In a purely elastic model, $d(\pm a) = 0$, i.e. the crack tip opening displacement is zero. We can obtain the displacements along the crack face by considering the equivalent inclusion

$$u_j(\mathbf{x}) = e_{ij}^* x_j$$

The displacement in the x direction is zero, and the displacement in the y direction on the crack face is

$$u_2 = e_{22}^* y$$

The equivalent inclusion is an ellipse with semi-axes a and b (with $b \to 0$). Thus on the crack surface, x and y are related by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

so the displacement field on the upper surface of the crack at $x \in [-a, a]$ is

$$u_2(x) = (e_{22}^* b)\frac{y}{b} = e^* \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

Therefore

$$\begin{aligned} u_2(x) &= \frac{\sigma_{22}^A a(1-\nu)}{\mu}\sqrt{1 - \left(\frac{x}{a}\right)^2} \\ &= \frac{\sigma_{22}^A (1-\nu)}{\mu}\sqrt{a^2 - x^2} \end{aligned}$$

Thus, the crack opening displacement in plane strain is

$$d(x) = 2\frac{\sigma_{22}^A (1-\nu)}{\mu}\sqrt{a^2 - x^2} \tag{6.1}$$

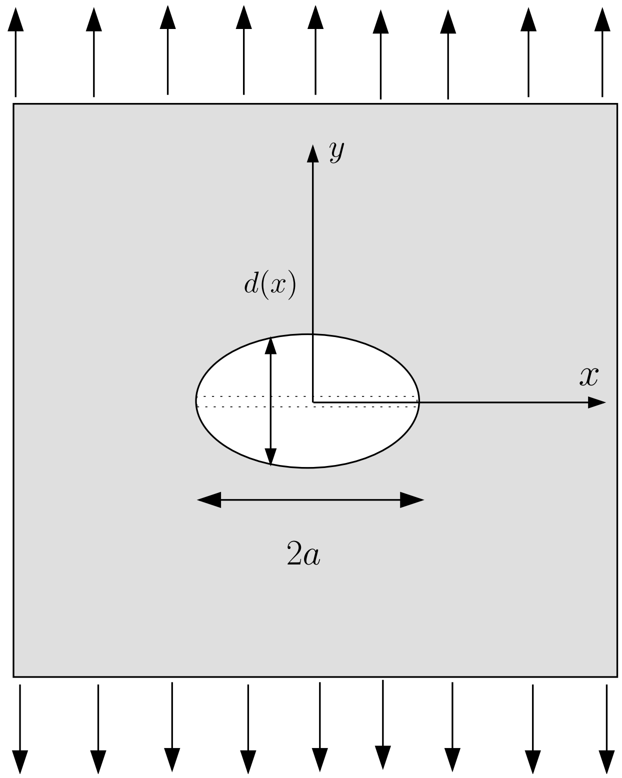


Figure 6.1: Opening displacement $d(x)$ of a slit like crack.

The crack opening displacement in plane stress is

$$d(x) = 2\frac{\sigma_{22}^A}{\mu(1+\nu)}\sqrt{a^2 - x^2} \tag{6.2}$$

With the expression for $d(x)$, we can calculate the enthalpy of the crack by measuring the work done while opening up the crack, i.e. (in plane stress),

$$\begin{aligned}
\frac{\Delta H}{c} &= -\frac{1}{2}\int_{-a}^{a} d(x)\sigma_{22}^A \,\mathrm{d}x \\
&= -\frac{1}{2}\sigma_{22}^A 2\frac{\sigma_{22}^A(1-\nu)}{\mu}\int_{-a}^{a}\sqrt{a^2 - x^2}\,\mathrm{d}x \\
&= \frac{1-\nu}{2\mu}(\sigma_{22}^A)^2\pi a^2
\end{aligned}$$

which is exactly the enthalpy calculated previously.

6.2 Stress Intensity Factors

We now consider the stress field in front of the crack tip. We will determine the nature of the stress field singularity in front of the crack tip. Let r be the distance to the crack tip.

We will show that as $r \to 0$, the stress field diverges as $\sigma(r) \to 1/\sqrt{r}$. To measure the "intensity" of this singularity, the *stress intensity factor* is defined to be,

$$K_I = \lim_{r \to 0} \sigma(r)\sqrt{2\pi r} \tag{6.3}$$

The subscript I denotes the mode of the crack. There are three crack opening modes as shown in Fig. 6.2: tensile (mode I), in-plane shear (mode II), and out-of-plane shear (mode III).

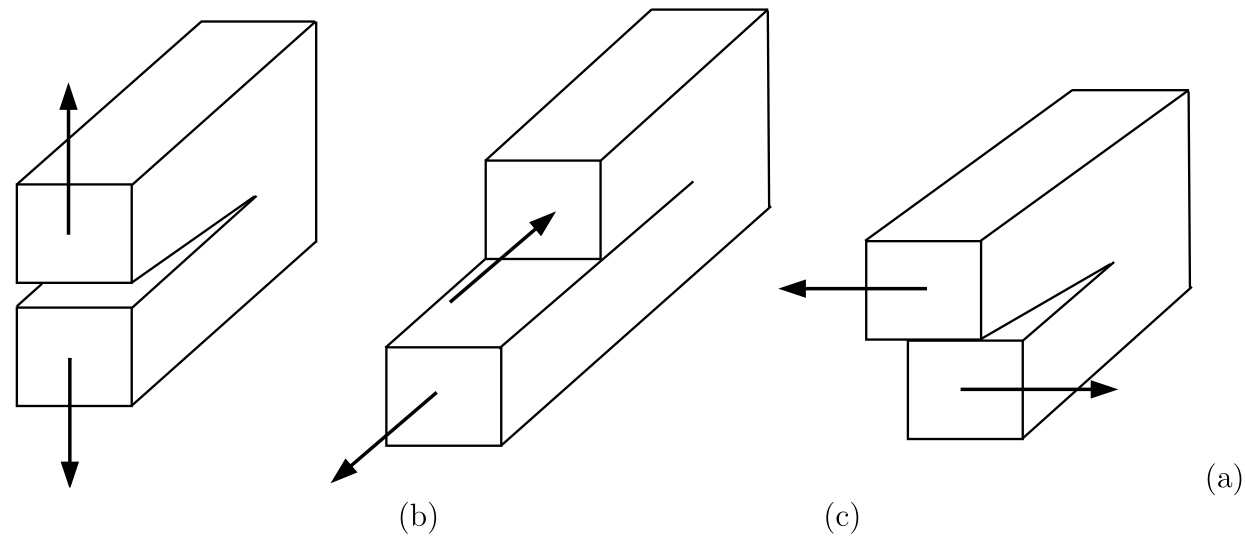


Figure 6.2: Crack opening modes: (a) mode I – tension, (b) mode II – in-plane shear, and (c) mode III – out-of-plane shear.

In order to determine the stress intensity factors of a slit like crack under tension, the stress field around the crack must be evaluated. This can be done by the Eshelby's tensor outside the equivalent inclusion. Previously we have introduced the auxiliary tensor $\mathcal{D}_{ijkl}$ to relate the constrained displacements inside the inclusion to the eigenstrain. For ellipsoidal inclusion, $\mathcal{D}_{ijkl}$ is a constant. Similarly, we can can define D^{∞}_{ijkl} as the tensor to relate the constrained displacements *outside* the inclusion to the eigenstrain inside the ellipsoid. D^{∞}_{ijkl} is no longer a constant but is a function of $\mathbf{x}$. Similarly, we can define a new Eshelby's tensor to relate the constrained strain outside the inclusion to the eigenstrain, $\mathcal{S}^{\infty}_{ijkl}$. The auxillary tensor $\mathcal{D}^{\infty}_{ijkl}$ for a two dimensional elliptical inclusion (elliptic cylinder) is

$$\mathcal{D}^{\infty}_{ijkl} = -\frac{ab}{2\pi}\int_0^{2\pi} (zz)^{-1}_{ij} z_k z_l \kappa(\gamma)\,\mathrm{d}\theta \tag{6.4}$$

where

$$\kappa(\gamma) = \frac{1}{\beta^2}\left(1 - \frac{|\gamma|}{\sqrt{\gamma^2 - \beta^2}}\right) \tag{6.5}$$

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = (k_1 a, k_2 b) \tag{6.6}$$

$$\beta = \lambda/k = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \tag{6.7}$$

$$\gamma = \mathbf{k}\cdot\mathbf{x}/k \tag{6.8}$$

In the following, we derive the stress field on the crack plane, i.e. $\mathbf{x} = (x, 0)$, $x > a$. In this case ($y = 0$), $\kappa(\gamma)$ can be written as

$$\kappa(\gamma) = \frac{1}{a^2\cos^2\theta + b^2\sin^2\theta}\left(1 - \frac{|x\cos\theta|}{(x^2-a^2)\cos^2\theta - b^2\sin^2\theta}\right) \tag{6.9}$$

In isotropic elasticity $(zz)^{-1}_{ij}$ is known analytically,

$$(zz)^{-1}_{ij} = \frac{1}{\mu}\left(\delta_{ij} - \frac{1}{2(1-\nu)}z_i z_j\right)$$

thus

$$(zz)^{-1}_{ij} z_k z_l = \frac{1}{\mu}\left(\delta_{ij} z_k z_l - \frac{1}{2(1-\nu)}z_i z_j z_k z_l\right)$$

$\mathcal{D}^{\infty}_{ijkl}$ can be written in terms of a second order and fourth order tensors H_{kl} and J_{ijkl}

$$\mathcal{D}^{\infty}_{ijkl} = \frac{1}{\mu}\left(\delta_{ij} H_{kl} - \frac{1}{2(1-\nu)}J_{ijkl}\right)\,\mathrm{d}\theta \tag{6.10}$$

where

$$\begin{aligned}
H_{kl} &= \int_0^{2\pi} \frac{-ab}{2\pi}\frac{1}{\beta^2} z_k z_l \left(1 - \frac{|\gamma|}{\sqrt{\gamma^2-\beta^2}}\right)\,\mathrm{d}\theta \\
J_{ijkl} &= \int_0^{2\pi} \frac{-ab}{2\pi}\frac{1}{\beta^2} z_i z_j z_k z_l \left(1 - \frac{|\gamma|}{\sqrt{\gamma^2-\beta^2}}\right)\,\mathrm{d}\theta
\end{aligned}$$

All of the components of the above tensors can be written in terms of a few integrals. Define the integrals

$$\begin{aligned}
I_k &= \int_0^{2\pi} \frac{\cos^{2k}\theta}{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta \\
J_k &= \int_0^{2\pi} \frac{\cos^{2k}\theta}{a^2\cos^2\theta + b^2\sin^2\theta}\frac{1}{\sqrt{p^2 - b^2\tan^2\theta}}\,\mathrm{d}\theta \\
L_k &= -\frac{ab}{2\pi}\left[I_k - |x| J_k\right]
\end{aligned}$$

where

$$p = \sqrt{x^2 - a^2}$$

Then

$$\begin{aligned}
J_{1111} &= L_2 \\
J_{2222} &= L_0 - 2L_1 + L_2 \\
J_{1122} &= J_{1212} = J_{1221} = J_{2112} = L_1 - L_2 \\
H_{11} &= L_1 \\
H_{22} &= L_0 - L_1
\end{aligned}$$

all other terms are zero. Evaluating the integrals gives

$$
\begin{aligned}
I_0 &= \frac{2\pi}{ab} \\
I_1 &= \frac{2\pi}{a^2} - \frac{2\pi b}{a^3} + \mathcal{O}(b^2) \\
I_2 &= \frac{\pi}{a^2} + \mathcal{O}(b^2) \\
J_0 &= \frac{2\pi}{ab|x|} + +\mathcal{O}(b^2) \\
J_1 &= \frac{2\pi}{a^2 p} - \frac{2\pi b}{a^3|x|} + +\mathcal{O}(b^2) \\
J_2 &= \frac{\pi}{a^2 p} + \mathcal{O}(b^2) \\
L_0 &= 0 + \mathcal{O}(b^2) \\
L_1 &= -\frac{b}{a}\left(1 - \frac{|x|}{p}\right) + \mathcal{O}(b^2) \\
L_2 &= -\frac{b}{2a}\left(1 - \frac{|x|}{p}\right) + \mathcal{O}(b^2)
\end{aligned}
$$

The Eshelby's tensor inside the matrix is

$$
\mathcal{S}^\infty_{ijkl} = -\lambda \mathcal{D}^\infty_{ikkj}\delta_{mn} - \frac{\mu}{2}\left(\mathcal{D}^\infty_{inmj} + \mathcal{D}^\infty_{jnmi} + \mathcal{D}^\infty_{jmni} + \mathcal{D}^\infty_{imnj}\right)
$$

and the non-zero terms of the Eshelby's tensor are

$$
\begin{aligned}
\mathcal{S}^\infty_{1111} &= \frac{3-2\nu}{1-\nu}\frac{\Delta}{2} \\
\mathcal{S}^\infty_{2222} &= \mathcal{S}^\infty_{1122} = -\frac{1-2\nu}{1-\nu}\frac{\Delta}{2} \\
\mathcal{S}^\infty_{2211} &= -\frac{1+2\nu}{1-\nu}\frac{\Delta}{2} \\
\mathcal{S}^\infty_{1212} &= -\frac{1}{1-\nu}\frac{\Delta}{2}
\end{aligned}
$$

where

$$
\Delta = \frac{b}{a}\left(1 - \frac{|x|}{\sqrt{x^2 - a^2}}\right)
$$

From previous analysis of a slit like crack under uniform tension using the equivalent eigenstrain method the eigenstrain was determined to be

$$
\begin{pmatrix} e^*_{11} \\ e^*_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1-\nu}{2\mu} \\ \frac{1-\nu}{\mu}\frac{a}{b} \end{pmatrix} \sigma^A_{22}
$$

