Midterm Exam

Issued: Feb. 9, 2005
Due: Feb. 16, 2005 (in class)

Problem M.1 (20') Plane strain.
Consider an anisotropic elastic medium with elastic stiffness tensor C_{ijkl} under plane strain deformation. This means that the z-component of the displacement field is zero everywhere. The displacement fields in x and y directions are also independent of z. Mathematically, this can be written as,

$$\begin{aligned} u_3 &= 0 && (1)\\ u_{1,3} &= 0 && (2)\\ u_{2,3} &= 0 && (3) \end{aligned}$$

(a) What are the non-zero components of the strain field? What are the non-zero components of the stress field?

(b) In 2-dimension, the Hooke's law can be expressed as,

$$\sigma_{ij} = c_{ijkl} e_{kl} \tag{4}$$

where the indices now only goes from 1 to 2. What is the expression of c_{ijkl} in terms of C_{ijkl}?

(c) Suppose the medium is subjected to body force b_j ($j = 1, 2$), which is independent of z. What is the equilibrium condition in terms of the displacement fields u_j?

Problem M.2 (10') Green's function in 2D.
(a) What is the equilibrium equation for the Green's function $G_{ij}(\mathbf{x} - \mathbf{x}')$ in terms of c_{ijkl}, where $\mathbf{x}$, $\mathbf{x}'$ are 2-dimensional vectors? Notice that a point force in 2D corresponds to a line force in 3D.

(b) Solve the Green's function in Fourier space, i.e. $g_{ij}(\mathbf{k})$. Again $\mathbf{k}$ is a 2-dimensional vector.

(c) Solve the Green's function in real space $G_{ij}(\mathbf{x})$. Express the result in terms of x and θ,

where $x_1 = x\cos\theta$, $x_2 = x\sin\theta$. The final result can be expressed in terms of an integral over a unit circle.
Hint:

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ikx}}{|k|}\,\mathrm{d}k = -2\ln|x| \qquad \text{(up to a constant)} \tag{5}$$

Problem M.3 (30') Inclusion in 2D.
Consider an elliptic inclusion in the 2D medium that can occupies the area,

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 \leq 1 \tag{6}$$

Let its eigenstrain be e^*_{ij} $(i,j = 1,2)$. Define Eshelby's tensor $\mathcal{S}_{ijkl}$ and auxiliary tensor $\mathcal{D}_{ijkl}$ similarly as in the lecture, but with $i,j,k,l = 1,2$.

(a) Show that $\mathcal{S}_{ijkl}$ and $\mathcal{D}_{ijkl}$ are constants inside the inclusion (use anisotropic elasticity).

(b) What is c_{ijkl} in terms of μ and ν in isotropic elasticity?

(c) Derive the expressions for $\mathcal{S}_{ijkl}$ and $\mathcal{D}_{ijkl}$ for a circular inclusion in an isotropic medium (plane strain).

Problem M.4 (20') Void in 2D.
Consider a 2D isotropic medium (plain strain) containing a circular void with radius a, with a uniform loading σ^A_{22}.

(a) What is the eigenstrain e^*_{ij} of the equivalent inclusion?

(b) Determine the location in the matrix where the maximum stress $\sigma^{\max}_{22}$ is reached. The stress concentration factor SCF is defined as $\sigma^{\max}_{22}/\sigma^A_{22}$. Determine SCF for a circular void.