Midterm Exam

Chris Weinberger and Wei Cai

Issued: Feb. 9, 2005
Due: Feb. 16, 2005 (in class)

Problem M.1 (20') Plane strain.
Consider an anisotropic elastic medium with elastic stiffness tensor C_{ijkl} under plane strain deformation. This means that the z-component of the displacement field is zero everywhere. The displacement fields in x and y directions are also independent of z. Mathematically, this can be written as,

$$u_3 = 0 \tag{1}$$
$$u_{1,3} = 0 \tag{2}$$
$$u_{2,3} = 0 \tag{3}$$

(a) What are the non-zero components of the strain field? What are the non-zero components of the stress field?

(b) In 2-dimension, the Hooke's law can be expressed as,

$$\sigma_{ij} = c_{ijkl}e_{kl} \tag{4}$$

where the indices now only goes from 1 to 2. What is the expression of c_{ijkl} in terms of C_{ijkl}?

(c) Suppose the medium is subjected to body force b_j ($j = 1, 2$), which is independent of z. What is the equilibrium condition in terms of the displacement fields u_j?

Solution:

(a) Since $u_3 = 0$ for all z, then $u_{3,1} = u_{3,2} = u_{3,3} = 0$. In combination with $u_{1,3} = 0$ and $u_{2,3} = 0$ we have $e_{33} = e_{13} = e_{23} = 0$ and the non-zero components of the strain are e_{11}, e_{12}, and e_{22}. For a generally anisotropic material, all of the stresses can be related to the strains, thus none of the stresses are necessarily zero. For isotropic material, $\sigma_{13} = \sigma_{23} = 0$. The non-zero stress components are σ_{11}, σ_{12}, σ_{22}, σ_{33}.

(b) Since whenever $k = 3$ or $l = 3$, $e_{kl} = 0$,

$$\sigma_{ij} = C_{ijkl}e_{kl} \qquad \text{for } i, j, k, l = 1, 2 \tag{5}$$

i.e.,

$$c_{ijkl} = C_{ijkl} \qquad \text{for } i, j, k, l = 1, 2 \tag{6}$$

(c) The equations of equilibrium can be derived just as in the notes, no change necessary

$$c_{ijkl}u_{k,il} + b_j = 0 \tag{7}$$

Problem M.2 (10') Green's function in 2D.

(a) What is the equilibrium equation for the Green's function $G_{ij}(\mathbf{x} - \mathbf{x}')$ in terms of c_{ijkl}, where $\mathbf{x}$, $\mathbf{x}'$ are 2-dimensional vectors? Notice that a point force in 2D corresponds to a line force in 3D.

(b) Solve the Green's function in Fourier space, i.e. $g_{ij}(\mathbf{k})$. Again $\mathbf{k}$ is a 2-dimensional vector.

(c) Solve the Green's function in real space $G_{ij}(\mathbf{x})$. Express the result in terms of x and θ, where $x_1 = x \cos \theta$, $x_2 = x \sin \theta$. The final result can be expressed in terms of an integral over a unit circle.
Hint:

$$\int_{-\infty}^{\infty} \frac{e^{-ikx}}{|k|} \, dk = -2 \ln |x| \qquad \text{(up to a constant)} \tag{8}$$

Solution:

(a) We can derive the equilibrium for the Green's function just as in the notes, however noting that the vector $\mathbf{x}$ is a 2-dimensional vector and the delta function is a 2-dimensional delta function:

$$c_{imns}G_{ij,sm}(\mathbf{x} - \mathbf{x}') + \delta_{nj}\delta(\mathbf{x} - \mathbf{x}') = 0$$

