Lecture Note 4. Eshelby’s Inhomogeneity

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1 Introduction

We can apply Eshelby’s solution of inclusions to other problems such as inhomogeneities, cracks, and dislocations to name a few. These solutions are modelled using a technique called the equivalent inclusion method, where the eigenstrain is chosen to model the specific problem. This is possible for ellipsoidal inhomogeneities because the stress and strain inside ellipsoidal inclusions are constant.

To start, let us consider a simple example. Suppose we cut a volume $V_0$ out of an infinite solid and fill it with a liquid to a pressure $p_0$. What are the stress, strain, and displacement fields inside the matrix? In principle, we could use the Green’s function as a direct method to solve this problem. Because the liquid exerts a force $T_j = p_0 \delta_{kj} n_k$ on the surface of the void, the displacement field inside the matrix should be,

$$u_i(x) = \int p_0 \delta_{kj} n_k \tilde{G}_{ij}(x, x') \, dx'$$

where $\tilde{G}_{ij}(x, x')$ is the Green’s function for an infinite body with a cavity. However, we do not know the expression for $\tilde{G}_{ij}(x, x')$, which is different from the Green’s function of an infinite body (without the cavity). Therefore, the formal solution in Eq. (1) is not very helpful in practice. Thus the remaining question is, can we express the displacement field in terms of infinite medium Green’s function, $G_{ij}(x, x') = G_{ij}(x - x')$? This turns out to be possible if the shape of the cavity is an ellipsoid.

The solution can be constructed using Eshelby’s equivalent inclusion method. The idea is to replace the liquid with an inclusion whose eigenstrain is chosen such that the stress field inside exactly matches that in the liquid, i.e., $\sigma^I_{ij} = -p_0 \delta_{ij}$. This is possible because we know the stress and strain in both the inclusion and liquid are constant. The required eigenstrain $e^*_ij$ of the equivalent inclusion can be obtained from Eshelby’s tensor $S_{ijkl}$. Because

$$\sigma^I_{ij} = \sigma^c_{ij} - \sigma^*_{ij} = C_{ijkl}(e^c_{kl} - e^*_{kl}) = C_{ijkl}(S_{klmn}e^*_{mn} - e_{kl})$$

Therefore

$$C_{ijkl}(S_{klmn} - \delta_{km}\delta_{ln})e^*_{mn} = -p_0 \delta_{ij}$$

from which we can solve for the equivalent eigenstrain $e^*_ij$. From this set of six equations we can solve for the six unkown equivalent eigenstrains. Once the eigenstrain is known, the displacements on the void surface $S_0$ can be calculated from

$$u_i = u^c_i = S_{ijkl}e^*_{kl}x_j$$

What is the elastic energy inside the matrix? It must be the same as the elastic energy inside the matrix when it contains the equivalent inclusion, instead of the liquid. The total elastic energy inside the matrix and the inclusion is,

$$E = E^I + E^M = -\frac{1}{2} \sigma^I_{ij} e^I_{ij} V_0$$

and the energy in the inclusion is

$$E^I = \frac{1}{2} \sigma^I_{ij} e^I_{ij} V_0 = \frac{1}{2} \sigma^I_{ij} (e^c_{ij} - e^*_{ij}) V_0$$
Therefore, the energy in the matrix is

\[ E^M = E - E^l = -\frac{1}{2} \sigma_{ij}^e e_{ij}^c V_0 \]
\[ = -\frac{1}{2} (-p_0 \delta_{ij}) S_{ijkl} e_{kl}^* V_0 \]
\[ = \frac{1}{2} p_0 S_{ijkl} e_{kl}^* V_0 \]

(7)

2 Transformed inhomogeneity

Let us now apply the same idea to solve the transformed inhomogeneity problem. A transformed inhomogeneity is the same as a transformed inclusion, except that it has a different elastic constant \( C'_{ijkl} \) from the matrix. Let us assume that the inhomogeneity is ellipsoidal in shape and has a volume \( V_0 \) bounded by a surface \( S_0 \). Suppose it undergoes a permanent transformation described by eigenstrain \( e_{ij}^* \). Our problem is to determine the stresses and strains distribution in the solid as well as its total elastic energy. Notice that we use \( ' \) to express all properties related to the inhomogeneity.

