Lecture Note 3. Eshelby’s Inclusion II

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# 1 Inclusion energy in an infinite solid

So far we have obtained the expressions for the stress, strain and displacement field both inside and outside the inclusion. An important question is: what is the total elastic energy \( E \) of the solid containing an inclusion? In this and subsequent sections, we derive the expressions for \( E \), which we refer to as the *inclusion energy* for brevity. However, we emphasize that \( E \) is the total elastic energy of the solid containing an inclusion. \( E \) includes the elastic energy stored both inside *and* outside the inclusion. For example, if we obtain \( E \) as a function of the inclusion size, then the derivative of \( E \) provides the driving force for the expansion (or shrinkage) of the inclusion. Notice that this is the case only if \( E \) is the total elastic energy, not just the energy stored inside the inclusion.

There are two ways to obtain the expression for the total energy \( E \). First, we can integrate the elastic energy density both inside and outside the inclusion, using the field expressions we have already obtained. Second, we can obtain the elastic energy \( E \) by measuring the work done in a virtual experiment that transforms a solid system with zero elastic energy to the solid containing an inclusion. In this section, we take the first approach. The work method is discussed in the next section, which leads to identical results but may provide more physical insight.

For clarity, let us introduce some symbols to describe the elastic fields inside and outside the inclusion. Let the elastic (stress, strain, displacement) fields inside the inclusion be denoted by a superscript \( I \), and the elastic fields outside the inclusion (in the matrix) be denoted by a superscript \( M \). Notice that whenever the superscript \( I \) or \( M \) is used, the fields only include the elastic component. For a homogeneous infinite solid, the elastic fields in the matrix and the inclusion are,

<table>
<thead>
<tr>
<th>matrix</th>
<th>inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_{ij}^M = e_{ij}^c )</td>
<td>( e_{ij}^I = e_{ij}^c - e_{ij}^* )</td>
</tr>
<tr>
<td>( \sigma_{ij}^M = \sigma_{ij}^c )</td>
<td>( \sigma_{ij}^I = \sigma_{ij}^c - \sigma_{ij}^* )</td>
</tr>
<tr>
<td>( u_i^M = u_i^c )</td>
<td>( u_i^I = u_i^c - e_{ij}^* x_j )</td>
</tr>
</tbody>
</table>

Therefore, the total elastic energy is,

\[
E = \frac{1}{2} \int_{V_0} \sigma_{ij}^I e_{ij}^I \, dV + \frac{1}{2} \int_{V_\infty - V_0} \sigma_{ij}^M e_{ij}^M \, dV \tag{1}
\]

Rewriting \( E \) in terms of displacements, we have

\[
E = \frac{1}{4} \int_{V_0} \sigma_{ij}^I (u_{i,j}^I + u_{j,i}^I) \, dV + \frac{1}{4} \int_{V_\infty - V_0} \sigma_{ij}^M (u_{i,j}^M + u_{j,i}^M) \, dV \tag{2}
\]

and noting the symmetry of the stress tensor

\[
E = \frac{1}{2} \int_{V_0} \sigma_{ij}^I u_{j,i}^I \, dV + \frac{1}{2} \int_{V_\infty - V_0} \sigma_{ij}^M u_{j,i}^M \, dV \tag{3}
\]

Now, the derivative can be factored out using the following rule

\[
\sigma_{ij} u_{i,j} = (\sigma_{ij} u_j)_i - \sigma_{ij,i} u_j \tag{4}
\]
\begin{equation}
E = \frac{1}{2} \int_{V_0} (\sigma_{ij} u_j^I)_{,i} - \sigma_{ij,i} dV + \frac{1}{2} \int_{V_{\infty}-V_0} (\sigma_{ij} u_j^M)_{,i} - \sigma_{ij,i} u_j^M dV \tag{5}
\end{equation}

The body is assumed not to have any body forces acting on it, thus the divergence of the stress tensor, \(\sigma_{ij,i}\), is zero. Thus

\begin{equation}
E = \frac{1}{2} \int_{V_0} (\sigma_{ij} u_j^I)_{,i} dV + \frac{1}{2} \int_{V_{\infty}-V_0} (\sigma_{ij} u_j^M)_{,i} dV \tag{6}
\end{equation}