(b) Note that here all of the vectors are 2-dimensional, and thus we only need to take two dimensional fourier transforms. Define 2-dimensional Fourier transform and its inverse as,

$$
\begin{aligned}
g_{ij}(\mathbf{k}) &= \int_{-\infty}^{\infty} \exp(-i\mathbf{k} \cdot \mathbf{x})G_{ij}(\mathbf{x}) \, d\mathbf{x} \\
G_{ij}(\mathbf{x}) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \exp(-i\mathbf{k} \cdot \mathbf{x})g_{ij}(\mathbf{k}) \, d\mathbf{k} \\
1 &= \int_{-\infty}^{\infty} \exp(i\mathbf{k} \cdot \mathbf{x})\delta(\mathbf{x}) \, d\mathbf{x} \\
\delta(\mathbf{x}) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \exp(-i\mathbf{k} \cdot \mathbf{x}) \, d\mathbf{k}
\end{aligned}
$$

2

Plugging in these definitions into the above equilibrium equations gives:

$$
\begin{aligned}
0 &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left(c_{imns} \frac{\partial^2}{\partial x_s \partial x_m} g_{ij}(\mathbf{k}) + \delta_{nj} \right) \exp(i\mathbf{k} \cdot \mathbf{x}) \, d\mathbf{k} \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left(-c_{imns} z_m z_s k^2 g_{ij}(\mathbf{k}) + \delta_{nj} \right) \exp(i\mathbf{k} \cdot \mathbf{x}) \, d\mathbf{k}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\delta_{nj} &= c_{imns} z_m z_s k^2 g_{ij}(\mathbf{k}) \\
&= (zz)_{ni} k^2 g_{ij}(\mathbf{k})
\end{aligned}
$$

where

$$
(zz)_{ni} \equiv c_{imns} z_m z_s
$$

The solution for the Green's function in 2-dimension is

$$
g_{ij}(\mathbf{k}) = \frac{(zz)_{ij}^{-1}}{k^2}
$$

(c) Taking the inverse Fourier transform,

$$
\begin{aligned}
G_{ij}(\mathbf{x}) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \exp(-i\mathbf{k} \cdot \mathbf{x}) \frac{(zz)_{ij}^{-1}}{k^2} \, d\mathbf{k} \\
&= \frac{1}{4\pi^2} \int_0^{\infty} \int_0^{2\pi} \exp(-ikx \cos\theta) \frac{(zz)_{ij}^{-1}}{k^2} k \, dk \, d\theta \\
&= \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \exp(-ikx \cos\theta) \frac{(zz)_{ij}^{-1}}{|k|} \, dk \, d\theta \\
&= \frac{1}{8\pi} \int_0^{2\pi} (zz)_{ij}^{-1} (-2 \ln|x \cos\theta|) \, d\theta \\
&= -\frac{1}{4\pi} \int_0^{2\pi} (zz)_{ij}^{-1} \ln|x \cos\theta| \, d\theta \\
&= -\frac{1}{4\pi} \int_0^{2\pi} (zz)_{ij}^{-1} (\ln x + \ln|\cos\theta|) \, d\theta \\
&= C \ln x + \Phi(\theta_0)
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
C &= -\frac{1}{4\pi} \int_0^{2\pi} (zz)_{ij}^{-1} \, d\theta \\
\Phi(\theta_0) &= -\frac{1}{4\pi} \int_0^{2\pi} (zz)_{ij}^{-1} \ln|\cos\theta| \, d\theta \\
x &= |\mathbf{x}| \\
\mathbf{x} &= (x \cos\theta_0, x \sin\theta_0)
\end{aligned} \tag{10}
$$

3

Problem M.3 (30') Inclusion in 2D.

Consider an elliptic inclusion in the 2D medium that can occupies the area,

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 \leq 1 \tag{11}$$

Let its eigenstrain be e_{ij}^* $(i, j = 1, 2)$. Define Eshelby's tensor $\mathcal{S}_{ijkl}$ and auxiliary tensor $\mathcal{D}_{ijkl}$ similarly as in the lecture, but with $i, j, k, l = 1, 2$.

(a) Show that $\mathcal{S}_{ijkl}$ and $\mathcal{D}_{ijkl}$ are constants inside the inclusion (use anisotropic elasticity).