![Diagram of a linear elastic solid with a transformed inhomogeneity](image)

**Figure 1:** A linear elastic solid with volume \( V \) and a transformed inhomogeneity \( V_0 \), described by elastic constant \( C'_{ijkl} \) and eigenstrain \( e_{ij}^* \). While the problem can be defined when \( V_0 \) has a general shape, it can only be solved (elegantly) by Eshelby’s equivalent inclusion method when \( V_0 \) is an ellipsoid.

This problem is more complicated than the liquid-in-void problem in the previous section. This is because the inhomogeneity is a solid. To replace it with an equivalent inclusion, both the traction force and the displacement field on the interface \( S_0 \) should be matched. A sufficient condition is to match both the elastic stress and the total strain field inside the transformed inhomogeneity and inside the equivalent inclusion.

The stress inside the inhomogeneity is,

\[ \sigma_{ij}' = \sigma_{ij}^e - \sigma_{ij}^* = C'_{ijkl}(e_{kl}' - e_{kl}^*) \]

(8)
This should match the stress inside the equivalent inclusion,

\[ \sigma_{ij}^I = \sigma_{ij}^e - \sigma_{ij}^* = C_{ijkl}(e_{kl}^e - e_{kl}^*) \]  

(9)

The total strain inside the inhomogeneity is \( e_{ij}^e \), which must match the total strain side the equivalent inclusion \( e_{ij}^e \). Therefore,

\[ C_{ijkl}(e_{kl}^e - e_{kl}^*) = C_{ijkl}(e_{kl}^e - e_{kl}^*) \]  

(10)

Because \( e_{kl}^e = S_{klmn}e_{mn}^* \), we have,

\[ \left[(C_{ijkl} - C_{ijkl})S_{klmn} + C_{ijklmn}\right]e_{mn}^* = C_{ijkl}e_{kl} \]  

(11)

from which we can solve for the equivalent \( e_{mn}^* \) for the inclusion in terms of the eigenstrain \( e_{kl}^* \) of the transformed inhomogeneity.

The total strain inside the inhomogeneity is the same as the total strain inside the equivalent inclusion, i.e.,

\[ e_{ij}^e = e_{ij}^e = S_{ijkl}e_{kl} \]  

(12)

The stress inside the inhomogeneity is also the same as the stress inside the equivalent inclusion, i.e.,

\[ \sigma_{ij}' = \sigma_{ij}' = C_{ijkl}(e_{kl}^e - e_{kl}^*) = (C_{ijkl}S_{klmn} - C_{ijklmn})e_{mn}^* \]  

(13)

The elastic energy inside the matrix is the same in both the transformed inhomogeneity problem and the equivalent inclusion problem, i.e.,

\[ E^M = -\frac{1}{2}\sigma_{ij}^I e_{ij}^e V_0 \]  

(14)

However, the elastic energy inside the transformed inhomogeneity \( (E') \) and that inside the equivalent inclusion \( (E') \) are not the same. Specifically,

\[ E' = \frac{1}{2}\sigma_{ij}' e_{ij}' V_0 = \frac{1}{2}\sigma_{ij}'(e_{ij}^e - e_{ij}^*)V_0 = \frac{1}{2}\sigma_{ij}'(e_{ij}^e - e_{ij}^*)V_0 \]  

(15)

whereas,

\[ E = \frac{1}{2}\sigma_{ij}^I e_{ij}' V_0 = \frac{1}{2}\sigma_{ij}^I(e_{ij}^e - e_{ij}^*)V_0 \]  

(16)

Thus, the total energy for the solid with a transformed inhomogeneity is,

\[ E = E' + E^M = -\frac{1}{2}\sigma_{ij}^I e_{ij}' V_0 \]  