We wish to now use Gauss’s theorem on this equation. We need to careful about the sign of the unit normal vector that points outside the integration volume. Let the normal vector pointing out of the inclusion volume \(V_0\) be \(n_i^\text{out}\). Let the unit normal vector pointing out of the outer surface of the matrix \(V_{\infty}\) (at infinity) be \(n_i^\infty\). Applying Gauss’s theorem,

\begin{equation}
E = \frac{1}{2} \int_{S_0} \sigma_{ij} u_j^I n_i^\text{out} dS - \frac{1}{2} \int_{S_0} \sigma_{ij} u_j^M n_i^\text{out} dS + \frac{1}{2} \int_{S_{\infty}} \sigma_{ij} u_j^M n_i^\infty dS \tag{7}
\end{equation}

We expect that the surface integral over \(S_{\infty}\) should vanish as it approaches infinity. To show this, let \(S_{\infty}\) be a spherical surface whose radius \(R\) approaches infinity. Notice that

\begin{align}
D_{ijkl}(x) &= \int_{V_0} G_{ij,kl}(x-x') dV(x') \tag{8}
\end{align}

Because \(G_{ijkl}(x-x') \to R^{-3}\) where \(R = |x|\), for large \(R\), then \(D_{ijkl}(x) \to R^{-3}\). Therefore,

\begin{align}
e_{ij}^M &= \mathcal{O} \left( \frac{1}{R^3} \right) \tag{9} \\
\sigma_{ij}^M &= \mathcal{O} \left( \frac{1}{R^3} \right) \tag{10} \\
dS &= \mathcal{O} \left( R^2 \right) \tag{11}
\end{align}

Thus

\begin{equation}
\int_{S_{\infty}} \sigma_{ij}^M u_j^M n_i^\text{out} dV \to 0 \quad \text{as} \quad R \to \infty \tag{12}
\end{equation}

Combining the two integrals over \(S_0\),

\begin{equation}
E = \frac{1}{2} \int_{S_0} (\sigma_{ij} u_j^I - \sigma_{ij}^M u_j^M) n_i^\text{out} dS \tag{13}
\end{equation}

Although the stress across the inclusion interface \(S_0\) does not have to be continuous, the traction force across the interface must be continuous, i.e.,

\begin{equation}
\sigma_{ij}^I n_i^\text{out} = \sigma_{ij}^M n_i^\text{out} \tag{14}
\end{equation}

which leads to

\begin{equation}
E = \frac{1}{2} \int_{S_0} \sigma_{ij}^I (u_j^I - u_j^M) n_i^\text{out} dS \tag{15}
\end{equation}
From the definition of (elastic displacement fields) $u^I_j$ and $u^M_j$, we have

$$u^I_j - u^M_j = (u^c_j - e^{*}_{jk}x_k) - u^c_j = -e^{*}_{jk}x_k$$  \hspace{1cm} (16)

Thus

$$E = -\frac{1}{2} \int_{S_0} \sigma^I_{ij} n^\text{out}_i e^{*}_{jk} x_k \, dS$$  \hspace{1cm} (17)

Therefore, we have expressed the total elastic energy $E$ in terms of a surface integral over $S_0$, the inclusion interface. We can further simplify this expression by transforming the integral back into a volume integral (over the inclusion volume $V_0$).

$$E = -\frac{1}{2} \int_{V_0} (\sigma^I_{ij} e^{*}_{jk} x_k) \, dV = -\frac{1}{2} \int_{V_0} e^{*}_{jk} (\sigma^I_{ij,i} x_k + \sigma^I_{ij} x_{k,i}) \, dV = -\frac{1}{2} \int_{V_0} e^{*}_{jk} \sigma^I_{ij} \, dV = -\frac{1}{2} e^{*}_{ij} \int_{V_0} (\sigma^c_{ij} - \sigma^*_{ij}) \, dV$$  \hspace{1cm} (18)

For an ellipsoidal inclusion, the stress inside is a constant, thus

$$E = -\frac{1}{2} (\sigma^c_{ij} - \sigma^*_{ij}) e^{*}_{ij} V_0 = -\frac{1}{2} \sigma^I_{ij} e^{*}_{ij} V_0$$  \hspace{1cm} (19)

If the volume is not an ellipsoid, we can still write the energy in terms of the average stress in the inclusion

$$E = -\frac{1}{2} \overline{\sigma^I_{ij}} e^{*}_{ij} V_0$$  \hspace{1cm} (20)

where

$$\overline{\sigma^I_{ij}} \equiv \frac{1}{V_0} \int_{V_0} \sigma^c_{ij}(\mathbf{x}) \, dV(\mathbf{x}) - \sigma^*_{ij}$$  \hspace{1cm} (21)