(b) What is c_{ijkl} in terms of μ and ν in isotropic elasticity?

(c) Derive the expressions for $\mathcal{S}_{ijkl}$ and $\mathcal{D}_{ijkl}$ for a circular inclusion in an isotropic medium (plane strain).

Solution:

(a)

$$
\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= \int_{V_0} G_{ij,kl}(\mathbf{x} - \mathbf{x}')dV(\mathbf{x}') \\
&= \int_{V_0} \frac{\partial^2}{\partial x_k \partial x_l} \left[\frac{1}{(2\pi)^2} \int \exp\left[-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')\right] \frac{(zz)_{ij}^{-1}}{k^2} \, d\mathbf{k} \right] dV(\mathbf{x}') \\
&= -\frac{1}{(2\pi)^2} \int_{V_0} \int \exp\left[-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')\right] (zz)_{ij}^{-1} z_k z_l \, d\mathbf{k} \, dV(\mathbf{x}') \\
&= -\frac{1}{(2\pi)^2} \int \exp(-i\mathbf{k} \cdot \mathbf{x})(zz)_{ij}^{-1} z_k z_l Q(\mathbf{k}) \, d\mathbf{k}
\end{aligned} \tag{12}
$$

where

$$Q(\mathbf{k}) \equiv \int_{V_0} \exp(i\mathbf{k} \cdot \mathbf{x}')dV(\mathbf{x}') \tag{13}$$

Define

$$
\begin{aligned}
\boldsymbol{\lambda} &\equiv (\lambda_1, \lambda_2) = (k_1 a, k_2 b), \quad \lambda = |\boldsymbol{\lambda}| \\
\mathbf{R} &\equiv (R_1, R_2) = (x_1/a, x_2/b), \quad R = |\mathbf{R}| \\
\gamma &= (\mathbf{k} \cdot \mathbf{x})/k = (\boldsymbol{\lambda} \cdot \mathbf{R})/k \\
\beta &= \lambda/k
\end{aligned} \tag{14}
$$

4

Then

$$
\begin{aligned}
Q(\mathbf{k}) &\equiv \int_{V_0} \exp(i\mathbf{k} \cdot \mathbf{x}')dV(\mathbf{x}') \\
&= ab \int_{|\mathbf{R}| \leq 1} \exp(i\boldsymbol{\lambda} \cdot \mathbf{R}) \, \mathrm{d}\mathbf{R} \\
&= ab \int_0^1 \int_0^{2\pi} R \exp(i\lambda R \cos\theta) \, \mathrm{d}\theta \, \mathrm{d}R \\
&= 2\pi ab \int_0^1 R \, J_0(\lambda R) \, \mathrm{d}R \\
&= 2\pi ab \frac{J_1(\lambda)}{\lambda}
\end{aligned}
\tag{15}
$$

Therefore,

$$
\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{ab}{2\pi} \int \exp(-i\mathbf{k} \cdot \mathbf{x})(zz)_{ij}^{-1} z_k z_l \frac{J_1(\lambda)}{\lambda} \, \mathrm{d}\mathbf{k} \\
&= -\frac{ab}{2\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-ik\gamma)(zz)_{ij}^{-1} z_k z_l \frac{J_1(k\beta)}{k\beta} k \, \mathrm{d}k \, \mathrm{d}\theta \\
&= -\frac{ab}{2\pi} \int_0^{2\pi} (zz)_{ij}^{-1} z_k z_l \kappa(\gamma) \, \mathrm{d}\theta
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
\kappa(\gamma) &= \frac{1}{\beta} \int_0^{\infty} \exp(-ik\gamma) J_1(k\beta) \, \mathrm{d}k \\
&= \frac{1}{\beta^2} \left[1 - \frac{i\gamma}{\sqrt{\beta^2 - \gamma^2}} \right]
\end{aligned}
\tag{17}
$$