The constrained stresses can now be written using the stiffness tensor and Eshelby's tensor

$$
\begin{aligned}
\begin{pmatrix} \sigma^c_{11} \\ \sigma^c_{22} \end{pmatrix} &= C\mathcal{S} \begin{pmatrix} e^*_{11} \\ e^*_{22} \end{pmatrix} \\
&= \begin{bmatrix} C_{1111} & C_{1122} \\ C_{2211} & C_{2222} \end{bmatrix} \begin{bmatrix} \mathcal{S}_{1111} & \mathcal{S}_{1122} \\ \mathcal{S}_{2211} & \mathcal{S}_{2222} \end{bmatrix} \begin{pmatrix} e^*_{11} \\ e^*_{22} \end{pmatrix} \\
&= \begin{bmatrix} \lambda + 2\mu & \lambda \\ \lambda & \lambda + 2\mu \end{bmatrix} \begin{bmatrix} \frac{3-2\nu}{1-\nu}\frac{\Delta}{2} & -\frac{1-2\nu}{1-\nu}\frac{\Delta}{2} \\ -\frac{1+2\nu}{1-\nu}\frac{\Delta}{2} & -\frac{1-2\nu}{1-\nu}\frac{\Delta}{2} \end{bmatrix} \begin{pmatrix} e^*_{11} \\ e^*_{22} \end{pmatrix} \\
&= -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(1 - \frac{|x|}{\sqrt{x^2 - a^2}}\right) \sigma^A_{22}
\end{aligned}
$$

The total stress is

$$
\begin{aligned}
\begin{pmatrix} \sigma^{\rm tot}_{11} \\ \sigma^{\rm tot}_{22} \end{pmatrix} &= \begin{pmatrix} \sigma^c_{11} \\ \sigma^c_{22} + \sigma^A_{22} \end{pmatrix} \\
&= \begin{pmatrix} -1 + \frac{|x|}{\sqrt{x^2-a^2}} \\ \frac{|x|}{\sqrt{x^2-a^2}} \end{pmatrix}
\end{aligned}
$$

Now define the variable $r \equiv x - a$ as the distance from the crack tip. Then the leading term of total stress in the limit of $r \to 0$ is

$$
\lim_{r \to 0} \begin{pmatrix} \sigma^{\rm tot}_{11} \\ \sigma^{\rm tot}_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sqrt{\frac{a}{2r}}\, \sigma^A_{22} \tag{6.11}
$$

Thus, the stress intensity factor, K_I is

$$
K_I = \lim_{r \to 0} \sigma(r)\sqrt{2\pi r} = \sqrt{\pi a}\, \sigma^A_{22} \tag{6.12}
$$

6.3 Another derivation of crack extension force

Using the crack opening displacement $d(x)$ and the stress field $\sigma^{\rm tot}_{22}(x)$ in front of the crack tip, we can recompute the driving force for crack extension using yet another method. Consider the two dimensional crack under uniform tension σ^A_{22}, as shown in Fig. 6.3. Imagine that the crack half-size extends from a to $a + \delta a$. Initially we apply additional traction forces $T^{\pm}_j$ on the lower and upper surfaces of the crack in the region of $[a, a + \delta a]$ and $[-a - \delta a, -a]$ so that the shape of the crack remains the same as before. We then slowly remove the traction forces so that in the end we have a crack with half-size $a + \delta a$. The work done by the traction forces is the change of system enthalpy, i.e. $\delta H = \delta W$. The thermodynamic driving force on a is $f = -(\delta H/c)/\delta a$. Notice that

$$
\begin{aligned}
T^+_j(x) &= \sigma_{j2}(x) \\
T^-_j(x) &= -\sigma_{j2}(x) \\
d(x) &= u^-_2 - u^+_2
\end{aligned} \tag{6.13}
$$

Thus

$$
\frac{\delta H}{c} = 2 \cdot \frac{1}{2} \cdot \int_a^{a+\delta a} (T^+_j u^+_j + T^-_j u^-_j)\, \mathrm{d}x \tag{6.14}
$$

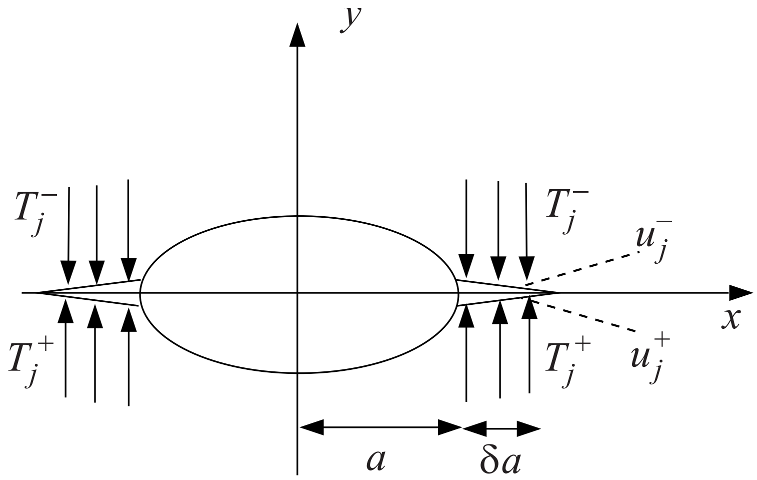


Figure 6.3: Reversibly opening up the crack by removing the traction force $T_j^{\pm}$ on the upper and lower surfaces, during which the surfaces experience a displacement of $u_j^{\pm}$.

The overall prefactor of 2 accounts for the simultaneous extension of both sides of the crack. Notice that T_j is evaluated when the crack half-size is a while $d(x)$ is evaluated when the crack half-size is $a + \delta a$. Thus

$$\begin{aligned} \frac{\delta H}{c} &= -\int_a^{a+\delta a} T_2^+(x)|_a \, d(x)|_{a+\delta a} \, \mathrm{d}x \\ &= -\int_a^{a+\delta a} \frac{x}{\sqrt{x^2-a^2}} \sigma_{22}^A \frac{2\sigma_{22}^A(a-\nu)}{\mu} \sqrt{(a+\delta a)^2 - x^2} \, \mathrm{d}x \\ &= -\frac{2(\sigma_{22}^A)^2(1-\nu)}{\mu} \int_a^{a+\delta a} \frac{\sqrt{(a+\delta a)^2 - x^2}\, x}{\sqrt{x^2-a^2}} \, \mathrm{d}x \\ &= -\frac{2(\sigma_{22}^A)^2(1-\nu)}{\mu} \frac{\delta a(2a+\delta a)}{4} \pi \end{aligned}$$

In the limit of $\delta a \ll 1$ and keeping only terms linear with δa, we have,

$$\frac{\delta H}{c} = -\frac{1-\nu}{\mu} (\sigma_{22}^A)^2 \pi a \delta a$$

and the driving force is

$$f = -\frac{\delta(\Delta H/c)}{\delta a} = \frac{1-\nu}{\mu} (\sigma_{22}^A)^2 \pi a \tag{6.15}$$

which is exactly the same as obtained before.

6.4 *J*-Integral

In 1951 Eshelby showed that an elastic singularity can be computed using the energy momentum tensor [11]. In 1968 Rice extended Eshelby's derivation to include crack driving force and called it the J-integral [12]. Because the J-integral is applicable for infinite as well as finite, homogeneous as well as inhomogeneous, linear as well as non-linear materials, it is a very powerful method for determining the crack extension force.

The J-integral in its three dimensional form states that the force on an elastic singularity in the x_i direction is

$$J_i = \int_S (w n_i - T_j u_{j,i}) \, \mathrm{d}S \tag{6.16}$$

In two dimensions, the J-integral is often written for the x direction (J_x) for a crack along x direction as

$$J = \int_\Gamma w \, \mathrm{d}y - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x} \, \mathrm{d}s \tag{6.17}$$

where Γ is a contour line going counter-clockwise from the bottom surface to the top surface of the crack. The J-integral has the following properties.

1. J_i is the driving force for the singularity along x_i direction.

2. J_i is invariant with respect to the shape of surface S or contour Γ as long as it contains the same singularity.

We will prove these properties in the following.

6.4.1 J-integral as driving force

In order to show that the J-integral is indeed the force on a crack, let us consider a finite elastic body shown in Fig. 6.4. The body is under constant load T_j^{ext} boundary condition on part of the surface S_T and constant displacement boundary condition on other part of the surface S_u. The total enthalpy of the system is

$$H = E - \int_{S_T} T_j^{\mathrm{ext}} u_j \, \mathrm{d}S$$

where

$$E = \int w \, \mathrm{d}V$$

and w is the strain energy density

$$w(e_{ij}) = \int_0^{e_{ij}} \sigma_{ij} \, \mathrm{d}e'_{ij}$$

The driving force on the singularity is

$$f_i = -\frac{\delta H}{\delta \xi_i}$$

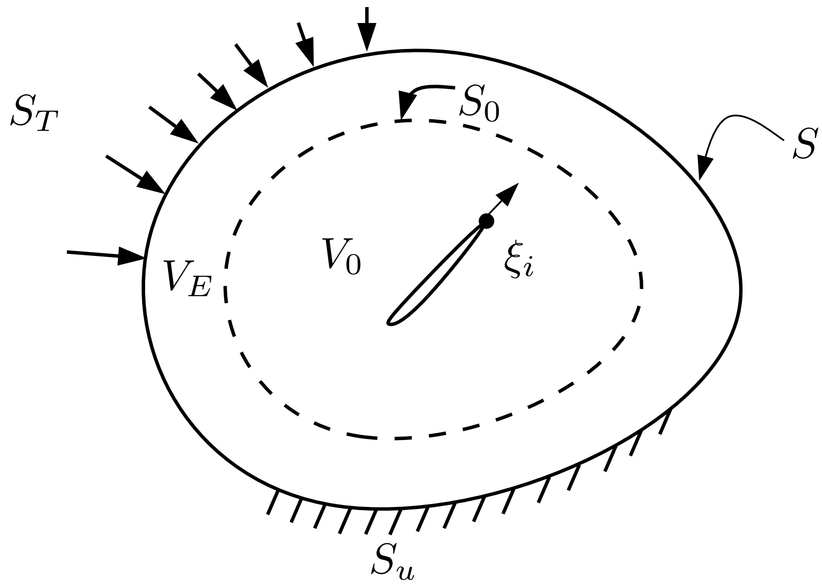


Figure 6.4: A finite solid under constant traction $T_j^{\rm ext}$ condition on S_T and constant displacement condition on S_u containing a crack tip at ξ_i. An arbitrary volume inside the solid V_0 contains the crack tip. S_0 is the surface of V_0. V_E is the volume outside V_0.

In order to determine f_i, we will first compute the change of total enthalpy δH when the crack tip moves by $\delta\xi_i$. Let δw and δu_j be the corresponding change of strain energy density field and displacement field. Then,

$$\delta H = \int_V \delta w \, \mathrm{d}V - \int_{S_T} T_j^{\rm ext} \delta u_j \, \mathrm{d}S \tag{6.18}$$

Let us now consider a sub-volume V_0 within the solid and the corresponding surface S_0. Let $V_E = V - V_0$. The change of elastic energy stored inside V_E is

$$\begin{aligned}
\int_{V_E} \delta w \, \mathrm{d}V &= \int_{V_E} \sigma_{ij} \delta e_{ij} \, \mathrm{d}V \\
&= \int_{V_E} \sigma_{ij} \delta u_{j,i} \, \mathrm{d}V \\
&= \int_{V_E} (\sigma_{ij} \delta u_j)_{,i} \, \mathrm{d}V
\end{aligned}$$

Apply Gauss's Theorem

$$\int_{V_E} \delta w \, \mathrm{d}V = \int_{S_T} T_j^{\rm ext} \delta u_j \, \mathrm{d}S - \int_{S_0} T_j \delta u_j \, \mathrm{d}S$$

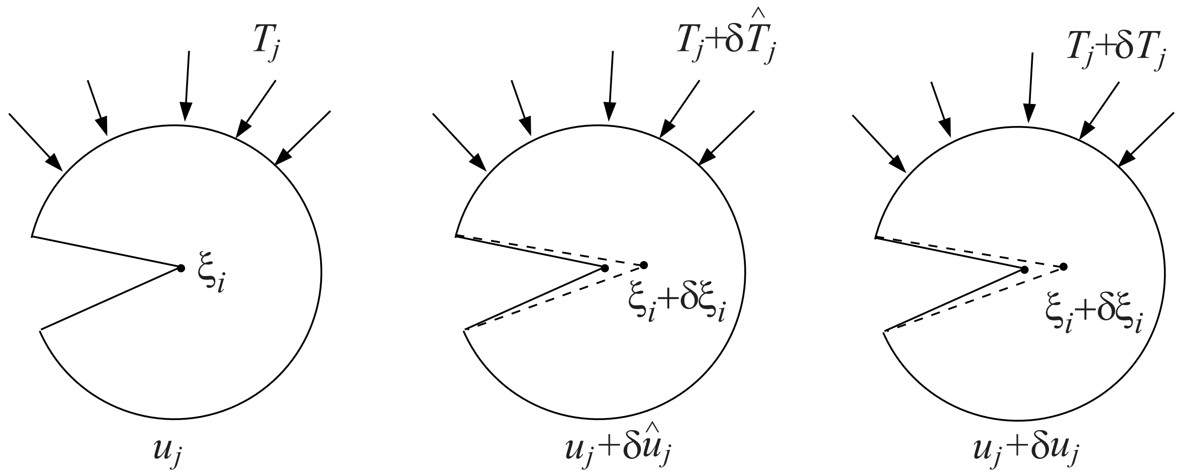


Figure 6.5: An intermediate state (middle) is introduced to facilitate the derivation of energy change as the singularity move from ξ_i (left) to $\xi_i + \delta\xi_i$ (right) (see text).

Thus, the enthalpy change for the total system becomes

$$\begin{aligned}
\delta H &= \int_{V_E} \delta w \, \mathrm{d}V + \int_{V_0} \delta w \, \mathrm{d}V - \int_{S_T} T_j^{\mathrm{ext}} \delta u_j \, \mathrm{d}S \\
&= \int_{V_0} \delta w \, \mathrm{d}V - \int_{S_T} T_j^{\mathrm{ext}} \delta u_j \, \mathrm{d}S + \int_{S_T} T_j^{\mathrm{ext}} \delta u_j \, \mathrm{d}S - \int_{S_0} T_j \delta u_j \, \mathrm{d}S \\
&= \int_{V_0} \delta w \, \mathrm{d}V - \int_{S_0} T_j \, \delta u_j \, \mathrm{d}S \qquad (6.19)
\end{aligned}$$

This means that the driving force for the crack can be computed based on the information within an arbitrary volume V_0 and its surface S_0, as long as V_0 contains the crack.

Now we wish to convert this equation into a similar form as the J-integral defined above. The key is to analyze the energy term $\delta E \equiv \int_{V_0} \delta w \, \mathrm{d}V$ in the above equation and to see how it depends on $\delta\xi_i$. Notice that before the motion of the singularity, the traction force and displacement field on S_0 are T_j and u_j respectively. After the singularity has moved to $\xi_i + \delta\xi_i$, they become $T_j + \delta T_j$ and $u_j + \delta u_j$ respectively. What we want is δE, the change of elastic energy stored V_0, caused by the singularity motion.