Suppose that we wish to account for how much of the energy is stored inside the inclusion and how much is stored in the matrix. The energy store inside the inclusion is

$$E^I = \frac{1}{2} \int_{V_0} \sigma^I_{ij} e^I_{ij} \, dV$$

For ellipsoidal inclusion, the stress and strain are constant inside, hence

$$E^I = \frac{1}{2} \sigma^I_{ij} e^I_{ij} V_0 = \frac{1}{2} \sigma^I_{ij} (e^c_{ij} - e^*_{ij}) V_0$$
Since the total elastic energy is
\[ E = -\frac{1}{2} \sigma_{ij} e_{ij}^* V_0 \]
the elastic energy stored inside the matrix must be,
\[ E^M = E - E^I = -\frac{1}{2} \sigma_{ij} e_{ij}^c V_0 \]

### 2 Inclusion energy by work method

In this section, we re-derive the expressions in the previous section concerning the inclusion energy using a different approach. Rather than integrating the strain energy density over the entire volume, we make use of the fact that the stored elastic (potential) energy in the solid must equal to the work done to it. By considering a virtual experiment that transforms a stress-free solid into a solid containing an inclusion and accounting for the work done along the way, we can derive the total elastic energy (or the elastic energy stored within the inclusion or the matrix) using considerably less math than before.

To better illustrate this method, let us consider a simple example. Consider a mass \( M \) attached to a linear spring with stiffness \( k \). Let \( E_0 \) be the equilibrium state of the system under no applied force. Obviously \( E_0 = 0 \).

Define the origin as the position of the mass at this state. Suppose we gradually apply a force to the mass until the force reaches \( F_1 \). At this point the mass must have moved by a distance \( x_1 = F_1/k \). Let the energy of this state be \( E_1 \). The work done in moving the mass from 0 to \( x_1 \) equals to the average force \( \overline{F} \) applied to the mass times the distance travelled (\( x_1 \)). Because the initial force is 0 and the final force is \( F_1 \), the average force is \( \overline{F} = F_1/2 \). Therefore, the work done in moving the mass from 0 to \( x_1 \) is,
\[ W_{01} = \overline{F} x_1 = \frac{1}{2} F_1 x_1 = \frac{1}{2} k x_1^2 \]  \( (22) \)

Hence
\[ E_1 = E_0 + W_{01} = \frac{1}{2} k x_1^2 \]  \( (23) \)

Suppose we further increase the force to \( F_2 \) and the system reaches a new state at \( x_2 = F_2/k \) with energy \( E_2 \). Since the initial force during this transformation is \( F_1 \) and the final force is \( F_2 \), the average force is \( \overline{F} = (F_1 + F_2)/2 \). The mass moves by a distance of \( x_2 - x_1 \) under this force. Therefore the work done is
\[ W_{12} = \frac{1}{2} (F_1 + F_2) (x_2 - x_1) = \frac{1}{2} k (x_2^2 - x_1^2) \]  
(24)

Hence

\[ E_2 = E_1 + W_{12} = \frac{1}{2} k x_2^2 \]  
(25)

Now, let's apply this method to Eshelby’s inclusion problem. Let us consider the four steps in Eshelby’s construction of a solid containing an inclusion. Recall that after step 1, the inclusion is outside the matrix. The inclusion has undergone a deformation due to its eigenstrain. No forces are applied to either the inclusion or the matrix. Obviously, the total elastic energy at this state is \( E_1 = 0 \).

In step 2, we apply a set of traction forces on the inclusion surface \( S_0 \). At the end of step 2, the traction forces are \( T_j = -\sigma^*_ij n_j \) and the displacements on the surface are \( u_j = -e^*_kj x_k \). Therefore, the work done in step 2 is

\[
W_{12} = \frac{1}{2} \int_{S_0} T_j(x) u_j(x) \, dS(x) \\
= \frac{1}{2} \int_{S_0} \sigma^*_ij n_i e^*_kj x_k \, dS(x) \\
= \frac{1}{2} \sigma^*_ij e^*_kj \int_{S_0} x_k n_i \, dS(x)
\]
(26)

and using Gauss’s theorem

\[
W_{12} = \frac{1}{2} \sigma^*_ij e^*_kj \int_{V_0} x_{k,j} \, dV(x) \\
= \frac{1}{2} \sigma^*_ij e^*_kj \int_{V_0} \delta_{ki} \, dV(x) \\
= \frac{1}{2} \sigma^*_ij e^*_ij V_0
\]
(27)