The derivations in Eq. (16) and (17) are carried out by `Mathematica`, whose outputs are given at the end of this solution for reference. Notice that $\mathcal{D}_{ijkl}(\mathbf{x})$ is real. Since $(zz)_{ij}^{-1} z_k z_l$ is also real, the imaginary part of $\kappa(\gamma)$ can be neglected. Therefore, as long as $\beta > |\gamma|$, we can write

$$
\kappa(\gamma) = \frac{1}{\beta^2}
\tag{18}
$$

which is independent of γ. Therefore $\mathcal{D}_{ijkl}(\mathbf{x})$ is independent of $\mathbf{x}$. $\beta > |\gamma|$ is satisfied if $\mathbf{x}$ is within the inclusion. This can be shown by the following. If $\mathbf{x}$ is inside the ellipse, then

$$
\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = R_1^2 + R_2^2 < 1
\tag{19}
$$

which means $R < 1$. Therefore,

$$
|\gamma| = |\boldsymbol{\lambda} \cdot \mathbf{R}|/k \leq |\boldsymbol{\lambda}| \cdot |\mathbf{R}|/k = \lambda R/k < \lambda/k = \beta
\tag{20}
$$

(b) The isotropic stiffness tensor for plane strain is

$$
\begin{aligned}
c_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&= \frac{2\mu\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&= \mu\left(\frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)
\end{aligned}
$$

(c) We have shown that inside an elliptic inclusion of an isotropic medium

$$
\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{ab}{2\pi}\int_0^{2\pi}(zz)_{ij}^{-1}z_k z_l\frac{1}{\beta^2}\,\mathrm{d}\theta
$$

For a circular inclusion, $a = b$, then $\beta = a$ and $\mathcal{D}$ becomes

$$
\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{1}{2\pi}\int_0^{2\pi}(zz)_{ij}^{-1}z_k z_l\,\mathrm{d}\theta
$$

Notice that

$$
\begin{aligned}
c_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
(zz)_{ij} &= \mu\delta_{ij} + (\lambda+\mu)z_i z_j \\
(zz)_{ij}^{-1} &= \frac{1}{\mu}\left(\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu}z_i z_j\right) = \frac{1}{\mu}\left(\delta_{ij} - \frac{1}{2(1-\nu)}z_i z_j\right)
\end{aligned}
$$

Therefore,

$$
\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\mu}\left(\delta_{ij} - \frac{1}{2(1-\nu)}z_i z_j\right)z_k z_l\,\mathrm{d}\theta
$$

Notice that $z_1 = \cos\theta$ and $z_2 = \sin\theta$, $\mathcal{D}_{ijkl}$ can be evaluated explicitly. Let us define

$$
H_{kl} \equiv \int_0^{2\pi} z_k z_l\,\mathrm{d}\theta
$$

and

$$
J_{ijkl} \equiv \int_0^{2\pi} z_i z_j z_k z_l\,\mathrm{d}\theta \tag{21}
$$

The only non-zero elements of H_{kl} are H_{11} and H_{22}, i.e.,

$$
H_{kl} = \delta_{kl}\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \pi\delta_{kl}
$$

Similarly J_{ijkl} is non-zero only when all four indices are the same or they come in pairs.

$$
J_{1111} = J_{2222} = \int_0^{2\pi}\cos^4\theta\,\mathrm{d}\theta = \frac{3\pi}{4}
$$

6

$$J_{1122} = J_{2211} = J_{1212} = J_{2121} = \int_0^{2\pi} \cos^2 \theta \sin^2 \theta = \frac{\pi}{4}$$

therefore

$$J_{ijkl} = \frac{\pi}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Thus

$$
\begin{aligned}
\mathcal{D}_{ijkl} &= -\frac{1}{2\pi\mu}\left(\delta_{ij}H_{kl} - \frac{1}{2(1-\nu)}J_{ijkl}\right)\\
&= -\frac{1}{2\pi\mu}\left(\delta_{ij}\delta_{kl}\pi - \frac{1}{2(1-\nu)}\frac{\pi}{4}(\delta_{ij}\delta_{kl}\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right)\\
&= -\frac{1}{16\mu(1-\nu)}\left((8-8\nu)\delta_{ij}\delta_{kl} - \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)\\
&= -\frac{1}{16\mu(1-\nu)}\left((7-8\nu)\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathcal{S}_{ijmn} &= -\frac{1}{2}c_{lkmn}(\mathcal{D}_{iklj} + \mathcal{D}_{jkli})\\
&= -\lambda\mathcal{D}_{ikkj}\delta_{mn} - \mu(\mathcal{D}_{inmj} + \mathcal{D}_{jnmi})
\end{aligned}
$$