(17)

whereas the total energy of the equivalent inclusion problem is,

\[ E_{eq, inc} = E^I + E^M = -\frac{1}{2}\sigma_{ij}^I e_{ij}^* V_0 \]  

(18)
3 Inhomogeneity under uniform applied loads

Let us consider another important inhomogeneity problem where the inhomogeneity has no eigenstrain by itself. Instead, the solid containing the inhomogeneity is subjected to external loads. The load is uniform meaning that if the solid were homogeneous (no inhomogeneity) the stress strain fields should be uniform throughout the solid. The question now is, what are the stress and strain fields when the solid does contain the inhomogeneity. We can solve this problem when the inhomogeneity is an ellipsoid.

![Figure 2: A solid containing an inhomogeneity under uniform loads. The total stress strain fields can be constructed as a superposition of two sets of fields. (a) Let the entire body have a uniform strain field $e_{ij}^A$. We need to apply a body force $T_j = (\sigma_{ij}^A - \sigma_{ij}^A)n_i$ on interior surface $S_0$ to maintain equilibrium. (b) Apply body force $F_j = -T_j$ on $S_0$ to cancel the extra body force. The resulting stress strain fields are called $\sigma_{ij}^C$ and $e_{ij}^C$. Notice that this problem has a simple solution only when the inhomogeneity is an ellipsoid.](image)

Let us construct the stress strain fields inside the solid by superimposing two sets of fields. First, imagine that the solid containing the inhomogeneity is subjected to a uniform strain $e_{ij}^A$, which is the strain through out the solid under the applied load if the entire solid has elastic constant $C_{ijkl}$. The stress field inside the matrix is $\sigma_{ij}^A = C_{ijkl}e_{kl}^A$ while the stress field inside the inhomogeneity is $\sigma_{ij}^A' = C_{ijkl}'e_{kl}^A$. The equilibrium condition would not be satisfied, unless a body force $T_j = (\sigma_{ij}^A' - \sigma_{ij}^A)n_i$ is applied to the surface $S_0$ of the inhomogeneity.

To obtain the solution of the original problem, this body force must be removed. Thus, for the second set of elastic fields, imagine that we apply a body force $F_j = -T_j$ on the surface $S_0$ of the inhomogeneity. The solid is not subjected to external loads in this case.
Let the stress and strain field due to $F_j$ be $\sigma_{ij}^c$ and $e_{ij}^c$. Superimposing these two sets of fields, the elastic stress field inside the inhomogeneity is,

$$\sigma_{ij}' = \sigma_{ij}^A + \sigma_{ij}^c = C_{ijkl}(e_{kl}^A + e_{kl}^c)$$  \hspace{1cm} (19)

The total strain field inside the inhomogeneity is the same as its elastic strain (since $e_{ij}^* = 0$),

$$e_{ij}' = e_{ij}^A + e_{ij}^c$$  \hspace{1cm} (20)

![Figure 3: An equivalent inclusion problem that gives the same stress and total strain fields as the inhomogeneity problem in Fig.2. The stress strain fields can be constructed as superpositions of two sets of fields: (a) A homogeneous solid (zero eigenstrain) under uniform strain $e_{ij}^A$. (b) A solid containing an inclusion with eigenstrain $e_{ij}^*$ and zero applied load.](image)

At the same time, we can construct the stress strain fields of an equivalent inclusion with eigenstrain $e_{ij}^*$ in a solid under a uniform applied load. The elastic stress field inside the inclusion is,

$$\sigma_{ij}^I = \sigma_{ij}^A + e_{ij}^c - \sigma_{ij}^* = C_{ijkl}(e_{kl}^A + e_{kl}^c - e_{kl}^*)$$  \hspace{1cm} (21)

The total strain field inside the inclusion is,

$$e_{ij}^A + e_{ij}^c$$  \hspace{1cm} (22)