In step 3, the inclusion is put inside the matrix with the traction force unchanged. No work is done in this step, i.e. \( W_{23} = 0 \). In step 4, the traction force \( T_j \) is gradually reduced to zero. Both the inclusion and the matrix displace over a distance of \( u^*_j \). Since the traction force is \( T_j \) at the beginning of step 4 and 0 at the end of step 4, the average traction force
is, again, $T_j/2$. The work done to the entire system (inclusion + matrix) is

$$W_{34} = \frac{1}{2} \int_{S_0} T_j(x) u_j^c(x) \, dS(x)$$

$$= -\frac{1}{2} \int_{S_0} \sigma^*_{ij} n_i u_j^c(x) \, dS(x)$$

$$= -\frac{1}{2} \sigma^*_{ij} \int_{V_0} u_{ji}^c(x) \, dS(x)$$

$$= -\frac{1}{2} \sigma^*_{ij} \int_{V_0} e_{ij}^c \, dS(x)$$

$$= -\frac{1}{2} \sigma^*_{ij} e_{ij}^c V_0$$

(28)

The total elastic energy at the end of step 4 is,

$$E = E_1 + W_{12} + W_{23} + W_{34}$$

$$= 0 + \frac{1}{2} \sigma^*_{ij} e_{ij}^c V_0 + 0 - \frac{1}{2} \sigma^*_{ij} e_{ij}^c V_0$$

$$= -\frac{1}{2} (\sigma_{ij} - \sigma^*_{ij}) e_{ij}^c V_0$$

$$= -\frac{1}{2} \sigma^*_{ij} e_{ij}^c V_0$$

(29)

which is exactly the same as Eq. (19).

The same approach can also be applied to obtain the elastic energy stored inside the inclusion ($E^I$) or inside the matrix ($E^M$). In step 4, the matrix also exert force on the inclusion, which does work as the interface $S_0$ moves. This leads to a transfer of elastic energy from the inclusion to the matrix.

## 3 Inclusion energy in a finite solid

Let us now consider a problem with an inclusion in a finite solid. Again, the stress-strain fields in this case can be solved by superpositions. Suppose the finite solid assumes the stress-strain fields of an infinite solid containing an inclusion, as solved previously. Then we must apply a set of traction forces $\tilde{T}_j$ to the outer surface $S_{\text{ext}}$ of the solid to maintain equilibrium. To obtain the solution of a finite solid with zero traction on its outer surface, we need to remove $\tilde{T}_j$ on $S_{\text{ext}}$. This is equivalent to apply a cancelling traction force $\tilde{F}_j = -\tilde{T}_j$ on the outer surface $S_{\text{ext}}$ of the finite solid. Let $\epsilon_{ij}^{\text{im}}, \sigma_{ij}^{\text{im}}$ and $u_{ij}^{\text{im}}$ be the strain, stress and displacement fields in response to the surface traction $\tilde{F}_j$. They are called image fields. In this case, the elastic fields inside the matrix and the inclusion are,

<table>
<thead>
<tr>
<th>matrix</th>
<th>inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{ij}^M = e_{ij}^c + e_{ij}^{\text{im}}$</td>
<td>$e_{ij}^c = e_{ij}^c - e_{ij}^* + e_{ij}^{\text{im}}$</td>
</tr>
<tr>
<td>$\sigma_{ij}^M = \sigma_{ij}^c + \sigma_{ij}^{\text{im}}$</td>
<td>$\sigma_{ij}^c = \sigma_{ij}^c - \sigma_{ij}^* + \sigma_{ij}^{\text{im}}$</td>
</tr>
<tr>
<td>$u_{ij}^M = u_{ij}^c + u_{ij}^{\text{im}}$</td>
<td>$u_{ij}^c = u_{ij}^c - e_{ij}^* x_j + u_{ij}^{\text{im}}$</td>
</tr>
</tbody>
</table>
Obviously, the image fields satisfy the condition,

\[
e^{\text{im}}_{ij}(x) = \frac{1}{2}(u^{\text{im}}_{i,j}(x) + u^{\text{im}}_{j,i}(x)) \tag{30}
\]

\[
\sigma^{\text{im}}_{ij}(x) = C_{ijkl} e^{\text{im}}_{kl}(x) \tag{31}
\]

Notice that the image fields are generally not uniform within the inclusion. The free traction boundary condition on the outer surface \( S_{\text{ext}} \) can be expressed as,

\[
\sigma^{\text{M,ext}}_{ij} = 0 \quad (\text{on } S_{\text{ext}}) \tag{32}
\]