$\mathcal{D}_{ikkj}$ can be evaluated by

$$\mathcal{D}_{ikkj} = -\frac{1}{16\mu(1-\nu)}\left((7-8\nu)\delta_{ik}\delta_{kj} - \delta_{ik}\delta_{kj} - \delta_{ij}\delta_{kk}\right)$$

Note, that now in two dimensions, $\delta_{kk} = 2$

$$
\begin{aligned}
\mathcal{D}_{ikkj} &= -\frac{1}{16\mu(1-\nu)}\left((7-8\nu)\delta_{ij}\delta_{kj} - \delta_{ij} - 2\delta_{ij}\right)\\
&= -\frac{(4-8\nu)}{16\mu(1-\nu)}\delta_{ij}\\
\lambda &= \frac{2\mu\nu}{1-2\nu}\\
\lambda\mathcal{D}_{ikkj} &= -\frac{\nu}{2(1-\nu)}\delta_{ij}
\end{aligned}
\tag{22}
$$

Thus

$$
\begin{aligned}
\mathcal{S}_{ijmn} &= \frac{\nu}{2(1-\nu)}\delta_{ij}\delta_{mn} + \frac{1}{16(1-\nu)}\left((6-8\nu)(\delta_{in}\delta_{jm} + \delta_{jn}\delta_{im}) - 2\delta_{ij}\delta_{mn}\right)\\
&= \frac{4\nu-1}{8(1-\nu)}\delta_{ij}\delta_{mn} + \frac{3-4\nu}{8(1-\nu)}(\delta_{in}\delta_{jm} + \delta_{jn}\delta_{im})
\end{aligned}
$$

Mathematica:

```
In[1]:= Assuming[{L>0,R>0}, Integrate[Exp[I*L*R*Cos[Theta]],{Theta,0,2*Pi}]]

Out[1]= 2 Pi BesselJ[0, L R]

In[2]:= Assuming[ {L>0}, Integrate[ R*BesselJ[0,L*R], {R,0,1}] ]

          BesselJ[1, L]
Out[2]= -------------
              L
In[3]:= Assuming[{b>0,g>0}, Integrate[Exp[-I*k*g]*BesselJ[1,k*b],{k,0,Infinity}]]


                g
         1 - --------------
                 2    2
             Sqrt[-b  + g ]
Out[3]= -------------------
                b
```

Problem M.4 (20') Void in 2D.
Consider a 2D isotropic medium (plain strain) containing a circular void with radius a, with a uniform loading σ_{22}^A.

(a) What is the eigenstrain e_{ij}^* of the equivalent inclusion?

(b) Determine the location in the matrix where the maximum stress $\sigma_{22}^{\max}$ is reached. The stress concentration factor SCF is defined as $\sigma_{22}^{\max}/\sigma_{22}^A$. Determine SCF for a circular void.

Solution:

(a) To solve this problem, we require the stress inside the inclusion to be zero, ie

$$\sigma_{ij}^A + \sigma_{ij}^I = \sigma_{ij}^A + \sigma_{ij}^c - \sigma_{ij}^* = 0$$

For a uniform applied stress σ_{22}^A the equations reduce to

$$
\begin{aligned}
\sigma_{22}^A + \sigma_{22}^c - \sigma_{22}^* &= 0 \\
\sigma_{11}^c - \sigma_{11}^* &= 0 \\
\sigma_{12}^c - \sigma_{12}^* &= 0
\end{aligned}
$$

Because the material is isotropic, the shear terms are completely decoupled, thus we can choose $e_{12}^* = 0$ which means that $\sigma_{12}^* = \sigma_{12}^c = 0$. The problem now becomes a "simple" problem of two equations and two unknowns (e_{11} and e_{22}).