Similar to the problem in the previous section, both the elastic stress and the total strain have to match between the inhomogeneity and the inclusion problems. Therefore,

$$C_{ijkl}(e_{kl}^A + e_{kl}^c) = C_{ijkl}(e_{kl}^A + e_{kl}^c - e_{kl}^*)$$  \hspace{1cm} (23)

$$e_{ij}^A + e_{ij}^c = e_{ij}^A + e_{ij}^c$$  \hspace{1cm} (24)
Eq. (24) simply leads to \( e'_{ij} = e^e_{ij} \). Plug it into Eq. (23), we get,

\[
C'_{ijkl}(e^A_{kl} + e^e_{kl}) = C_{ijkl}(e^A_{kl} + e^e_{kl} - e^s_{kl})
\]

\[
\left[(C'_{ijkl} - C_{ijkl}) S_{lm} + C_{ijkl} S_{lm}ight] e^e_{mn} = (C_{ijkl} - C'_{ijkl}) e^A_{kl} \tag{26}
\]

From this we can solve for the equivalent eigenstrain \( e^e_{mn} \). Notice that \( e^e_{mn} \) is proportional to the difference in the elastic constants \( C_{ijkl} - C'_{ijkl} \) and the applied field \( e^A_{kl} \), as it should. Once the equivalent eigenstrain is known, the stress and strain fields can be easily obtained.

Now, let us determine the total elastic energy and enthalpy of the inhomogeneity problem. To compute total elastic energy, we measure the work done during a reversible path that creates the final configuration. Let system 1 be the solid with inhomogeneity under uniform strain \( e^A_{ij} \), as shown in Fig. 2(a). The elastic energy of this state is,

\[
E_1 = \frac{1}{2} \sigma^A_{ij} e^A_{ij} V_M + \frac{1}{2} \sigma^e_{ij} e^e_{ij} V_0 = \frac{1}{2} \sigma^A_{ij} e^A_{ij} V + \frac{1}{2} (\sigma^A_{ij} - \sigma^e_{ij}) e^e_{ij} V_0 \tag{27}
\]

where \( V_M \) is the volume of the matrix, \( V_0 \) is the volume of the inhomogeneity, and \( V \) is the total volume of the solid. In system 1, a body force \( T_j = (\sigma^A_{ij} - \sigma^e_{ij}) n_i \) is applied on \( S_0 \) to maintain equilibrium. We then gradually remove this body force and go to system 2, whose energy \( E_2 \) is the desired solution. Let \( \Delta W_{12} \) be the work done to the solid during this transformation, then \( E = E_2 = E_1 + \Delta W_{12} \). Notice that during this transformation, both the internal force on \( S_0 \) and the external force on \( S_{ext} \) do work. Let these two work contribution be \( \Delta W_{12}^{ext} \) and \( \Delta W_{12}^{int} \) respectively.

Let us first compute \( \Delta W_{12}^{int} \). During the transformation from \( E_1 \) to \( E_2 \), the body force on \( S_0 \) decreases from \( T_j \) to 0, so that the average body force is \( T_j / 2 \). The additional displacement on \( S_0 \) is \( u^e_j \). Thus,

\[
\Delta W_{12}^{int} = \frac{1}{2} \int_{S_0} T_j u^e_j \, dS
\]

\[
= \frac{1}{2} \int_{S_0} (\sigma^A_{ij} - \sigma^e_{ij}) n_i u^e_j \, dS
\]

\[
= \frac{1}{2} (\sigma^A_{ij} - \sigma^e_{ij}) \int_{S_0} n_i u^e_j \, dS
\]

\[
= \frac{1}{2} (\sigma^A_{ij} - \sigma^e_{ij}) \int_{V_0} e^e_{ij} \, dV
\]

\[
= \frac{1}{2} (\sigma^A_{ij} - \sigma^e_{ij}) e^e_{ij} V_0 \tag{28}
\]

Because the applied load does not change, the factor of \( \frac{1}{2} \) does not appear in \( \Delta W_{12}^{ext} \). Let \( T_j^A = \sigma^A_{ij} n_i \) be the traction force on the outer surface \( S_{ext} \), then

\[
\Delta W_{12}^{ext} = \int_{S_{ext}} T_j^A u^e_j \, dS \tag{29}
\]