Similar to Eq. (7), the total elastic energy in the solid can be expressed in terms of surface integrals,

\[
E = \frac{1}{2} \int_{S_0} (\sigma^I_{ij} u^I_j - \sigma^M_{ij} u^M_j) n^\text{out}_i \, dS + \int_{S_{\text{ext}}} \sigma^M_{ij} u^M_j n^\text{ext}_i \, dS \tag{33}
\]

Because of Eq. (32), the second integral does not contribute. Using the traction continuity argument \((\sigma^I_{ij} n^\text{out}_i = \sigma^M_{ij} n^\text{out}_i)\) as before, we get

\[
E = \frac{1}{2} \int_{S_0} \sigma^I_{ij} (u^I_j - u^M_j) n^\text{out}_i \, dS \tag{34}
\]

Again, using \(u^I_j - u^M_j = -e^{*}_{ik} x_k\), we get

\[
E = -\frac{1}{2} \int_{S_0} \sigma^I_{ij} e^{*}_{jk} x_k n^\text{out}_i \, dS \tag{35}
\]

This is the same as Eq. (17) except that the stress field inside the inclusion now contains the image component. Define

\[
\sigma^{I,\infty}_{ij} \equiv \sigma^c_{ij} - \sigma^*_{ij} \tag{36}
\]

as the stress field inside the inclusion in an infinite medium. Then

\[
\sigma^I_{ij}(x) = \sigma^{I,\infty}_{ij} + \sigma^{\text{im}}_{ij}(x) \tag{37}
\]

Similarly, define

\[
E_\infty \equiv -\frac{1}{2} \int_{S_0} \sigma^{I,\infty}_{ij} e^{*}_{jk} x_k n^\text{out}_i \, dS = -\frac{1}{2} \sigma^{I,\infty}_{ij} e^{*}_{ij} V_0 \tag{38}
\]

as the inclusion energy in an infinite solid. Then the inclusion energy in a finite solid is,

\[
E = E_\infty - \frac{1}{2} \int_{S_0} \sigma^{\text{im}}_{ij} e^{*}_{jk} x_k n^\text{out}_i \, dS \tag{39}
\]
Converting the second integral into volume integral, we have

\[
E = E_\infty - \frac{1}{2} \int_{V_0} (\sigma_{ij}^{im} e_{jk}^* x_k)_i dV
\]

\[
= E_\infty - \frac{1}{2} \int_{V_0} \sigma_{ij}^{im} e_{ij}^* dV
\]

\[
= E_\infty - \frac{1}{2} e_{ij}^* \int_{V_0} \sigma_{ij}^{im} dV
\]

\[
= E_\infty - \frac{1}{2} \sigma_{ij}^{im} e_{ij}^* V_0
\]

\[
= E_\infty + E_{im}
\]

where

\[
\sigma_{ij}^{im} \equiv \frac{1}{V_0} \int_{V_0} \sigma_{ij}^{im}(x) dV(x)
\]

(40)

is the averaged image stress inside the inclusion.

\[
E_{im} \equiv -\frac{1}{2} \sigma_{ij}^{im} e_{ij}^* V_0
\]

(41)

is the “image” contribution to the total inclusion energy. The average stress inside the inclusion is,

\[
\sigma_{ij}^I \equiv \sigma_{ij}^{I,\infty} + \sigma_{ij}^{im}
\]

(42)

Thus the total inclusion energy is still related to the averaged stress inside the inclusion as

\[
E = -\frac{1}{2} \sigma_{ij}^I e_{ij}^* V_0
\]

(43)

The results of the total inclusion energy for ellipsoidal inclusion under various boundary conditions are summarized below.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Total Elastic Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infinite solid</td>
<td>(E = -\frac{1}{2} \sigma_{ij}^I e_{ij}^* V_0)</td>
</tr>
<tr>
<td>Finite solid with zero traction</td>
<td>(E = -\frac{1}{2} \sigma_{ij}^I e_{ij}^* V_0) (\sigma_{ij}^I = \sigma_{ij}^{I,\infty} + \sigma_{ij}^{im})</td>
</tr>
</tbody>
</table>

4 Colonetti’s theorem

We now wish to study the energy of a solid containing an inclusion and also subjecting to applied forces at its outer surface. Before we do that, let us first prove Colonetti’s theorem, which is very helpful to study such problems. Colonetti’s theorem [1] states that