Because energy is a state variable, i.e. it does not depend on how the state is reached, we can derive δE by imagining that the system goes from the initial state to the final state through an intermediate state, as shown in Fig. 6.5. In the intermediate state, the singularity has moved to $\xi_i + \delta\xi_i$, but the traction force and displacement field on S_0 are $T_j + \delta\hat{T}_j$ and $u_j + \delta\hat{u}_j$, different from the final state. The intermediate state is chosen (i.e. adjusting $\delta\hat{T}_j$) such that the elastic fields inside V_0 is a simple translation of the fields in the initial state by $\delta\xi_i$, i.e. rigidly following the singularity. This means that

$$
\begin{aligned}
\delta \hat{T}_j &= -\frac{\partial T_j}{\partial x_i}\delta\xi_i \\
&= -\frac{\partial \sigma_{kj}}{\partial x_i} n_k \delta\xi_i \\
\delta \hat{u}_j &= -\frac{\partial u_j}{\partial x_i}\delta\xi_i
\end{aligned}
$$

Let the energy of the initial, intermediate and final state be E_1, E_2 and E_3. Then the energy change from initial to intermediate state is,

$$
E_2 - E_1 = \int_{V_0} -\frac{\partial w}{\partial x_i}\delta\xi_i \,\mathrm{d}V
$$

The change in energy from intermediate state to final state can be obtained by measuring the reversible work done on the surface S_0. The average tractions in this process is $T_j + \frac{1}{2}\delta T_j + \frac{1}{2}\delta\hat{T}_j$. Hence

$$
E_3 - E_2 = \int_{S_0} \left(T_j + \frac{1}{2}\delta T_j + \frac{1}{2}\delta\hat{T}_j\right)(\delta u_j - \delta\hat{u}_j)\,\mathrm{d}S \tag{6.20}
$$

Neglecting $\mathcal{O}(\delta\xi_i^2)$ terms, we have

$$
E_3 - E_2 = \int_{S_0} T_j(\delta u_j - \delta\hat{u}_j)\,\mathrm{d}S \tag{6.21}
$$

Hence,

$$
\begin{aligned}
\delta E &= E_3 - E_1 \\
&= \int_{V_0} -\frac{\partial w}{\partial x_i}\delta\xi_i\,\mathrm{d}V + \int_{S_0} T_j(\delta u_j - \delta\hat{u}_j)\,\mathrm{d}S \\
\delta H &= \int_{V_0} \delta w\,\mathrm{d}V - \int_{S_0} T_j\delta u_j\,\mathrm{d}S \\
&= -\int_{V_0}\frac{\partial w}{\partial x_i}\delta\xi_i\,\mathrm{d}V + \int_{S_0} T_j(\delta u_j - \delta\hat{u}_j)\,\mathrm{d}S - \int_{S_0} T_j\delta u_j\,\mathrm{d}S \\
&= -\int_{V_0}\frac{\partial w}{\partial x_i}\delta\xi_i\,\mathrm{d}V - \int_{S_0} T_j\delta\hat{u}_j\,\mathrm{d}S \\
&= -\int_{V_0}\frac{\partial w}{\partial x_i}\delta\xi_i\,\mathrm{d}V + \int_{S_0} T_j\frac{\partial u_j}{\partial x_i}\delta\xi_i\,\mathrm{d}S
\end{aligned}
$$

Therefore, the driving force on the singularity is

$$
\begin{aligned}
f_i = -\frac{\delta H}{\delta\xi_i} &= \int_{V_0}\frac{\partial w}{\partial x_i}\,\mathrm{d}V - \int_{S_0} T_j\frac{\partial u_j}{\partial x_i}\,\mathrm{d}S \\
&= \int_{S_0}(w_i - T_j u_{j,i})\,\mathrm{d}S \\
&= J_i
\end{aligned}
$$

6.4.2 Invariance of J-integral

Since the driving force on a singularity is unique, the J-integral must be invariant with respect to the surface S_0 on which it is evaluated, as long as S_0 always contains the same singularity. But the invariance of J-integral can also be proved more rigorously. In order to prove this, we first show that over a closed surface S_0 containing no defect, the J-integral is zero. Recall that

$$J_k = \int_{V_0} \frac{\partial w}{\partial x_k}\,\mathrm{d}V - \int_{S_0} T_j \frac{\partial u_j}{\partial x_k}\,\mathrm{d}S$$

The derivative of strain energy density is

$$\begin{aligned}
\frac{\partial w}{\partial x_k} &= \frac{\partial w}{\partial e_{ij}}\frac{\partial e_{ij}}{\partial x_k} \\
&= \sigma_{ij}\frac{\partial e_{ij}}{\partial x_k} \\
&= \sigma_{ij}\frac{\partial^2 u_j}{\partial x_k \partial x_i} \\
&= \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u_j}{\partial x_k}\right)
\end{aligned}$$

The equilibrium condition $\sigma_{ij,i} = 0$ was used in the last step. Thus the J-integral becomes

$$\begin{aligned}
J_k &= \int_{V_0} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u_j}{\partial x_k}\right)\mathrm{d}V - \int_{S_0} T_j \frac{\partial u_j}{\partial x_k}\,\mathrm{d}S \\
&= \int_{S_0} n_i\sigma_{ij}\frac{\partial u_j}{\partial x_k} - T_j \frac{\partial u_j}{\partial x_k}\,\mathrm{d}S \\
&= 0
\end{aligned}$$

Now consider two contour lines Γ_1 and Γ_2 around a crack tip in a 2-dimensional problem. As shown in Fig. 6.6, there exist a complete contour: $\Gamma = \Gamma_1 + B_+ - \Gamma_2 + B_-$ that contains no singularity, so that the J-integral evaluated on Γ is zero, i.e.,

$$J(\Gamma) = J(\Gamma_1) - J(\Gamma_2) + J(B_+) + J(B_-)$$

Noticing that

$$J(B_+) = J(B_-) = 0$$

since $\mathrm{d}y = 0$ and $\mathbf{T} = 0$ on the crack faces, we have

$$J(\Gamma_1) = J(\Gamma_2)$$

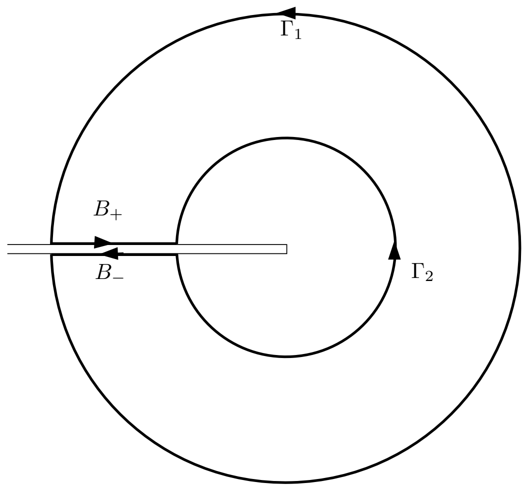


Figure 6.6: Γ_1 and Γ_2 are two different contours around the crack tip. $\Gamma = \Gamma_1 + B_+ - \Gamma_2 + B_-$ form a complete contour containing no defects.

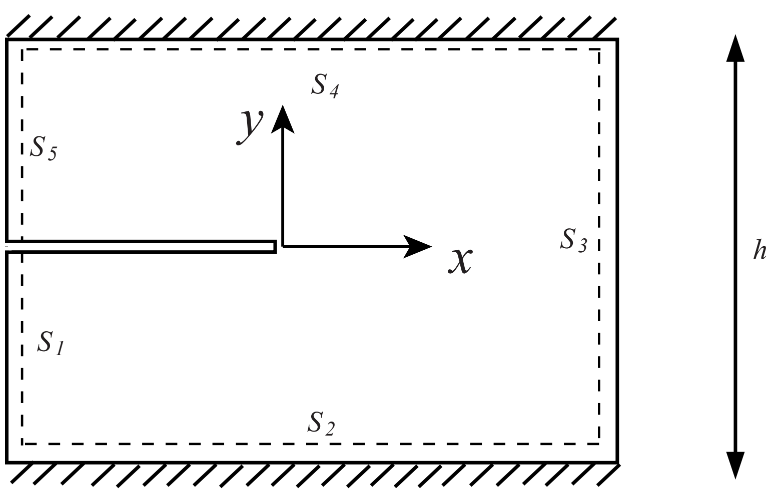


Figure 6.7: Contours Used to Evaluate The J Integral For Rice's Example Problem. A slit like crack in a long slab with fixed displacements at the top and bottom. The dashed lines S_1 ,S_2, S_3, S_4 and S_5 form the contour to evaluate the J-integral.

6.4.3 Applications of J-integral

We now apply the J-integral formula to a few examples and demonstrate how it can facilitate the calculation of crack extension driving forces.

Example 1. Let us first look at an example considered by Rice [12]. Consider the crack in a very long solid slab as shown in Fig. 6.7. The top and bottom surface are subjected to constant displacement boundary conditions and the left and right ends are subjected to zero surface traction boundary conditions. In this case, the most convenient contour goes around the out-most boundary of the solid: $\Gamma = S_1 + S_2 + S_3 + S_4 + S_5$. The 2-dimensional J integral is

$$J = \int w \, \mathrm{d}y - \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial x} \, \mathrm{d}S$$

Notice that on S_2 and S_4, $dy = 0$ and $\partial \mathbf{u}/\partial x = 0$. On S_1 and S_5, $w = 0$ and $\partial \mathbf{u}/\partial x = 0$. On S_3, $w = w_\infty$ and and $\partial \mathbf{u}/\partial x = 0$. Therefore, the total J-integral becomes,

$$J = w_\infty h$$

Example 2. Consider a contour around a two dimensional crack with a blunt tip. Since J does not depend on which contour is used, we can shrink the contour all the way to the tip of the crack such that [12]

$$J = \int_\Gamma w \, \mathrm{d}y \tag{6.22}$$

Thus, the J integral can be thought of as the average strain energy density around the crack tip.

Example 3. For the third application consider a mode-I crack with stress intensity factor K_I as shown in Fig. 6.8. We will derive the relationship between J and K_I. Because the J-integral is invariant with respect to contour shape (as long as it contains the crack tip), we choose the contour Γ to be a circle of radius r in the limit of $r \to 0$. In this limit, the leading singular field dominates the J-integral.

The stress fields around this crack can be calculated using isotropic elasticity stress functions in two dimensions. The leading singular terms are [4]

$$\begin{aligned}
\sigma_{rr} &= \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos\frac{\theta}{2} - \frac{1}{4} \cos\frac{3\theta}{2} \right) + \ldots \\
\sigma_{\theta\theta} &= \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos\frac{\theta}{2} + \frac{1}{4} \cos\frac{3\theta}{2} \right) + \ldots \\
\sigma_{r\theta} &= \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin\frac{\theta}{2} + \frac{1}{4} \sin\frac{3\theta}{2} \right) + \ldots
\end{aligned}$$

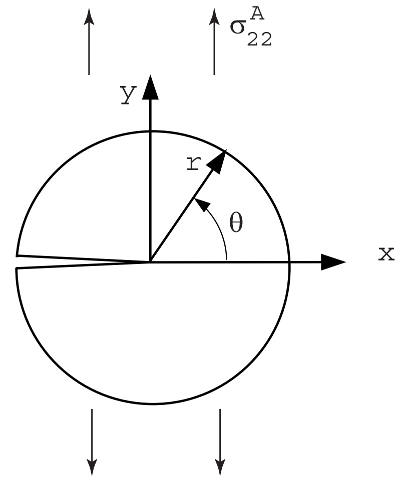


Figure 6.8: A model of a crack in 2D. The circle with radius r is the contour Γ used to evaluate the J-integral

Notice that

$$\begin{aligned} \sigma &\propto \frac{1}{\sqrt{r}} \\ e &\propto \frac{1}{\sqrt{r}} \\ w &\propto \frac{1}{r} \end{aligned}$$

Hence the strain energy of the solid should be finite. The stress intensity factor can be calculated using the leading terms of the stress. The strain energy density is

$$w = \frac{1}{2}\left(\sigma_{\theta\theta}e_{\theta\theta} + \sigma_{rr}e_{rr} + 2\sigma_{r\theta}e_{r\theta}\right) \tag{6.23}$$

and

$$\begin{aligned} T_r &= \sigma_{rr} \\ T_\theta &= \sigma_{r\theta} \end{aligned}$$

Thus

$$\int_\Gamma w\,\mathrm{d}y = \int_{-\pi}^{\pi} wr\cos\theta\,\mathrm{d}\theta = \frac{1-2\nu}{8\mu}K_I^2$$

and

$$\int \mathbf{T}\cdot\frac{\partial\mathbf{u}}{\partial x}\,\mathrm{d}S = \int_{-\pi}^{\pi} \mathbf{T}\cdot\frac{\partial\mathbf{u}}{\partial x} r\,\mathrm{d}\theta = -\frac{3-2\nu}{8\mu}K_I^2$$

Thus the J-integral is

$$J = \frac{1-\nu}{2\mu}K_I^2 \tag{6.24}$$

Recall that for a slit like double crack with half-width a, the stress intensity factor was derived previously as $K_I = \sqrt{\pi a}\sigma_{22}^A$. Therefore

$$J = \frac{1-2\nu}{2\mu}\pi a(\sigma_{22}^A)^2 \tag{6.25}$$

The enthalpy of this crack is

$$\frac{\Delta H}{c} = -\frac{1-\nu}{2\mu}\pi(\sigma_{22}^A)^2 a^2$$

and its derivative is the driving force on a

$$\begin{aligned} f_a = -\frac{\partial}{\partial a}\frac{\Delta H}{c} &= \frac{1-\nu}{\mu}\pi a(\sigma_{22}^A)^2 \\ &= 2J \end{aligned}$$

$f_a = 2J$ because when a increases by δa, both cracks move ahead (in opposite directions) by δa.

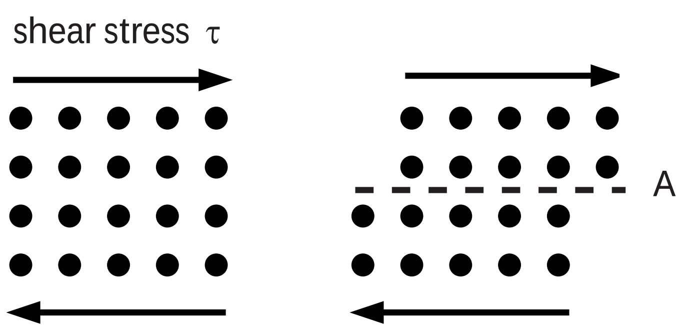
Although in the examples considered above, the materials are always linear elastic, the J-integral is also applicable to non-linear elastic materials. Because we may use a non-linear elastic material as a model for a elasto-plastic material (provided our load always increase monotonically, i.e. do not unload), J-integral has been applied to elasto-plastic material as well.