Notice that \( u^e_j \) is the displacement field due to body force \( F_j = -T_j \) on \( S_0 \) (see Fig. 2(b)). By Betti’s theorem,

\[
\Delta W_{12}^{ext} = \int_{S_0} F_j (u^A_j + u^e_j) \, dS \tag{30}
\]
where \( u_j^A + u_j^{c'} \) is the displacement field due to applied force \( T_j^A \). Thus,

\[
\Delta W_{12}^{\text{ext}} = -\int_{S_0} (\sigma_{ij}^A - \sigma_{ij}^A)n_i(u_j^A + u_j^{c'})\,dS
\]

\[
= -(\sigma_{ij}^A - \sigma_{ij}^A)(e_{ij}^A + e_{ij}^c)V_0
\]

Therefore,

\[
E = E_1 + \Delta W_{12}^{\text{int}} + \Delta W_{12}^{\text{ext}}
\]

\[
= \frac{1}{2}\sigma_{ij}^A e_{ij}^A V + \frac{1}{2}(\sigma_{ij}^A - \sigma_{ij}^A)e_{ij}^AV_0 + \frac{1}{2}(\sigma_{ij}^A - \sigma_{ij}^A)e_{ij}^cV_0 - (\sigma_{ij}^A - \sigma_{ij}^A)(e_{ij}^A + e_{ij}^c)V_0
\]

\[
= \frac{1}{2}\sigma_{ij}^A e_{ij}^A V - \frac{1}{2}(\sigma_{ij}^A - \sigma_{ij}^A)(e_{ij}^A + e_{ij}^c)V_0
\]

\[
= \frac{1}{2}\sigma_{ij}^A e_{ij}^A V - \frac{1}{2}(\sigma_{ij}^A - \sigma_{ij}^A)e_{ij}^'V_0
\]

\[
\Delta H = H - E
\]

\[
= -E
\]

\[
= -\frac{1}{2}\sigma_{ij}^A e_{ij}^A V + \frac{1}{2}(\sigma_{ij}^A - \sigma_{ij}^A)e_{ij}^'V_0
\]

\[
= -\frac{1}{2}\sigma_{ij}^A e_{ij}^A V
\]

as the enthalpy of the solid without the inhomogeneity under applied load. Then

\[
\Delta H = H - H_0
\]

\[
= \frac{1}{2}(\sigma_{ij}^A - \sigma_{ij}^A)e_{ij}^'V_0
\]

\[
= \frac{1}{2}(C_{ijkl} - C_{ijkl})e_{ij}^A e_{ij}^'V_0
\]
In the limit of \( \delta C_{ijkl} \equiv C'_{ijkl} - C_{ijkl} \) very small, then

\[
e_{ij}' = e_{ij}^A + O(\delta C_{ijkl})
\]

\[
\Delta H = \frac{1}{2} \delta C_{ijkl} e_{ij}^A e_{kl}^A V_0 + O(\delta C_{ijkl})^2
\]

(38)

(39)

Eshelby calls the expression \( \Delta H = \frac{1}{2} \delta C_{ijkl} e_{ij}^A e_{kl}^A V_0 \) the *Feynman-Hellman theorem*.

In the above derivation, the volume \( V \) of the solid is assumed to be large enough so that the image effects at \( S_{\text{ext}} \) are ignored. When the image effects are accounted for, the above results can be rewritten as,

\[
\Delta H = \frac{1}{2} (C'_{ijkl} - C_{ijkl}) \int_{V_0} e_{ij}^A e_{ij}' \, dV
\]

(40)

where \( e_{ij}' = e_{ij}^A + e_{ij}^c + e_{ij}^{im} \), and \( e_{ij}^{im} \) accounts for the image contribution. Note that the identity \( H = -E \) and the *Feynman-Hellman theorem* holds independent of the boundary condition on \( S_{\text{ext}} \).

**References**