There is no cross term in the total elastic energy of a solid, between the internal stress field and the applied stress field.
However, there is an interaction energy term between the internal and applied fields when the energy of the applied loads is included. Colonetti’s theorem can greatly simplify the energy expressions when we apply stress to a finite solid containing an inclusion. To appreciate Colonetti’s theorem, we need to be specific about the meaning of \textit{internal} and \textit{applied} stress fields. Let us start with a stress-free homogenous solid with outer surface $S_{\text{ext}}$. Define \textit{internal} stress fields as the response to a heterogeneous field of eigenstrain inside the solid with zero traction on $S_{\text{ext}}$. Define \textit{applied} stress fields as the response to a set of tractions on $S_{\text{ext}}$ when there is no eigenstrain inside the solid.

Let us consider two states of stress. State 1 is purely internal, and state 2 is “applied”. The total elastic energy inside the solid for these two states are,

\begin{align*}
E^{(1)} &= \frac{1}{2} \int_V \sigma_{ij}^{(1)} e_{ij}^{(1)} \, dV \\
E^{(2)} &= \frac{1}{2} \int_V \sigma_{ij}^{(2)} e_{ij}^{(2)} \, dV
\end{align*}

Now consider a state 1+2 which is the superposition of state 1 and 2. Its total elastic energy should be,

\begin{align*}
E^{(1+2)} &= \frac{1}{2} \int_V (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)})(e_{ij}^{(1)} + e_{ij}^{(2)}) \, dV \\
&= \frac{1}{2} \int_V (\sigma_{ij}^{(1)} e_{ij}^{(1)} + \sigma_{ij}^{(1)} e_{ij}^{(2)} + \sigma_{ij}^{(2)} e_{ij}^{(1)} + \sigma_{ij}^{(2)} e_{ij}^{(2)}) \, dV \\
&= E^{(1)} + E^{(2)} + E^{(1-2)}
\end{align*}

where

\begin{equation}
E^{(1-2)} \equiv \frac{1}{2} \int_V (\sigma_{ij}^{(2)} e_{ij}^{(1)} + \sigma_{ij}^{(1)} e_{ij}^{(2)}) \, dV \tag{44}
\end{equation}

is the “interaction term” between state 1 and state 2. Colonetti’s theorem states that $E^{(1-2)}$ must be zero, which we will prove below. First, we note that

\begin{align*}
\sigma_{ij}^{(2)} e_{ij}^{(1)} &= C_{ijkl} e_{kj}^{(2)} e_{ij}^{(1)} \\
\sigma_{ij}^{(1)} e_{ij}^{(2)} &= C_{ijkl} e_{kj}^{(1)} e_{ij}^{(2)}
\end{align*}

so that

\begin{equation}
\sigma_{ij}^{(2)} e_{ij}^{(1)} = \sigma_{ij}^{(1)} e_{ij}^{(2)} \tag{45}
\end{equation}

\begin{equation}
E^{(1-2)} = \int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} \, dV = \int_V \sigma_{ij}^{(2)} e_{ij}^{(1)} \, dV \tag{46}
\end{equation}

Since there is no body force, $\sigma_{ij,i}^{(1)} = 0$. Therefore,

\begin{equation}
E^{(1-2)} = \int_V (\sigma_{ij}^{(1)} u_{ij}^{(2)})_i \, dV \tag{47}
\end{equation}
Now, we wish to apply Gauss’s theorem to convert the volume integral into a surface integral. However, to use Gauss’s theorem, the integrand must be continuous inside the entire volume \( V \). However, this is not necessarily the case if the eigenstrain field \( e_{ij}^*(x) \) is not sufficiently smooth. For example, in Eshelby’s transformed inclusion problem, \( e_{ij}^*(x) \) is not continuous at the inclusion surface. As a result, the internal stress field \( \sigma_{ij}^{(1)}(x) \) is not continuous at the inclusion surface either.

However, for clarity, let us assume for the moment that the eigenstrain field \( e_{ij}^*(x) \) and the internal stress field \( \sigma_{ij}^{(1)}(x) \) are sufficiently smooth for the Gauss’s theorem to apply. This corresponds to the case of thermal strain induced by a smooth variation of temperature inside the solid. In this case,

\[
E^{(1-2)} = \int_V (\sigma_{ij}^{(1)} u_j^{(2)})_i \, dV = \int_{S_{\text{ext}}} n_i^{\text{ext}} \sigma_{ij}^{(1)} u_j^{(2)} \, dS \tag{48}
\]

Since \( \sigma_{ij}^{(1)} \) is a purely internal stress state,

\[
\sigma_{ij}^{(1)} n_i^{\text{ext}} = 0 \quad (\text{on } S_{\text{ext}}) \tag{49}
\]

Hence

\[
E^{(1-2)} = 0 \tag{50}
\]

which is Colonetti’s theorem.