Chapter 7

Dislocations

7.1 Introduction

The idea of a dislocation was originally introduced by mathematician Volterra in 1907 [13]. In his paper, Volterra introduced several types of "dislocations" by the displacement of a cut cylinder. The types of dislocations proposed by Volterra cover the class of modern elasticity models of dislocations and disclinations. However, the importance of Volterra's dislocations in elasticity were not appreciated until 1934, when three scientists, Taylor, Orowan and Polanyi independently proposed that dislocations are responsible for crystal plasticity [14]. They postulated that these types of defects could exist in crystals and that their motion under stress (much lower than previous theoretical predictions) can explain the actual yield stress of metals. Dislocations remained a theoretical model until the 1950's, when it was first observed in experiments. The most common method of observing dislocations is Transmission Electron Microscopy (TEM) [15].



(a) (b)

Figure 7.1: (a) A perfect crystal consisting of a periodic array of atoms subject to external loading. (b) The crystal has undergone permanent shear deformation. The upper half of the crystal has slipped to the right by one lattice vector with respect to the lower half.

To see how dislocations could explain the low yield stress of metals, let us first consider the theoretical strength of a perfect crystal against plastic shear deformation [15]. Let τ be the shear stress needed to cause the spontaneous shearing of all the bonds across the plane A, such that the upper half crystal is shifted to the right by x with respect to the lower half. Because of the periodicity of the crystal structure, τ is a periodic function of x with periodicity b, see Fig. 7.1(b). The simplest model (see Section 7.7) would give us the expression,

$$\tau(x) = \frac{\mu b}{2\pi a} \sin \frac{2\pi x}{b} \tag{7.1}$$

The maximum of function $\tau(x)$ gives us the *theoretical critical shear stress*,

$$\tau_{\text{th}} = \frac{\mu b}{2\pi a} \tag{7.2}$$

This is the stress under which the crystal is unstable against spontaneous shear deformation shown in Fig. 7.1(b). Using various models of $\tau(x)$, the *theoretical critical shear stress* is found to be between $\mu/3$ and $\mu/30$, which is more than 3 orders of magnitudes higher than the experimentally measured *yield stress* in real crystals. The yield stress is the stress at which macroscopic plastic deformation is observed.

The apparent discrepancy between theory and experiments can be resolved by noticing that crystals are not perfect, as shown in Fig. 7.1(a), but contain defects such as dislocations, which can move and introduce plastic deformation at much lower stress than τ_{th}. A model of edge dislocation is shown in Fig. 7.2. Imagine that only part of the atoms above plane A has slipped with respect to those below the plane by a lattice vector b. The area over which the slip has occurred is shown in the dashed line in Fig. 7.2. The configuration is equivalent to inserting an extra half plane of atoms inside the crystal (plus the surface step on the left side of the crystal). The boundary line between the slipped and un-slipped area is a dislocation, and is represented by the $\perp$ symbol. It represents a dislocation line going perpendicular to the paper. Notice that the local bonding environment inside the crystal is close to that in a perfect crystal except near the dislocation line. If the dislocation moves to the right and travels across the entire crystal, we will end up at the same configuration as in Fig. 7.1(b). Because the dislocation can move at much lower stress than τ_{th}, this explains why the crystal has much lower yield stress than τ_{th}.

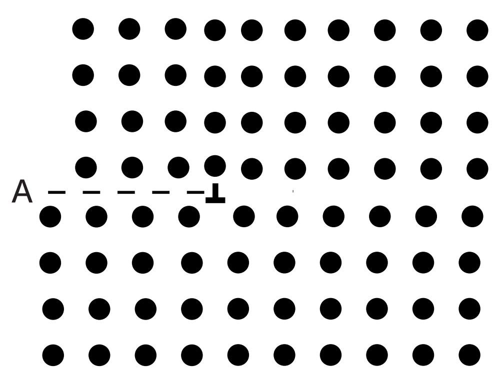


Figure 7.2: An end-on view of an edge dislocation $\perp$. It is the boundary between slipped (dashed line) and un-slipped area of plane A.

Before constructing a continuum model for dislocations, let us first introduce a few rules and terminology that will facilitate the discussion of dislocations. Consider a case where the material below a surface S has slipped with respect to the material above S by $\mathbf{b}$, as shown in Fig. 7.3. The boundary L of surface S is then a dislocation line. The slip vector $\mathbf{b}$ is related to the *Burgers vector* of the dislocation. To rigorously define the Burgers vector, we need to introduce the notion of the *Burgers circuit.* Imagine that we draw closed circuits (loops) inside the crystal before the dislocation is introduced. After introducing the dislocation, the circuit will no longer be closed if it encloses the dislocation line L. (The circuit will remain closed if it does not enclose the dislocation line L.) Choose a positive direction $\boldsymbol{\xi}$ for dislocation line L, and define the direction of the Burgers circuit with respect to $\boldsymbol{\xi}$ according to the right-hand rule. The vector that connects the starting point $\mathbf{S}$ and ending point $\mathbf{E}$ of the open Burgers circuit is the Burgers vector. In this case, the Burgers vector is exactly $\mathbf{b}$.

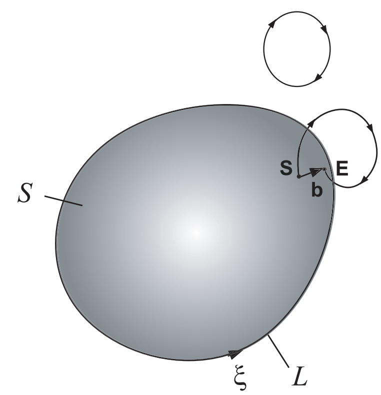


Figure 7.3: The direction of the Burgers circuit is defined through the dislocation line direction $\boldsymbol{\xi}$ according to the right-hand rule. The vector $\mathbf{b}$ connecting the starting point $\mathbf{S}$ and ending point $\mathbf{E}$ of the Burgers circuit is the Burgers vector.

From this definition, we see that the Burgers vector $\mathbf{b}$ is only defined with respect to a dislocation line direction $\boldsymbol{\xi}$. If the line direction of a dislocation is reversed, the Burgers vector should also be reversed (i.e $-\mathbf{b}$). This can be illustrated with the following example. Consider a dislocation dipole, i.e. two parallel infinite straight dislocations with opposite Burgers vectors. This dipole is exactly the same as two parallel dislocations with the same Burgers vector but opposite line directions, as shown in Fig. 7.4. Thus the two dislocations may also be regarded as opposite sides of the same (elongated) dislocation loop, as the length of the loop goes to infinity.

Let us now apply the Burgers circuit analysis to the dislocation in Fig. 7.2. As shown in Fig. 7.5, if we let the dislocation line direction $\boldsymbol{\xi}$ point out of the plane, then according to the right-hand rule, the Burgers circuit goes counter-clockwise. In this case, the Burgers vector $\mathbf{b}$ is one lattice spacing pointing to the right. If we choose the line direction to point into the plane, then the Burgers vector would point to the left.

Since the Burgers vector is constant along a dislocation loop, but the line direction may vary, the angle between the two may change over the loop. This angle is called the character angle θ. When $\mathbf{b}$ and $\boldsymbol{\xi}$ are parallel, the dislocation is called *screw* ($\theta = 0°$) and when they are

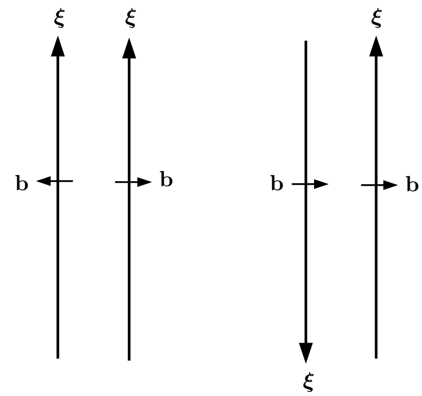


Figure 7.4: Equivalent representations of a dislocation dipole.

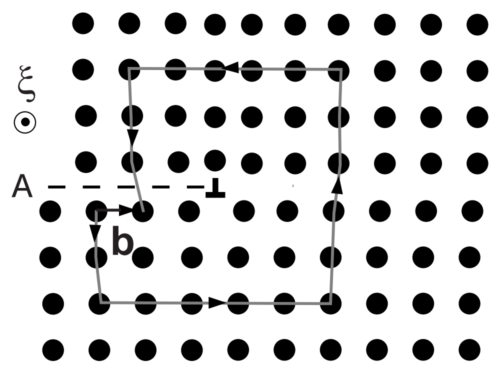


Figure 7.5: Burger's circuit around an edge dislocation. Choose the line direction to point out of the plane. According to the right-hand rule, the Burgers circuit goes counter-clockwise. In this case the Burgers vector **b** points to the right.

perpendicular it is called *edge* ($\theta = 90°$). Anything in between is called a *mixed* dislocation.

7.2 Dislocation's effects on mechanical properties

Dislocations are responsible for plastic deformation in crystals (e.g. metals and semiconductors). The stress strain curve of a crystal is linear up to the yield stress. At the yield stress, a large number of dislocations are able to move and the material deforms plastically. The total length of dislocations generally multiplies significantly during plastic deformation. Therefore, continued deformation of the material usually requires higher stress because dislocations themselves start to act as barriers to the motion of other dislocations. The plastic strain rate is related to the mobile dislocation density through the well known *Orowan's law*,

$$\dot{\epsilon}_{pl} = \rho b v \tag{7.3}$$

where ρ is the mobile dislocation density (in unit of m^{-2}), b is the Burgers vector, and v is the average dislocation velocity. Orowan's law can be proven using Betti's theorem.

Some stress-strain curves for body-centered-cubic (BCC) metal Molybdenum under uniaxial tension at a constant strain rate are shown in Fig. 7.6. The behavior at $T = 493°K$ shows a typical 3-stage behavior. Immediately after yield, there is *stage I* in which the plastic deformation proceeds easily without significant increase of applied stress. In *stage I*, dislocations are mostly gliding on parallel planes and their mutual interaction is weak. However, at higher deformation, the crystal enters *stage II* with the characteristic of a much higher but constant slope, i.e. hardening rate. This is because dislocations on several non-parallel slip planes have been activated and they started to block each other's motion. The dislocations start to form dense entangled structures. The total dislocation density keeps increasing during stage II. Eventually, the crystal enters *stage III* in which the hardening rate deviates from a constant due to recovery mechanisms that start to annihilate dislocations in the dense network.

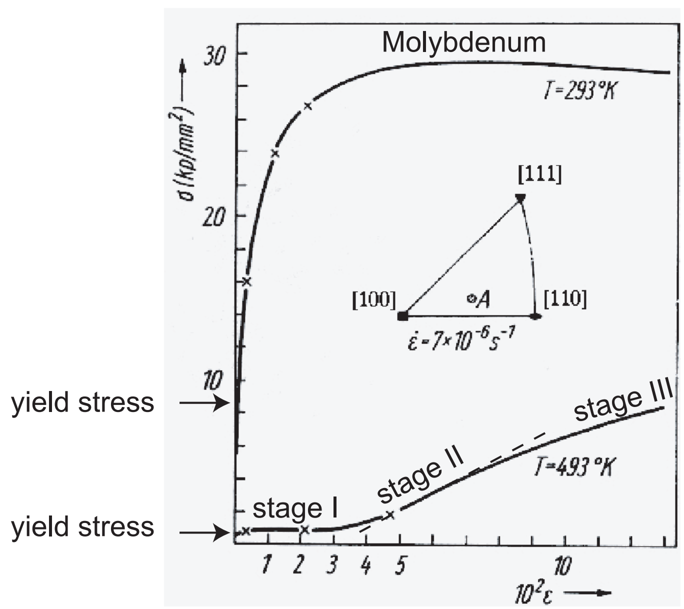


Figure 7.6: Tensile stress strain curve for Molybdenum at two temperatures [16]. The behavior at $T = 493°K$ exhibits a typical 3-stage behavior after initial yield (see text). The tensile axis A and strain rate $\dot{\varepsilon}$ are given in the inset. $1kp/mm^2=9.8MPa$.

Dislocations also play an important role in fracture, due to their interactions with cracks. For example, in ductile materials, a crack tip can nucleate many dislocations that shield and blunt the crack tip. This results in a higher critical strain energy release rate J_c for crack advancement and hence higher fracture toughness. A snapshot from Molecular Dynamics simulation of crack motion is shown in Fig. 7.7. A large number of dislocations are nucleated at the crack tip. Dislocations can also initiate fracture. In the fatigue process [18], the material is under cyclic loading. Dislocations keep multiplying during the cyclic loading and can form dense pile-up structures with very high local stresses that can lead to crack nucleation even in ductile materials.

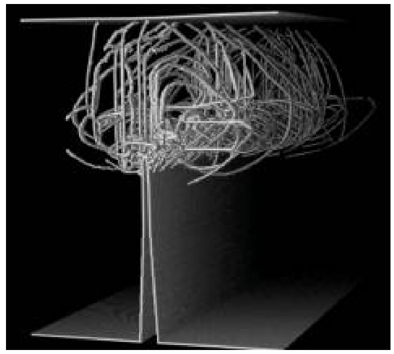


Figure 7.7: Snapshot of Molecular Dynamics simulation of dislocation nucleation in front of a crack tip [17].

7.3 Elastic fields of a dislocation loop

We now derive the elastic displacement and stress fields of a dislocation loop. Consider a dislocation loop L that is formed by displacing the lower side of surface S by $\mathbf{b}$ with respect to the upper side, as shown in Fig. 7.8. Notice that we have chosen the surface normal $\mathbf{n}$ of S and the line sense $\boldsymbol{\xi}$ of L to be consistent with the right-hand rule. To be more precise about the operation that introduces the dislocation, let us imagine that an infinitesimally thin layer of material around surface S is removed, so that the remaining material has two internal surfaces: S^+ and S^-. The lower surface is S^+ with normal vector $\mathbf{n}^+ = \mathbf{n}$ and the upper surface is S^- with normal vector $\mathbf{n}^- = -\mathbf{n}$. The dislocation is introduced by displacing the surface S^+ by $\mathbf{b}$ with respect to S^- and then gluing the two surfaces together. If this creates a gap or an overlap, then material must be added or removed to eliminate it.