8

Let us first express the stress $\sigma_{ij}^c - \sigma_{ij}^*$ in terms of e_{mn}^*.

$$
\begin{aligned}
\sigma_{ij}^c - \sigma_{ij}^* &= (c_{ijkl}\mathcal{S}_{klmn} - c_{ijmn})e_{mn}^* \\
&= c_{ijkl}\left(\mathcal{S}_{klmn} - \frac{1}{2}\delta_{km}\delta_{ln} - \frac{1}{2}\delta_{kn}\delta_{lm}\right)e_{mn}^* \\
&= [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\left[\frac{4\nu-1}{8(1-\nu)}\delta_{kl}\delta_{mn} - \frac{1}{8(1-\nu)}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})\right]e_{mn}^* \\
&= -\frac{\mu}{4(1-\nu)}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})e_{mn}^*
\end{aligned}
$$

This allows us to write the two equations of this problem as:

$$
\begin{aligned}
-\sigma_{22}^A &= -\frac{3\mu}{4(1-\nu)}e_{22}^* - \frac{\mu}{4(1-\nu)}e_{11}^* \\
0 &= -\frac{3\mu}{4(1-\nu)}e_{11}^* - \frac{\mu}{4(1-\nu)}e_{22}^*
\end{aligned}
$$

Solving the second equation gives

$$
e_{11}^* = -\frac{1}{3}e_{22}^*
$$

Plugging this result into the first equation gives

$$
\begin{aligned}
e_{22}^* &= \frac{3}{2}\frac{1-\nu}{\mu}\sigma_{22}^A \\
e_{11}^* &= -\frac{1}{2}\frac{1-\nu}{\mu}\sigma_{22}^A
\end{aligned}
$$

Now, σ_{11}^* and σ_{22}^* can be computed as

$$
\begin{aligned}
\sigma_{11}^* &= (\lambda + 2\mu)e_{11}^* + \lambda e_{22}^* \\
\sigma_{22}^* &= (\lambda + 2\mu)e_{22}^* + \lambda e_{11}^*
\end{aligned}
$$

which results in

$$
\begin{aligned}
\sigma_{11}^* &= \frac{(4\nu-1)(1-\nu)}{1-2\nu}\sigma_{22}^A \\
\sigma_{22}^* &= \frac{(4\nu-3)(1-\nu)}{1-2\nu}\sigma_{22}^A
\end{aligned}
$$

(b) The solution can be obtained by computing the jump of the stress over the interface. Since the stress inside the void is zero, the total jump in the stress should be equal to the stress at the inside surface of the matrix. The jump in the stress is

$$
[[\sigma_{ij}]] = \sigma_{ij}^* - c_{ijkl}(nn)_{km}^{-1}\sigma_{mn}^* n_n n_l
$$

First, lets write the jump in the constrained field $[[\sigma_{ij}^c]]$ as

$$
\begin{aligned}
[[\sigma_{ij}^c]] &= -c_{ijkl}(nn)_{km}^{-1}\sigma_{mn}^* n_n n_l \\
&= -\left[\frac{2\mu\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\frac{1}{\mu}\left[\delta_{km} - \frac{1}{2(1-\nu)}n_k n_m\right]\sigma_{mn}^* n_n n_l \\
&= -\left[\frac{\nu}{1-\nu}\delta_{ij}n_m n_n + \delta_{im}n_n n_j + \delta_{jm}n_n n_i - \frac{1}{1-\nu}n_i n_j n_m n_n\right]\sigma_{mn}^*
\end{aligned}
$$