Let us now consider the case where the eigenstrain field \( e_{ij}^*(x) \) and the internal stress field \( \sigma_{ij}^{(1)}(x) \) are piecewise smooth inside various inclusion volumes \( V_K \) as well as in the matrix \( V - \sum_K V_K \). Let \( n_i^{\text{out},K} \) be the outward normal vector of inclusion volume \( V_K \). We can apply Gauss’s theorem in each inclusion and the matrix separately, which gives,

\[
E^{(1-2)} = \int_{S_{\text{ext}}} \sigma_{ij}^{(1)} u_j^{(2)} n_i^{\text{ext}} \, dS + \sum_K \int_{S_K} (\sigma_{ij}^{(1),K} - \sigma_{ij}^{(1)}) u_j^{(2)} n_i^{\text{out},K} \, dS \tag{51}
\]

where \( \sigma_{ij}^{(1),K} \) is the stress inside the volume \( V_K \) and \( \sigma_{ij}^{(1)} \) is the stress in matrix. The traction force across the inclusion interface must be continuous, i.e.,

\[
(\sigma_{ij}^{K} - \sigma_{ij}^{(1)}) n_i^{\text{out},K} = 0 \quad \text{for any } K, \tag{52}
\]

where the summation is not implied over \( K \). Therefore, again we have,

\[
E^{(1-2)} = \int_{S_{\text{ext}}} \sigma_{ij}^{(1)} u_j^{(2)} n_i^{\text{ext}} \, dS = 0 \tag{53}
\]

which is Colonetti’s theorem.

Colonetti’s theorem only deals with the elastic strain energy that is stored inside the solid (internal energy). When the system is under applied load, its evolution proceeds
towards minimizing its *enthalpy*, which is the internal energy subtracting the work done by the external force. For example, the enthalpy of a system under external pressure \( p \) is 
\[
H = E + pV.
\]
The enthalpy for the solid under study is,
\[
H = E^{(1+2)} - \Delta W_{LM}
\] (54)
\( \Delta W_{LM} \) is the work done by the loading mechanism,
\[
\Delta W_{LM} = \int_{S_{ext}} \sigma_{ij}^{(2)} (u_{j}^{(1)} + u_{j}^{(2)}) n_{i}^{ext} \, dS
\] (55)
which can be written as
\[
\Delta W_{LM} = \Delta W_{LM}^{(1-2)} + \Delta W_{LM}^{(2)}
\] (56)
where
\[
\Delta W_{LM}^{(1-2)} = \int_{S} \sigma_{ij}^{(2)} u_{j}^{(1)} n_{i}^{ext} \, dS
\]
\[
\Delta W_{LM}^{(2)} = \int_{S} \sigma_{ij}^{(2)} u_{j}^{(2)} n_{i}^{ext} \, dS
\]
\( \Delta W_{LM}^{(1-2)} \) can be regarded as the cross term between the two stress states in the total enthalpy.

5 Finite solid with applied tractions

We now apply Colonetti’s theorem to our problem of an inclusion in a finite solid under a set of applied tractions. We will use superscript \( A \) to denote the fields in response to the applied tractions when the eigenstrain vanishes (no inclusion). Let superscript \( F \) denote the fields of an inclusion in a finite solid under zero external tractions (as in section 3). From Colonetti’s theorem,
\[
E = E^{A} + E^{F}
\] (57)
where
\[
E^{A} = \frac{1}{2} \int_{V} \sigma_{ij}^{A} e_{ij}^{A} \, dV
\] (58)
\[
E^{F} = -\frac{1}{2} (\sigma_{ij}^{I,\infty} + \sigma_{ij}^{lm}) e_{ij}^{*} V_{0}
\] (59)
The enthalpy of the system is
\[
H = E - \Delta W_{LM}
\] (60)
where the \( A \) and \( F \) fields do have interaction terms in the work term \( \Delta W_{LM} \), i.e.,
\[
\Delta W_{LM} = \Delta W_{LM}^{A} + \Delta W_{LM}^{A-F}
\] (61)
\[
\Delta W_{LM}^{A} = \int_{S_{ext}} \sigma_{ij}^{A} u_{j}^{A} n_{i}^{ext} \, dS = \int_{V} \sigma_{ij}^{A} e_{ij}^{A} \, dV = 2E^{A}
\] (62)
\[
\Delta W_{LM}^{A-F} = \int_{S_{ext}} \sigma_{ij}^{A} u_{j}^{F} n_{i}^{ext} \, dS
\] (63)
We would like to express $\Delta W_{LM}^{A-F}$ in terms of an integral over the inclusion volume $V_0$. The result is,