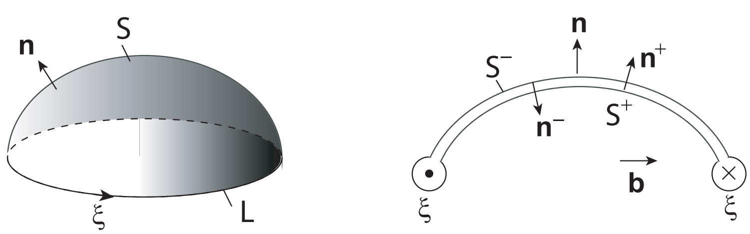


Figure 7.8: Continuum model of a dislocation. Imagine that a thin layer of material around surface S is removed, creating two internal surfaces S^+ and S^-. The dislocation is introduced by displacing S^+ by $\mathbf{b}$ with respect to S^-.

The elastic fields of this dislocation loop in a homogeneous infinite medium can be solved analytically by modeling this configuration as an equivalent inclusion. The inclusion occupies the space between S^+ and S^-. Let h be the separation between S^+ and S^-, i.e. the thickness of the inclusion. Then the equivalent eigenstrain to model the dislocation loop is,

$$e^*_{ij} = -\frac{n_i b_j + n_j b_i}{2h}$$

Now, in the limit as the separation h goes to zero the eigenstrain becomes

$$e^*_{ij} = -\frac{n_i b_j + n_j b_i}{2}\,\delta(\mathbf{S}-\mathbf{x})$$

Where $\delta(\mathbf{S}-\mathbf{x})$ is a shorthand notation for

$$\delta(\mathbf{S}-\mathbf{x}) \equiv \int_S \delta(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}')$$

i.e., $\delta(\mathbf{S}-\mathbf{x})$ is zero when $\mathbf{x}$ is not on $\mathbf{S}$ and infinite when it is. Therefore,

$$\int_V \delta(\mathbf{S}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') = \int_S \mathrm{d}S(\mathbf{x}')$$

Now, the eigenstress associated with the inclusion is

$$\sigma^*_{ij} = -C_{ijmn} n_m b_n \delta(\mathbf{S}-\mathbf{x})$$

The constrained displacement field is

$$\begin{aligned}
u^c_i(\mathbf{x}) &= \int_S F_j(\mathbf{x}')G_{ij}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \\
&= \int_S \sigma^*_{jk} n_k G_{ij}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \\
&= -\int_V \sigma^*_{jk} G_{ij,k}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\
&= \int_V C_{jkmn} b_m n_n \delta(\mathbf{S}-\mathbf{x}) G_{ij,k}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \qquad (7.4) \\
&= \int_S C_{jkmn} b_m n_n G_{ij,k}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}')
\end{aligned}$$

This is the *Volterra's formula* for displacement field of a dislocation loop. The constrained field is the displacement field everywhere in the solid, both in the inclusion and the matrix. It contains both elastic and plastic components. If one wishes to write down the elastic displacement gradients (to compute stress) everywhere in the solid, it is

$$u^{elastic}_{i,j}(\mathbf{x}) = \int_S C_{klmn} b_m n_n G_{ik,lj}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') + b_i n_j \delta(\mathbf{S}-\mathbf{x})$$

The second term is to account for the removal of the plastic distortions (i.e., the eigenstrain). The stress follows from Hooke's Law:

$$\sigma_{ij}(\mathbf{x}) = \int_S C_{ijkl} C_{pqrs} b_r n_s G_{kp,ql}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') + C_{ijkl} b_k n_l \delta(\mathbf{S}-\mathbf{x})$$

Suppose we wish to compute the stress of a continuous distribution of dislocations using Volterra's formula (for example, see section 7.6), where every point on the original Volterra's dislocation is spread out according to a distribution function $w(\mathbf{x})$, then the stress would just be a convolution of the original stress field with the distribution function, as in,

$$\tilde{\sigma}_{ij}(\mathbf{x}) = \sigma_{ij}(\mathbf{x}) * w(\mathbf{x}) \equiv \int_V \sigma_{ij}(\mathbf{x}'')\, w(\mathbf{x} - \mathbf{x}'')\, \mathrm{d}V(\mathbf{x}'')$$

The resulting stress field becomes

$$\begin{aligned} \tilde{\sigma}_{ij}(\mathbf{x}) &= \int_V w(\mathbf{x} - \mathbf{x}'') \int_S C_{ijkl} C_{pqrs} b_r n_s G_{kp,ql}(\mathbf{x}'' - \mathbf{x}')\, \mathrm{d}S(\mathbf{x}')\, \mathrm{d}V(\mathbf{x}'') \\ &\quad + \int_V C_{ijkl} b_k n_l w(\mathbf{x} - \mathbf{x}'') \delta(\mathbf{S} - \mathbf{x}'')\, \mathrm{d}V(\mathbf{x}'') \\ &= \int_S C_{ijkl} C_{pqrs} b_r n_s G^a_{kp,ql}(\mathbf{x} - \mathbf{x}')\, \mathrm{d}S(\mathbf{x}') + \int_S C_{ijkl} b_k n_l w(\mathbf{x} - \mathbf{x}')\, \mathrm{d}S(\mathbf{x}') \end{aligned}$$

where $G^a_{kp,ql}(\mathbf{x} - \mathbf{x}') \equiv G_{kp,ql}(\mathbf{x} - \mathbf{x}') * w(\mathbf{x}) \equiv \int_V w(\mathbf{x} - \mathbf{x}'') G_{kp,ql}(\mathbf{x}'' - \mathbf{x}')\, \mathrm{d}V(\mathbf{x}'')$ and the Burgers vector is assumed to be a constant over surface S. Notice that the second term on the right hand side corresponds to the eigenstress of an inclusion, whose eigenstrain distribution is,

$$e^*_{ij}(\mathbf{x}) = -\frac{b_i n_j + b_j n_i}{2} \int_S w(\mathbf{x} - \mathbf{x}') dS(\mathbf{x}') \tag{7.5}$$

So that the stress field of this continuously distributed dislocation is,

$$\tilde{\sigma}_{ij}(\mathbf{x}) = \int_S C_{ijkl} C_{pqrs} b_r n_s G^a_{kp,ql}(\mathbf{x} - \mathbf{x}')\, \mathrm{d}S(\mathbf{x}') - C_{ijkl} e^*_{kl}(\mathbf{x}) \tag{7.6}$$

Volterra's Formula for General Eigenstrain

In the above, Volterra's was derived for dislocations with a constant Burger's vector $\mathbf{b}$ on the slip plane S. However, Volterra's formula has the same form as above even if $\mathbf{b}$ is a non-uniform function of $\mathbf{x}$ in the slip plane. To show this, let's consider an inclusion as in Chapter 2, but now with a non-uniform eigenstrain distribution. The eigenstrain is assumed to be function of position in the inclusion. The displacement formula is then

$$u_i(\mathbf{x}) = \int_S F_j(\mathbf{x}')G_{ij}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') + \int_V b_j(\mathbf{x}')G_{ij}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')$$

where F_j and b_j are surface traction and body forces such that, if they were applied to the elastic medium, the total displacement of the body is zero everywhere (in this case, the stress field would be minus the eigenstress σ^*_{ij}). Therefore, F_j is simply related to the eigenstress by $F_j = -\sigma^*_{jk}n_k$ and equilibrium condition gives $-\sigma^*_{jk,k} + b_j = 0$. Therefore,

$$\begin{aligned}
u_i(\mathbf{x}) &= -\int_S \sigma^*_{jk}(\mathbf{x}')n_k(\mathbf{x}')G_{ij}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \\
&\quad + \int_V b_j(\mathbf{x}')G_{ij}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\
&= \int_V [-\sigma^*_{jk}(\mathbf{x}')G_{ij,k}(\mathbf{x}-\mathbf{x}') - \sigma^*_{jk,k}(\mathbf{x}')G_{ij}(\mathbf{x}-\mathbf{x}') \\
&\qquad + b_j(\mathbf{x}')G_{ij}(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}V(\mathbf{x}') \\
&= -\int_V \sigma^*_{jk}(\mathbf{x}')G_{ij,k}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \qquad (7.7)
\end{aligned}$$

Now, let the thickness of the eigenstrain go to zero as before, but let the Burgers vector be a function of $\mathbf{x}$, i.e.,

$$e^*_{ij}(\mathbf{x}) = -\frac{n_i b_j(\mathbf{x}) + n_j b_i(\mathbf{x})}{2}\,\delta(\mathbf{S}-\mathbf{x})$$

and substituting this into the Eq. (7.7) we get

$$u_i(\mathbf{x}) = \int_V C_{jkmn} b_m(\mathbf{x}') n_n \delta(\mathbf{S}'-\mathbf{x}')G_{ij,k}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')$$

which reduces to

$$u_i(\mathbf{x}) = \int_S C_{jkmn} b_m(\mathbf{x}') n_n G_{ij,k}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \qquad (7.8)$$

which is exactly Volterra's formula as previously stated earlier. Thus, Volterra's formula holds for any eigenstrain that is arbitrarily distributed over a surface S (so that it can be used to model a crack as well). However, Mura's formula, which will be derived shortly, only holds for uniform eigenstrains on surface S.

If we assume that $\mathbf{b}$ is a constant, then the displacement gradients, strains, and stresses can all be written as line integrals around L. The elastic displacement gradients can be written as

$$\begin{aligned} u_{i,j}^{elastic} &= \int_S \left[C_{klmn} b_m n_n G_{ik,lj}(\mathbf{x}-\mathbf{x}') + b_i n_j \delta(\mathbf{x}-\mathbf{x}')\right] \mathrm{d}S(\mathbf{x}') \\ &= \int_S \left[C_{klmn} G_{ik,lj}(\mathbf{x}-\mathbf{x}') + \delta_{im}\delta_{jn}\delta(\mathbf{x}-\mathbf{x}')\right] b_m n_n \, \mathrm{d}S(\mathbf{x}') \end{aligned}$$

Using the equilibrium of the Green's function from Chapter 1

$$\delta(\mathbf{x}-\mathbf{x}')\delta_{im} = -C_{mlpq} G_{pi,ql}(\mathbf{x}-\mathbf{x}') = -C_{klmn} G_{ki,ln}(\mathbf{x}-\mathbf{x}')$$

the elastic displacement gradients become

$$\begin{aligned} u_{i,j}^{elastic} &= \int_S \left[C_{klmn} G_{ik,lj}(\mathbf{x}-\mathbf{x}') - \delta_{jn} C_{klmn} G_{ki,ln}(\mathbf{x}-\mathbf{x}')\right] b_m n_n \, \mathrm{d}S(\mathbf{x}') \\ &= \int_S C_{klmn} b_m \left[n_n G_{ik,lj}(\mathbf{x}-\mathbf{x}') - n_j G_{ik,ln}(\mathbf{x}-\mathbf{x}')\right] \mathrm{d}S(\mathbf{x}') \qquad (7.9) \end{aligned}$$

The Stoke's Theorem,

$$\oint_L f v_h \, \mathrm{d}L = \int_S \epsilon_{ihk} f_{,i} n_k \, \mathrm{d}S \qquad (7.10)$$

can be re-written as,

$$\oint_L f v_h \epsilon_{jnh} \, \mathrm{d}L = \int_S \epsilon_{ihk}\epsilon_{jnh} f_{,i} n_k \, \mathrm{d}S = \int_S (\delta_{kj}\delta_{in} - \delta_{kn}\delta_{ij}) f_{,i} n_k \, \mathrm{d}S = \int_S (n_j f_{,n} - n_n f_j) \, \mathrm{d}S$$

So that the displacement gradients becomes

$$u_{i,j}^{elastic} = \oint_L \epsilon_{jnh} C_{klmn} b_m v_h G_{ik,l}(\mathbf{x}-\mathbf{x}') \, \mathrm{d}S(\mathbf{x}') \qquad (7.11)$$

In this last step there were two sign changes that cancel each other — one for turning $(n_j f_n - n_n f_j)$ into $-(n_n f_j - n_j f_n)$ (where $f = G_{ik,l}$) and the other for turning $\partial/\partial x'_n G_{ik,l}(\mathbf{x}-\mathbf{x}')$ into $-G_{ik,ln}(\mathbf{x}-\mathbf{x}')$. Notice that in changing the surface integral to a line integral, the surface delta function has completely disappeared and the displacement gradients are continuous everywhere. If the contribution from the surface delta function was originally ignored in the surface integral, this would not be the case. The stress field is

$$\sigma_{ij} = C_{ijkl} \oint_L \epsilon_{lnh} C_{pqmn} b_m v_h(\mathbf{x}') G_{kp,q}(\mathbf{x}-\mathbf{x}') \, \mathrm{d}L(\mathbf{x}') \qquad (7.12)$$

Eq. (7.12) is called *Mura's formula.* v_h is the unit vector along the local line direction, i.e. it is the same vector as $\boldsymbol{\xi}$ and we will use them interchangeably. Note that the above line integral forms of the stress field and displacement gradients are meaningful only when they are evaluated around a complete loop. Since any function that gives zero integral around a closed loop L can be added to these formulas without changing the final result, the stress field of a finite dislocation segment is not unique. This mathematical argument agrees with the physical model of a dislocation because dislocations in crystalline solids cannot end inside a crystal (although they can terminate at the crystal surface).

For numerical simulations, dislocation lines are usually represented by a connected set of straight dislocations segments, as shown in Fig. 7.9. The stress field from each segment only has physical meaning when they are summed over the entire loop. The stress field of a straight dislocation segment can be obtained analytically in isotropic elasticity, i.e. there exist a function $\sigma_{ij}^{\text{seg}}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{b}^{(12)})$, where $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are two end points of the segment and $\mathbf{b}^{(12)}$ is the Burgers vector. The stress field of the dislocation loop shown in Fig. 7.9 can then be obtained by summing over the stress fields of individual segments,

$$\sigma_{ij}^{\text{Loop}} = \sum_{n=1}^{N} \sigma_{ij}^{\text{seg}}(\mathbf{x}^{(n)}, \mathbf{x}^{(n+1)}, \mathbf{b})\,, \qquad \mathbf{x}^{(N+1)} \equiv \mathbf{x}^{(1)} \tag{7.13}$$

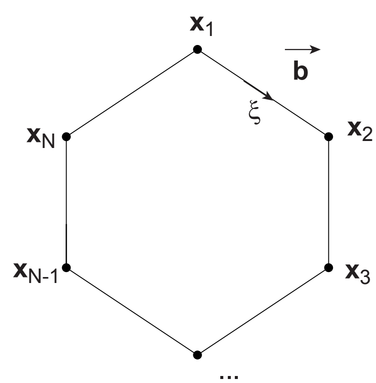


Figure 7.9: A dislocation loop with line direction $\boldsymbol{\xi}$ and Burgers vector $\mathbf{b}$ is represented by N straight dislocation segments.