The total jump in stress is

$$
\begin{aligned}
[[\sigma_{ij}]] &= \sigma_{ij}^* + [[\sigma_{ij}^c]] \\
&= \left[\delta_{im}\delta_{jn} - \frac{\nu}{1-\nu}\delta_{ij}n_m n_n - \delta_{im}n_n n_j - \delta_{jm}n_n n_i + \frac{1}{1-\nu}n_i n_j n_m n_n\right]\sigma_{mn}^*
\end{aligned}
$$

For this problem, we need to compute the SCF which is only dependent on $[[\sigma_{22}]]$.

$$
\begin{aligned}
[[\sigma_{22}]] &= \left(\delta_{2m}\delta_{2n} - \frac{\nu}{1-\nu}\delta_{22}n_m n_n - \delta_{2m}n_n n_2 - \delta_{2m}n_n n_2 + \frac{1}{1-\nu}n_2 n_2 n_m n_n\right)\sigma_{mn}^* \\
&= \sigma_{22}^* - \frac{\nu}{1-\nu}\sigma_{mn}^* n_m n_n - 2\sigma_{2n}^* n_n n_2 + \frac{1}{1-\nu}\sigma_{mn}^* n_m n_n n_2 n_2
\end{aligned}
$$

Because only σ_{11}^* and σ_{22}^* are non-zero,

$$
\begin{aligned}
[[\sigma_{22}]] &= \sigma_{22}^* - \frac{\nu}{1-\nu}\sigma_{mn}^* n_m n_n - 2\sigma_{22}^* n_2 n_2 + \frac{1}{1-\nu}\sigma_{mn}^* n_m n_n n_2 n_2 \\
&= \sigma_{22}^* - \frac{\nu}{1-\nu}(\sigma_{11}^* n_1 n_1 + \sigma_{22}^* n_2 n_2) - 2\sigma_{22}^* n_2 n_2 + \frac{1}{1-\nu}(\sigma_{11}^* n_1 n_1 + \sigma_{22}^* n_2 n_2)n_2 n_2
\end{aligned}
$$

Let $n_1 = \cos\theta$ and $n_2 = \sin\theta$, the above expression can be greatly simplified,

$$
\begin{aligned}
[[\sigma_{22}]] &= \left(4\cos^4\theta - \cos^2\theta\right)\sigma_{22}^A = f(\theta)\sigma_{22}^A \\
f(\theta) &\equiv 4\cos^4\theta - \cos^2\theta
\end{aligned}
\tag{23}
$$

Notice that $[[\sigma_{22}]]$ is independent of ν.

As shown in the figure, the maximum of $f(\theta)$ occurs at $\theta = 0$, with $f(0) = 3$. Therefore, the stress concentration factor is SCF $= \sigma_{22}^{\max}/\sigma_{22}^A = 3$.
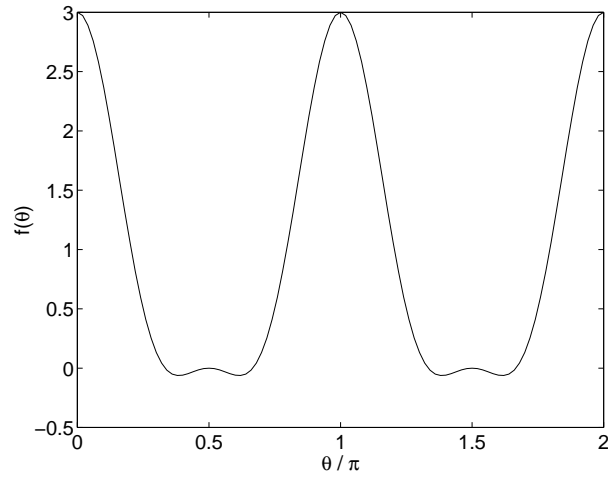
Figure 1: Stress jump $[[\sigma_{22}]]$ as a function of θ. $f(\theta) = [[\sigma_{22}]]/\sigma_{22}^{A} = 4\cos^4\theta - \cos^2\theta$.