7.4 Self energy of a dislocation loop

In the previous section the elastic fields of stress and strain of a dislocation loop were reduced to line integrals. We have shown that mathematically, this can be done but we should also expect this because of the line structure of the dislocation. From this physical argument, we should also expect that the self energy of, and interaction energies between, dislocations can be written as line integrals. However, the actual realization of these formulas will prove to be much more difficult. To see why this causes problems, lets attempt to calculate the self energy of a dislocation loop.

Now that we have the stress and strain field of a dislocation loop, the self energy can be evaluated in a very straight forward method by integrating the strain energy density over the volume of the crystal.

$$E = \int_V w \, \mathrm{d}V$$

where

$$w(e_{ij}) = \frac{1}{2}\sigma_{ij}e_{ij}$$

for linear elastic materials. However, a more elegant and arguably easier method is to use the work method. This method measures the amount of reversible work done when creating a dislocation loop. Both methods will give the same result. However, the energy of a dislocation loop obtained from linear elasticity theory is in fact singular (infinite), unless a certain truncation scheme is applied. The singularity problem will be discussed in Section 7.6. For now let us simply ignore the singularity.

Imagine that we create the dislocation loop shown in Fig. 7.8 by applying traction forces F_j^+ and F_j^- on S^+ and S^- and very slowly displace S^+ with respect to S^- by $\mathbf{b}$. The traction forces can be written in terms of the stress field,

$$\begin{aligned} F_j^+ &= \sigma_{kj}n_k^+ \\ F_j^- &= \sigma_{kj}n_k^- \end{aligned}$$

Let the displacements on S^+ and S^- be u_j^+ and u_j^-. We have

$$u_j^+ - u_j^- = b_j \tag{7.14}$$

The work done to create the dislocation loop is

$$\begin{aligned} W &= \frac{1}{2}\int_{S^+} F_j^+ u_j^+ \,\mathrm{d}S + \frac{1}{2}\int_{S^-} F_j^- u_j^- \,\mathrm{d}S \\ &= \frac{1}{2}\int_S \sigma_{kj}n_k^+(u_j^+ - u_j^-)\,\mathrm{d}S \\ &= \frac{1}{2}\int_S \sigma_{kj}n_k b_j \,\mathrm{d}S \end{aligned} \tag{7.15}$$

This is the same as the self energy E of the dislocation loop. Substituting in Mura's formula for the stress, we have,

$$E = \frac{1}{2}\int_S \oint_L b_i n_j(\mathbf{x}) C_{ijkl}\epsilon_{lnh}C_{pqmn}b_m v_h(\mathbf{x}')G_{kp,q}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}L(\mathbf{x}')\,\mathrm{d}S(\mathbf{x}) \tag{7.16}$$

A first reaction would be to use Stoke's theorem on this integral, but that gives back both a line integral and a surface integral which will not reduce further. In fact, no one has been able to reduce this formula to a line integral in its present form for general anisotropic materials. For isotropic material, substituting the analytic expression for the Green's function, the equation for the self energy can be reduced to

$$E = \oint_L \oint_L \frac{\mu}{16\pi} b_i b_j R_{,pp}\,\mathrm{d}x_i\,\mathrm{d}x_j' + \frac{\mu}{8\pi(1-\nu)}\epsilon_{ikl}\epsilon_{jmn}b_k b_m R_{,ij}\,\mathrm{d}x_l\,\mathrm{d}x_n' \tag{7.17}$$

and the interaction energy between two dislocation loops is (in vector form) [19]

$$\begin{aligned} W_{12} = &-\frac{\mu}{2\pi}\oint_{L_1}\oint_{L_2} \frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\,\mathrm{d}\mathbf{L}_1\times\,\mathrm{d}\mathbf{L}_2)}{R} \\ &+\frac{\mu}{4\pi}\oint_{L_1}\oint_{L_2} \frac{(\mathbf{b}_1\cdot\,\mathrm{d}\mathbf{L}_1)(\mathbf{b}_2\cdot\,\mathrm{d}\mathbf{L}_2)}{R} \\ &+\frac{\mu}{4\pi}\oint_{L_1}\oint_{L_2} (\mathbf{b}_1\times\,\mathrm{d}\mathbf{L}_1)\cdot\nabla\nabla R\cdot(\mathbf{b}_2\times\,\mathrm{d}\mathbf{L}_2) \end{aligned} \tag{7.18}$$

Full derivations of these equations can be found in [19]. For anisotropic elastic medium, while Eq. (7.16) has not been reduced to a double line integral, Lothe [20] has reduced the interaction energy to following integral form,

$$E = \frac{1}{8\pi^2} \oint_{L_1} \oint_{L_2} \mathrm{d}L(\mathbf{x})\, \mathrm{d}L(\mathbf{x}') \frac{1}{R} \int_0^{2\pi} \mathbf{b}_1 (\boldsymbol{\xi}_1 \times \mathbf{m}, \boldsymbol{\xi}_2 \times \mathbf{m})^{p,m} \mathbf{b}_2 \, \mathrm{d}\phi \tag{7.19}$$

where $(\mathbf{a}, \mathbf{b})^{p,m} \equiv (\mathbf{a}, \mathbf{b}) - (\mathbf{a}, \mathbf{m})(\mathbf{m}, \mathbf{m})^{-1}(\mathbf{m}, \mathbf{b})$, $(\mathbf{a}, \mathbf{b})_{jk} \equiv a_i C_{ijkl} b_l$, $\mathbf{m}$ is a unit vector perpendicular to $\mathbf{R} \equiv \mathbf{x}' - \mathbf{x}$, and ϕ specifies the angle between $\mathbf{m}$ and an arbitrary reference direction in the plane perpendicular to $\mathbf{R}$. The reader is directed to Lothe's 1982 paper [20] for a complete explanation of this equation which is to long to reproduce here.

7.5 Force on a dislocation

In order to determine the force exerted on a dislocation line, let us first look at the virtual displacement of a dislocation loop. Consider the dislocation L with line direction $\mathbf{v}$ as shown in Fig. 7.10. Notice that $\mathbf{v}$ and $\boldsymbol{\xi}$ mean the same thing and we will use them interchangeably. Let the loop move by a small amount $\delta\mathbf{r}(\mathbf{x})$, with $\delta\mathbf{r}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = 0$ because a line moving along itself has no physical consequence. Let the change of energy be δE. If δE can be expressed in the form of

$$\delta E = -\oint_L \mathbf{f}(\mathbf{x}) \cdot \delta r(\mathbf{x}) \, \mathrm{d}L(\mathbf{x}) \tag{7.20}$$

then $\mathbf{f}(\mathbf{x})$ is the line force (per unit length) on L. Because $\delta\mathbf{r}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = 0$, if $\mathbf{f}(\mathbf{x})$ is along $\mathbf{v}(\mathbf{x})$, it contributes zero to δE. This means that we may add this function to any solution $\mathbf{f}(\mathbf{x})$ of Eq. (7.20) and we obtain yet another solution. For uniqueness, we will enforce the intuitive constraint that $\mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = 0$.

The energy of a set of N dislocation loops can be written as the sum of the loop self energies and the interaction energies between the loops,

$$E = \sum_{i=1}^{N} E_i + \sum_{i=1}^{N} \sum_{j=i+1}^{N} W_{ij} \tag{7.21}$$

Let us consider the force on loop 1. We need to calculate the variation of the total energy with respect to the virtual displacement of the loop, $\delta\mathbf{r}_1(\mathbf{x})$, i.e.,

$$\mathbf{f}_1 = -\frac{\delta E}{\delta \mathbf{r}_1(\mathbf{x})} = -\frac{\delta E_1}{\delta \mathbf{r}_1(\mathbf{x})} - \sum_{j=2}^{N} \frac{W_{1j}}{\delta \mathbf{r}_1(\mathbf{x})} \tag{7.22}$$

The first term of this equation is divergent, since the self energy is singular (we will deal with this problem in Section 7.6). The second term is the force do to the interaction energy between the dislocations.

For brevity, let us consider a system with only two dislocations, so that we only have one interaction term,

$$W_{12} = \int_{S_1} \sigma_{ij}^{(2)}(\mathbf{x}) n_i^{(1)} b_j^{(1)} \, \mathrm{d}S(\mathbf{x})$$

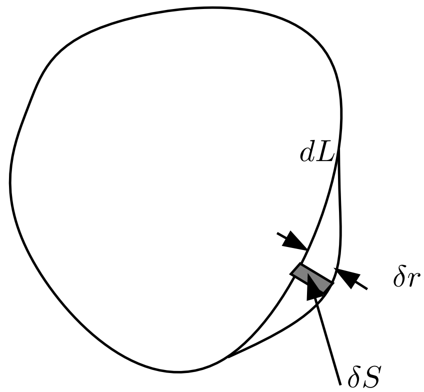


Figure 7.10: Virtual shape change of a dislocation loop.

Note that $\sigma_{ij}^{(2)}(\mathbf{x})$ is invariant with respect to $\delta\mathbf{r}(\mathbf{x})$ since the virtual motion is only for dislocation 1. Therefore, the only change of W_{12} is induced by the change of the integration area S_1, i.e.,

$$\delta W_{12} = \int_{\delta S_1} \sigma_{ij}^{(2)}(\mathbf{x}) n_i^{(1)} b_j^{(1)} \, \mathrm{d}S(\mathbf{x}) \tag{7.23}$$

and

$$\mathbf{n}\delta S = \delta\mathbf{r} \times \mathbf{v} \, \mathrm{d}L$$

$$n_i \, \mathrm{d}S = \epsilon_{imn} \delta r_m v_n \, \mathrm{d}L$$

Thus

$$\begin{aligned}
\delta W_{12} &= \oint_L \sigma_{ij}^{(2)}(\mathbf{x}) b_j^{(1)} \epsilon_{imn} \delta r_m(\mathbf{x}) v_n^{(1)}(\mathbf{x}) \, \mathrm{d}L(\mathbf{x}) \\
&= -\oint_L [\sigma_{ij}^{(2)}(\mathbf{x}) b_j^{(1)} \epsilon_{inm} v_n^{(1)}(\mathbf{x})] \delta r_m(\mathbf{x}) \, \mathrm{d}L(\mathbf{x})
\end{aligned}$$

which leads us to the self force

$$f_m(\mathbf{x}) = \epsilon_{inm} \sigma_{ij}^{(2)}(\mathbf{x}) b_j^{(1)} v_n^{(1)}(\mathbf{x}) \tag{7.24}$$

This is often written in the vector form as

$$\mathbf{f} = (\boldsymbol{\sigma} \cdot \mathbf{b}) \times \boldsymbol{\xi} \tag{7.25}$$

This is called the *Peach-Koehler formula*. Even though we have pictured $\sigma_{ij}^{(2)}$ to be the stress due to another dislocation loop, it could come from any stress source and the resulting force can be obtained from the Peach-Koehler formula in the same way. The total force on the dislocation should also include the effect of the stress field on itself. However, this contribution is infinite, unless some truncation scheme is applied (see Section 7.6).

7.6 Non-singular dislocation model

In the previous discussions we have introduced a model for a dislocation that has both a stress singularity and a self energy singularity. The nature of this singularity presents problem to define self forces on the dislocation. While several approaches have been proposed to define a finite self-force on dislocations, in this section we will discuss the model proposed in [21] which is relatively easy to explain. This model removes the singularity for dislocations while maintaining the simplest analytic expressions for the stress, energy and force formulas. It lets each point on the dislocation line become the center of a distribution of dislocations which spreads out the dislocation core. Let the spreading (distribution) function be $\tilde{w}(\mathbf{x})$. Recall that the stress field of a dislocation loop according the Mura's formula (singular) is

$$\sigma_{\alpha\beta}(\mathbf{x}) = \oint_L C_{\alpha\beta kl}\epsilon_{lnh}C_{pqmn}b_m v_n(\mathbf{x}')G_{kp,q}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}L(\mathbf{x}') \tag{7.26}$$

In the non-singular theory, the stress field should be the convolution of the above expression with $\tilde{w}(\mathbf{x})$, i.e.,

$$\begin{aligned}\tilde{\sigma}^{ns}_{\alpha\beta}(\mathbf{x}) &= \sigma_{\alpha\beta}(\mathbf{x}) * \tilde{w}(\mathbf{x}) \\ &= \oint_L \int C_{\alpha\beta kl}\epsilon_{lnh}C_{pqmn}b_m v_n(\mathbf{x}')G_{kp,q}(\mathbf{x}-\mathbf{x}')\tilde{w}(\mathbf{x}''-\mathbf{x}')\,\mathrm{d}L(\mathbf{x}')\,\mathrm{d}\mathbf{x}''\end{aligned}$$

However, to compute the force on the spread-out dislocation line, what is relevant is not the stress at a single point $\mathbf{x}$, but the stress field convoluted with a spreading function centered at $\mathbf{x}$. Both the stress source point $\mathbf{x}'$ and the field point $\mathbf{x}$ are spread out because they are both points on the dislocation line. Therefore, the more relevant stress field is,

$$\sigma^{ns}_{\alpha\beta}(\mathbf{x}) = \tilde{w}(\mathbf{x}) * \sigma_{\alpha\beta}(\mathbf{x}) * \tilde{w}(\mathbf{x})$$

Define

$$w(\mathbf{x}) = \tilde{w}(\mathbf{x}) * \tilde{w}(\mathbf{x})$$

The nonsingular stress field becomes

$$\sigma^{ns}_{\alpha\beta}(\mathbf{x}) = \sigma_{\alpha\beta}(\mathbf{x}) * w(\mathbf{x})$$

In isotropic elasticity the Green's function is expressible in terms of third derivatives of $R \equiv |\mathbf{x}-\mathbf{x}'|$. For example, Mura's formula for the singular stress field is

$$\begin{aligned}\sigma^{ns}_{\alpha\beta}(\mathbf{x}) &= \frac{\mu}{8\pi}\oint_L \partial_i\partial_p\partial_p R\left[b_m\epsilon_{im\alpha}\,\mathrm{d}x'_\beta + b_m\epsilon_{im\beta}\,\mathrm{d}x'_\alpha\right] \\ &\quad + \frac{\mu}{4(1-\nu)}\oint_L b_m\epsilon_{imk}(\partial_i\partial_\alpha\partial_\beta R - \delta_{\alpha\beta}\partial_i\partial_p\partial_p R)\,\mathrm{d}x'_k\end{aligned} \tag{7.27}$$

If we choose

$$w(\mathbf{x}) = \frac{15a^4}{8\pi(|\mathbf{x}|^2+a^2)^{7/2}} \tag{7.28}$$

then

$$R * w(\mathbf{x}) = R_a \equiv \sqrt{R^2 + a^2} \tag{7.29}$$

Therefore,

$$\begin{aligned}\sigma^{ns}_{\alpha\beta}(\mathbf{x}) &= \frac{\mu}{8\pi}\oint_L \partial_i\partial_p\partial_p R_a \left[b_m\epsilon_{im\alpha}\,\mathrm{d}x'_\beta + b_m\epsilon_{im\beta}\,\mathrm{d}x'_\alpha\right] \\ &\quad + \frac{\mu}{4(1-\nu)}\oint_L b_m\epsilon_{imk}(\partial_i\partial_\alpha\partial_\beta R_a - \delta_{\alpha\beta}\partial_i\partial_p\partial_p R_a)\,\mathrm{d}x'_k\end{aligned} \tag{7.30}$$

This completely removes the singularity from the stress field. Because the spatial derivatives of R_a and R are very similar, the analytic structures of the original singular theory is maintained in the non-singular theory. For example the stress field of a straight dislocation segment in isotropic elasticity can be obtained and the results are very similar to the original (singular) expressions.

Following the same derivation as before, the self energy of a dislocation loop now becomes,

$$E = \oint_L\oint_L \frac{\mu}{16\pi} b_i b_j R_{a,pp}\,\mathrm{d}x_i\,\mathrm{d}x'_j + \frac{\mu}{8\pi(1-\nu)}\epsilon_{ikl}\epsilon_{jmn}b_k b_m R_{a,ij}\,\mathrm{d}x_l\,\mathrm{d}x'_n \tag{7.31}$$

and the interaction energy is (in vector form) [19]

$$\begin{aligned}W_{12} &= -\frac{\mu}{4\pi}\oint_{L_1}\oint_{L_2}(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\,\mathrm{d}\mathbf{L}_1\times\,\mathrm{d}\mathbf{L}_2)\,\nabla^2 R_a \\ &\quad + \frac{\mu}{8\pi}\oint_{L_1}\oint_{L_2}(\mathbf{b}_1\cdot\,\mathrm{d}\mathbf{L}_1)(\mathbf{b}_2\cdot\,\mathrm{d}\mathbf{L}_2)\,\nabla^2 R_a \\ &\quad + \frac{\mu}{4\pi}\oint_{L_1}\oint_{L_2}(\mathbf{b}_1\times\,\mathrm{d}\mathbf{L}_1)\cdot\nabla\nabla R_a\cdot(\mathbf{b}_2\times\,\mathrm{d}\mathbf{L}_2)\end{aligned} \tag{7.32}$$

The self energy is now finite and the stress field is smooth and finite everywhere (including on the dislocation line itself). The Peach-Koehler formula can now be safely applied to obtain the self force on the dislocation without ambiguity. The total force on the dislocation can be simply obtained from the Peach-Koehler formula using the total stress field, from Eq. (7.30), on the dislocation itself.

7.7 Peierls-Nabarro model

The displacement jump as introduced in Volterra's singular dislocation model is a discontinuous function on the slip plane. For example, consider an infinite straight dislocation along the z-axis and let the cut plane S be the $x < 0$ portion of the x-z plane. Let u^- and u^+ be the displacement field on S^- and S^+, i.e. the upper and lower side of surface S, respectively, similar to Fig. 7.8. Define $[[u]] \equiv u^+ - u^-$ as the displacement jump across the cut plane. In Volterra's model, $[[u]]$ is a step function, as shown in Fig. 7.11. If the dislocation line direction is chosen to be along the positive z-axis (out of plane), then the Burgers vector of this dislocation is b.

We can define the derivative of $[[u]](x)$ as the dislocation core density $\rho(\mathbf{x})$. In this case, $\rho(x)$ is a delta function, i.e. $\rho(\mathbf{x}) = b\delta(\mathbf{x})$, as shown in Fig.7.11. The concentrated Burgers vector distribution is responsible for the singularity we experienced earlier. However, in a real crystal, no such singularity exists and we cannot define the position of a dislocation more accurately than the lattice spacing between atoms. Therefore, a more realistic model would be to let the core density be a spread-out smooth function of x, as shown in Fig. 7.12.

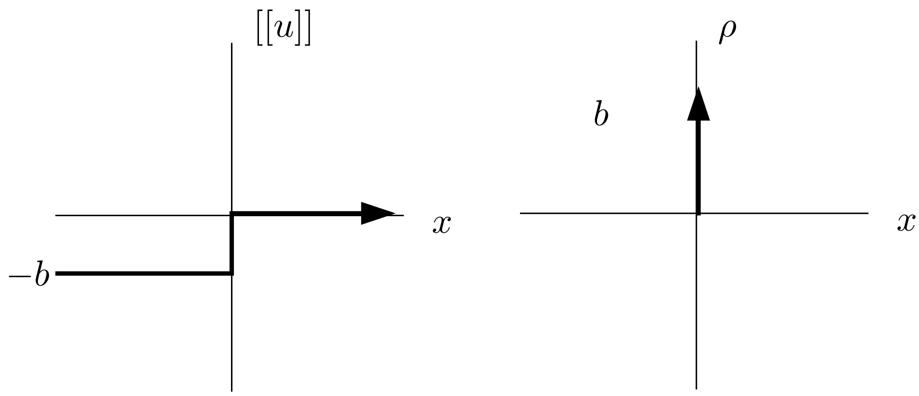


Figure 7.11: Displacement jump $[[u]](x)$ and dislocation core distribution $\rho(x) = d[[u]](x)/dx$ for a Volterra's dislocation.

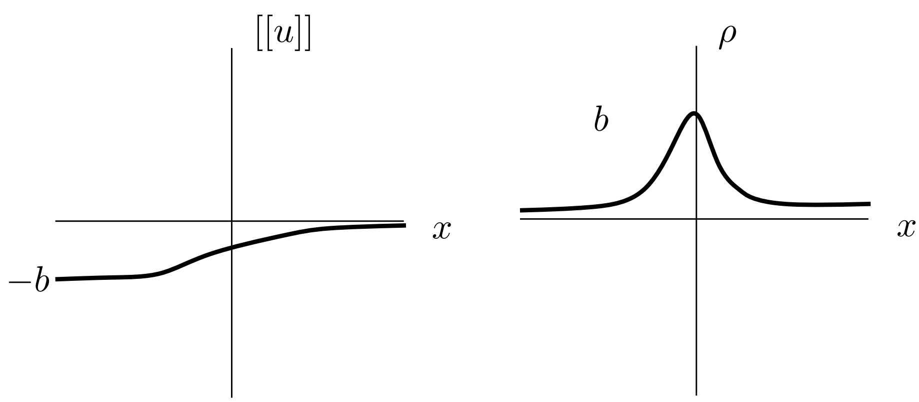


Figure 7.12: Displacement jump $[[u]](x)$ and dislocation core distribution $\rho(x) = d[[u]](x)/dx$ for a more realistic model which allows dislocation core to spread out.

To obtain the actual spreading function $\rho(x)$, the strategy is to obtain the total energy E_{tot} as a functional of $\rho(x)$ and find the $\rho(x)$ that minimizes E_{tot}. Obviously, E_{tot} should include the elastic energy contribution. The elastic energy of a dislocation can be evaluated by finding the reversible work done while creating the dislocation. The elastic energy for an infinite straight dislocation is

$$\begin{aligned}
E_{\mathrm{el}} &= \frac{1}{2}\int_{-\infty}^{\infty}(\mathbf{T}^{+}\cdot\mathbf{u}^{+}-\mathbf{T}^{-}\cdot\mathbf{u}^{-})\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty}(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot[[\mathbf{u}]](x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\sigma_{xy}(x)[[\mathbf{u}]](x)\,\mathrm{d}x
\end{aligned}$$

For an edge Volterra's dislocation

$$\sigma_{xy} = \frac{\mu b}{2\pi(1-\nu)}\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

On the plane $y = 0$

$$\sigma_{xy} = \frac{\mu b}{2\pi(1-\nu)}\frac{1}{x} \tag{7.33}$$

For this dislocation the displacement jump is

$$[[u]](x) = \begin{cases} -b & x < 0 \\ 0 & x > 0 \end{cases}$$

The energy is

$$E_{\mathrm{el}} = -\frac{1}{2}\int_{-\infty}^{0}\frac{\mu b^2}{2\pi(1-\nu)}\frac{1}{x}\,\mathrm{d}x \tag{7.34}$$

which is infinite. However, for a dislocation with core density $\rho(x)$ other than a delta function, the stress field is the convolution of Eq. (7.33) with $\rho(x)$, i.e.,

$$\sigma_{xy}(x) = \frac{\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{\rho(x')}{x-x'}\,\mathrm{d}x'$$

The corresponding elastic energy is

$$\begin{aligned}
E_{el} &= \frac{1}{2}\int_{-\infty}^{\infty}\sigma_{xy}(x)\,[[u]](x)\,\mathrm{d}x \\
&= \frac{\mu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\rho(x')\,[[u]](x)}{x-x'}\,\mathrm{d}x\,\mathrm{d}x' \\
&= -\frac{\mu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(x)\rho(x')\ln|x-x'|\,\mathrm{d}x\,\mathrm{d}x' + C
\end{aligned}$$

where C is a constant from integration by parts, which is independent of the shape of $[[u]](x)$ as long as the boundary conditions at $x = \pm\infty$ are fixed. This solution is for edge dislocations. The solution for screw dislocations only differs by a constant and the general solution can be written as

$$E_{\mathrm{el}} = -K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(x)\rho(x')\ln|x-x'|\,\mathrm{d}x\,\mathrm{d}x' + C$$

where

$$K = \begin{cases} \frac{\mu}{4\pi} & \text{screw} \\ \frac{\mu}{4\pi(1-\nu)} & \text{edge} \end{cases}$$

As the function $\rho(x)$ becomes more widely distributed (subjected to the normalization condition $\int_{-\infty}^{\infty} \rho(x)dx = b$), the elastic energy E_{el} becomes smaller. If the elastic energy is the only contribution to the total energy, the dislocation would spread out completely (in the end there will be no dislocation to speak of). In reality, the dislocation core is stabilized by the non-linear interfacial misfit energy between the two surfaces S^+ and S^-. This misfit energy is also called the generalized stacking fault energy γ. Due to the periodic nature of the crystal structure, γ is a periodic function of $[[u]]$. The simplest model for $\gamma(\cdot)$ is,

$$\gamma(u) = U_0 \sin\left(\frac{2\pi u}{b}\right)$$

and the corresponding misfit energy would be

$$E_{\mathrm{s}} = \int_{-\infty}^{\infty} \gamma([[u]](x))\,\mathrm{d}x$$

Therefore the total energy the dislocation is

$$E_{\mathrm{tot}} = -K \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x)\rho(x')\ln|x-x'|\,\mathrm{d}x\,\mathrm{d}x' + \int_{-\infty}^{\infty} \gamma([[u]](x))\,\mathrm{d}x + C$$

The function that minimizes E_{tot} describes the physical shape of the dislocation core. The minimizing function $[[u]](x)$ satisfies the condition

$$\begin{aligned} 0 &= \frac{\delta E_{tot}}{\delta[[u]]} \\ &= 2K \int_{-\infty}^{\infty} \frac{\rho(x')}{x-x'}\,\mathrm{d}x' + \frac{\mathrm{d}\gamma}{\mathrm{d}[[u]]} \end{aligned}$$

More explicitly,

$$-2K \int_{-\infty}^{\infty} \left.\frac{\mathrm{d}[[u]]/\,\mathrm{d}x}{x-x'}\right|_{x=x'} \mathrm{d}x' = \frac{U_0\pi}{b}\sin\left(\frac{2\pi[[u]]}{b}\right)$$

The analytic solution to this differential-integral equation was given by Rudolf Peierls as

$$[[u]](x) = \frac{b}{\pi}\arctan\left(\frac{x}{\xi}\right) - \frac{b}{2} \tag{7.35}$$

where

$$\xi = \frac{Kb^2}{U_0\pi} \tag{7.36}$$

ξ is called the half width of the dislocation core. We notice that the core half-width ξ represents the competition between the elastic stiffness K (which tends to spread the dislocations out) and the non-linear misfit potential U_0(which tends to localize the dislocation core). The dislocation distribution function $\rho(x)$ is

$$\rho(x) = \frac{b}{\pi}\frac{\xi}{x^2+\xi^2} \tag{7.37}$$

and the stress field along the x-axis ($y = 0$) becomes

$$\sigma_{xy}(x) = \frac{\mu b}{2\pi(1-\nu)}\frac{x}{x^2+\xi^2} \tag{7.38}$$

When an external stress field is applied, the optimal dislocation shape should minimize the Gibb's free energy, which also includes the negative of the work done by the external stress.

$$\Delta G = -K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(x)\rho(x')\ln|x-x'|\,\mathrm{d}x\,\mathrm{d}x' + \int_{-\infty}^{\infty}\gamma([[u]](x))\,\mathrm{d}x - \int_{-\infty}^{\infty}\sigma_{xy}^{A}(x)[[u]](x)\,\mathrm{d}x + C$$

This model can be generalized to model the nucleation of a dislocation dipole (in 1D) or a dislocation loop (in 2D).

Appendix A

Exercise Problems

A.1 Index Notation and Gauss's Theorem

Problem 1.1 (10') Index notation.

(a) Show that $\epsilon_{mkq}\epsilon_{nkq} = 2\delta_{mn}$.

(b) Consider a rank-two tensor $p_{ij} = a\delta_{ij} + bz_iz_j$, where $\mathbf{z}$ is a unit vector ($z_iz_i = 1$). Find the inverse q_{ij} of p_{ij}, which is defined through $q_{ij}p_{jk} = \delta_{ik}$.
[Hint: suppose q_{ij} also has the form of $q_{ij} = c\delta_{ij} + dz_iz_j$.]

Problem 1.2 (10') Tensor symmetry.

Any second rank tensor A_{ij} can be decomposed into its symmetric and antisymmetric parts

$$A_{ij} = A_{(ij)} + A_{[ij]}$$

where

$$A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right)$$

is the symmetric part and

$$A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right)$$

is the antisymmetric part.

(a) Show that if A_{ij} is a symmetric tensor, and B_{ij} is an arbitrary tensor, then,

$$A_{ij}B_{ij} = A_{ij}B_{(ij)} \tag{A.1}$$

(b) Show that if A_{ij} is an antisymmetric tensor, then

$$A_{ij}a_ia_j = 0$$

Problem 1.3 (10') Gauss's Theorem.

(a) For a elastic body V with surface S in equilibrium under surface traction T_i and zero body force ($b_i = 0$), show that

$$\int_S T_i u_i \mathrm{d}S = \int_V \sigma_{ij} e_{ij} \mathrm{d}V$$

where u_i, σ_{ij}, e_{ij} are displacement, stress and strain fields.
[Hint: Use the result in Problem 1.2.]

(b) Show that the average stress in the elastic body under zero body force is,

$$\overline{\sigma}_{ij} = \frac{1}{2V} \int_S (T_i\, x_j + T_j\, x_i) \mathrm{d}S$$

A.2 Elasticity in one and two dimensions

Problem 2.1 (10') Elastic constants.
The elastic stiffness tensor for the isotropic medium is $C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Determine the compliance tensor, S_{ijkl}, which is the inverse of C_{ijkl}, i.e.,

$$C_{ijkl} S_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{A.2}$$

[Hint: assume that S_{ijkl} has the form $\alpha \delta_{ij}\delta_{kl} + \beta\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.]

Problem 2.2 (10') 1D elasticity.
Determine the displacement, strain and stress field of a long rod of length L standing vertically in a gravitational field g. Assume the rod is an isotropic elastic medium with shear modulus μ and Poisson's ratio ν.

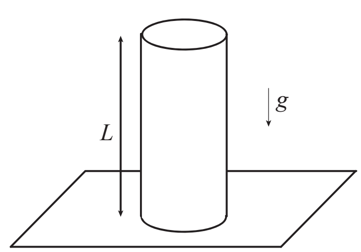


Figure A.1: A rod of length L standing vertically in a gravitational field g.

Problem 2.3 (10') 2D elaticity.

Lets look at equilibrium in 2-D elasticity using x-y cartesian coordinates under zero body force. Assume the 2-d body is in a state of plane stress, i.e.,

$$\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$$

which corresponds to a free standing thin film. The equilibrium equations reduce to

$$\sigma_{xx,x} + \sigma_{yx,y} = 0 \tag{A.3}$$
$$\sigma_{yy,y} + \sigma_{xy,x} = 0 \tag{A.4}$$

And the compatability equations reduce to

$$e_{xx,yy} - 2e_{xy,xy} + e_{yy,xx} = 0 \tag{A.5}$$

One popular method to solve such problems is to introduce the Airy's stress function ϕ such that,

$$\sigma_{xx} = \phi_{,yy} \tag{A.6}$$
$$\sigma_{yy} = \phi_{,xx} \tag{A.7}$$
$$\sigma_{xy} = -\phi_{,xy} \tag{A.8}$$

(a) Show that this particular choice of stress function automatically satisfies equilibrium.

(b) Assuming that Hooke's Law is of the form

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} \tag{A.9}$$
$$e_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} \tag{A.10}$$
$$e_{xy} = \frac{\sigma_{xy}(1+\nu)}{E} \tag{A.11}$$

show that the compatability equation reduces to

$$\phi_{,xxxx} + 2\phi_{,xxyy} + \phi_{,yyyy} = 0 \tag{A.12}$$

This is the biharmonic equation, which is often written as $\nabla^4\phi = 0$.

(c) What is the relation between E and the shear modulus μ and Poisson's ration ν?

(d) Note that the solution of Eq.(A.12) does not depend on elastic constants. Let's use this solution to solve a very simple stress problem. Consider a square of length a under hydrostatic pressure P. What are the stress components inside the box? (guess!) What is the stress function ϕ?

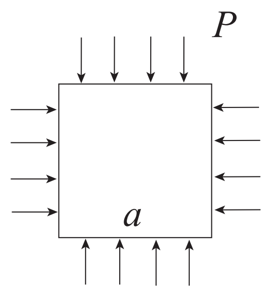


Figure A.2: A square of length a under hydrostatic pressure P.

A.3 Elastic Green Function

Problem 3.1 (10') Numerical calculation of Green's function.
(a) Write a Matlab program that returns C_{ijkl} given C_{11}, C_{12}, and C_{44} of an anisotropic elastic medium with cubic symmetry.

(b) Write a Matlab program that computes $(zz)_{ij}$ and $(zz)^{-1}_{ij}$ given C_{ijkl} and z_i. The elastic constants of Silicon are C_{11} = 161.6GPa, C_{12} = 81.6GPa, C_{44} = 60.3GPa. What are the values for all components of $g_{ij}(\mathbf{k})$ for $\mathbf{k} = [112]$ ($\mathbf{k}$ in unit of μm^{-1})?

(c) Write a Matlab program that computes $G_{ij}(\mathbf{x})$ given C_{ijkl} and $\mathbf{x}$. What are the values for all components of $G_{ij}(\mathbf{x})$ for $\mathbf{x} = [112]$ ($\mathbf{x}$ in unit of μm)? Plot $G_{33}(x, y)$ on plane $z = 1$.

Include a print out of your source code in your report. You may feel free to use other softwares (e.g. Mathematica) instead of Matlab if you prefer to do so.

Problem 3.2 (10') Reciprocal Theorem.
Use Betti's theorem (under zero body force),

$$\int_S \mathbf{t}^{(1)} \cdot \mathbf{u}^{(2)} dS = \int_S \mathbf{t}^{(2)} \cdot \mathbf{u}^{(1)} dS \tag{A.13}$$

to show that, the volume change of an isotropic medium with Young's modulus E and Possion's ratio ν under surface traction $\mathbf{t}^{(1)}$ is,

$$\delta V_1 = \int_S \frac{1-2\nu}{E} x_i t_i^{(1)} dS \tag{A.14}$$

Notice that the traction force satisfies,

$$\int_S t_i^{(1)} dS \;=\; 0 \tag{A.15}$$

$$\int_S \epsilon_{ijk} x_j t_k^{(1)} dS \;=\; 0 \tag{A.16}$$

[Hint: use auxiliary solution $\sigma_{ij}^{(2)} = \delta_{ij}$, i.e. the medium under unit hydrostatic tension.]

Problem 3.3 (10') Contact problem.
Consider a semi-infinite isotropic elastic medium filling the half space $x_3 \geq 0$. Let the shear modulus be μ and Poisson's ratio be ν. The Green's function for the half space is $G_{ij}^{\mathrm{h}}(\mathbf{x}, \mathbf{x}')$. If the force is only applied to the surface, i.e. $x_3' = 0$, then the Green's function can be written as,

$$G_{ij}^{\mathrm{h}}(\mathbf{x}, \mathbf{x}') = G_{ij}^{\mathrm{h}}(\mathbf{x} - \mathbf{x}') \tag{A.17}$$

Introduce function $F(\mathbf{x}) = x_3 \ln(x_3 + R) - R$ where $R = |\mathbf{x}|$. Then the surface Green's function can be expressed as (when the surface force is applied at $\mathbf{x}' = 0$),

$$G_{ij}^{\mathrm{h}}(\mathbf{x}) = \frac{1}{4\pi\mu} \left[\delta_{ij} \nabla^2 R - \partial_i \partial_j R - (-1)^{\delta_{i3}} (1 - 2\nu) \partial_i \partial_j F \right] \tag{A.18}$$

(a) What is the explicit form of $G_{33}^{\mathrm{h}}(\mathbf{x})$, i.e. the normal displacement in response to a normal surface force? What is the normal displacement $G_{33}^{\mathrm{h}}(x, y)$ on the surface ($x_3 = 0$)?

(b) Consider a spherical indentor with radius of curvature ρ punching on the surface along the x_3 axis. Let a be the radius of the contact area. The indentor is much stiffer than the substrate so that we can assume the substrate conforms to the shape of the indentor in the contact area, i.e.,

$$u_3(x, y) = d - \frac{x^2 + y^2}{2\rho} \tag{A.19}$$

where d is the maximum displacement on the surface and $r \equiv \sqrt{x^2 + y^2}$. Let the total indenting force be F. What is the pressure distribution on the surface $p(x, y)$?
[Hint: try the form $p(x, y) = B\sqrt{1 - (x/a)^2 - (y/a)^2}$ and determine B in terms of F. Show that $p(x, y)$ indeed gives rise to displacement according to Eq. (A.19).]

(c) What is the expression for the contact radius a in terms of indenting force F and indentor radius of curvature ρ?

(d) What is the expression for the maximum displacement d in terms of indenting force F and indentor radius of curvature ρ?

Note: you may find the following identity useful,

$$\int_{x'^2 + y'^2 \leq 1} \frac{\sqrt{1 - x'^2 - y'^2}\, dx' dy'}{\sqrt{(x - x')^2 + (y - y')^2}} = \frac{\pi^2}{2} \left(1 - \frac{x^2 + y^2}{2} \right) \tag{A.20}$$

A.4 Eshelby's Inclusion I

Problem 4.1 (15') Spherical inclusion.

(a) Derive the expressions for the auxiliary tensor $\mathcal{D}_{ijkl}$ for a spherical inclusion in an isotropic medium with shear modulus μ and Poisson's ratio ν.
[Hint: many components of $\mathcal{D}_{ijkl}$ are zero, unless there are repeated indices.]

(b) Derive the corresponding expressions for Eshelby's tensor $\mathcal{S}_{ijkl}$.

Problem 4.2 (15') Dilation field.

The "constrained" dilation of a transformed inclusion (not necessarily ellipsoidal) is,

$$\begin{aligned} u^{\rm c}_{i,i} &= \int_{S_0} \sigma^*_{kj} n_k(\mathbf{x}') G_{ij,i}(\mathbf{x}-\mathbf{x}') dS(\mathbf{x}') \\ &= -\int_{V_0} \sigma^*_{kj} G_{ij,ik}(\mathbf{x}-\mathbf{x}') dV(\mathbf{x}') \end{aligned} \tag{A.21}$$

(a) Show that if $e^*_{ij} = \varepsilon\delta_{ij}$ (pure dilational eigenstrain), then in isotropic elasticity the constrained dilation is constant inside the inclusion and independent of inclusion shape.

(b) What is $u^{\rm c}_{i,i}$ inside the inclusion in terms of ε?

Hint: The Green's function $G_{ij}(\mathbf{x})$ can be expressed in terms of second derivatives of $R = |\mathbf{x}|$.

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi\mu}\left[\delta_{ij}\nabla^2 R - \frac{1}{2(1-\nu)}\partial_i\partial_j R\right] \tag{A.22}$$

Notice that

$$\nabla^2 R = \frac{2}{R} \tag{A.23}$$

$$\nabla^2 \frac{1}{R} = -4\pi\delta(\mathbf{x}) \tag{A.24}$$

A.5 Eshelby's Inclusion II

Problem 5.1 (15') Use work method to derive the energy inside the inclusion E^I and inside the matrix E^M for an ellipsoidal inclusion in an infinite matrix. Follow the Eshelby's 4 steps to construct the inclusion.

(a) What are the forces applied to the inclusion and to the matrix in all 4 steps?

(b) What are the work done to the inclusion and to the matrix in all 4 steps?

(c) What is the elastic energy inside the inclusion E^I, and what is the elastic energy inside the matrix E^M at the end of step 4?

Problem 5.2 (15') Spherical inclusion. The Eshelby's tensor of a spherical inclusion inside an infinite medium is (see Lecture Note 2),

$$\mathcal{S}_{ijkl} = \frac{5\nu - 1}{15(1-\nu)}\delta_{ij}\delta_{kl} + \frac{4-5\nu}{15(1-\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{A.25}$$

Consider a spherical inclusion of radius R with a pure shear eigenstrain $e^*_{12} = \varepsilon$ (other components of $e^*_{ij} = 0$).

(a) What is the total elastic energy of the system E as a function of R?

(b) Now apply a uniform stress field $\sigma^A_{12} = \tau$ to the solid (other stress components are zero). What is the total elastic energy $E(R)$?

(c) What is the enthalpy of the system $H(R)$? What is the driving force for inclusion growth, i.e. $f(R) = -dH(R)/dR$?

[Hint: Consider the solid has a finite but very large volume V. The external stress is applied at the external surface. Volume V is so large that the Eshelby's solution in infinite solid remains valid.]

A.6 Cracks

Problem 6.1 (15') Plane strain and plain stress equivalence.
Let the elastic stiffness tensor of a homogeneous solid be C_{ijkl} and its inverse (compliance tensor) be S_{ijkl}. In the plane strain problem, $e_{13} = e_{23} = e_{33} = 0$. Let the 2-dimensional elastic stiffness tensor be c_{ijkl}, i.e.,

$$\sigma_{ij} = c_{ijkl}e_{kl} \quad \text{for } i,j,k,l, = 1,2 \quad \text{(plane strain)} \tag{A.26}$$

Obviously, $c_{ijkl} = C_{ijkl}$ for $i,j,k,l = 1,2$.

For a plain stress problem, $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. Let the 2-dimensional elastic compliance tensor be $\tilde{s}_{ijkl}$, i.e.,

$$e_{ij} = \tilde{s}_{ijkl}\sigma_{kl} \quad \text{for } i,j,k,l, = 1,2 \tag{A.27}$$

Obviously, $\tilde{s}_{ijkl} = S_{ijkl}$ for $i,j,k,l = 1,2$. The inverse of $\tilde{s}_{ijkl}$ (in 2-dimension) is the effective elastic stiffness tensor in plain stress, $\tilde{c}_{ijkl}$.

(a) For isotropic elasticity, write down the explicit expression for c_{ijkl} and $\tilde{c}_{ijkl}$.

(b) The Kolosov's constant is defined as

$$\kappa = \begin{cases} 3 - 4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases}$$

Express c_{ijkl} and $\tilde{c}_{ijkl}$ in terms of μ and κ. (They should have the same expression now.)

Problem 6.2 (15') Mode II crack
(a) Derive the eigenstrain of equivalent inclusion for a slit-like crack (width $2a$) under uniform shear σ_{12}^A in plane strain.

(b) Derive the stress distribution in front of the crack tip. What is the stress intensity factor $K_{II} = \lim_{r\to 0} \sigma_{12}(r)\sqrt{2\pi r}$, where $r = x - a$ is the distance from the crack tip?

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