$$\Delta W_{LM}^{A-F} = e^*_i \sigma^A_{ij} V_0$$

(64)

where

$$\sigma^A_{ij} \equiv \frac{1}{V_0} \int_{V_0} \sigma^A_{ij} dV$$

(65)

To show that this is the case, first note that $\sigma^F_{ij} n^\text{ext}_i = 0$ on the surface $S_{\text{ext}}$. Thus

$$\Delta W_{LM}^{A-F} = \int_{S_{\text{ext}}} (\sigma^A_{ij} u^F_{ij} - \sigma^F_{ij} u^A_{ij}) n^\text{ext}_i dS$$

where the superscript $M$ denotes the fields in the matrix. Consider a volume integral of the same integrand over the matrix volume $V_M = V - V_0$,

$$0 = \int_{V_M} (\sigma^A_{ij} e^F_{ij} - \sigma^F_{ij} e^A_{ij}) dV$$

$$= \int_{S_{\text{ext}}} (\sigma^A_{ij} u^F_{ij} - \sigma^F_{ij} u^A_{ij}) n^\text{ext}_i dS - \int_{S_0} (\sigma^A_{ij} u^F_{ij} - \sigma^F_{ij} u^A_{ij}) n^\text{out}_i dS$$

which means

$$\int_{S_{\text{ext}}} (\sigma^A_{ij} u^F_{ij} - \sigma^F_{ij} u^A_{ij}) n^\text{ext}_i dS = \int_{S_0} (\sigma^A_{ij} u^F_{ij} - \sigma^F_{ij} u^A_{ij}) n^\text{out}_i dS$$

(66)

Hence

$$\Delta W_{LM}^{A-F} = \int_{S_0} (\sigma^A_{ij} u^F_{ij} - \sigma^F_{ij} u^A_{ij}) n^\text{out}_i dS$$

(67)

Notice that the integral is on the matrix side of the inclusion interface. We can similarly write out the volume integral inside the inclusion

$$0 = \int_{V_0} (\sigma^A_{ij} e^F_{ij} - \sigma^F_{ij} e^A_{ij}) dV$$

$$= \int_{S_0} (\sigma^A_{ij} u^F_{ij} - \sigma^F_{ij} u^A_{ij}) n^\text{out}_i dS$$

which means that

$$\int_{S_0} \sigma^A_{ij} u^F_{ij} n^\text{out}_i dS = \int_{S_0} \sigma^F_{ij} u^A_{ij} n^\text{out}_i dS$$

(68)
Substituting this into Eq. (67) and noting the traction continuity condition 
\( \sigma_{ij}^{F,I} n_i^{\text{out}} = \sigma_{ij}^{F,M} n_i^{\text{out}} \), we have,

\[
\Delta W_{LM}^{A-F} = \int_{S_0} \sigma_{ij}^A (u_{j}^{F,M} - u_{j}^{F,I}) n_i^{\text{out}} \, dS
\]
\[
= \int_{S_0} \sigma_{ij}^A e_{ij}^* x_k n_i^{\text{out}} \, dS
\]  \( (69) \)

Using Gauss’s theorem, we get,

\[
\Delta W_{LM}^{A-F} = \int_{V_0} \sigma_{ij}^A e_{ij}^* \, dV
\]
\[
= -e_{ij}^* \int_{V_0} \sigma_{ij}^A \, dV
\]
\[
= -e_{ij}^* \sigma_{ij}^A V_0
\]  \( (70) \)

The major results of this lecture note are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>total elastic energy</th>
<th>total enthalpy</th>
</tr>
</thead>
<tbody>
<tr>
<td>infinite solid</td>
<td>( E = -\frac{1}{2} \sigma_{ij}^I e_{ij}^* V_0 )</td>
<td></td>
</tr>
<tr>
<td>finite solid with zero traction</td>
<td>( E = -\frac{1}{2} \sigma_{ij}^I e_{ij}^* V_0 )</td>
<td>( \bar{\sigma}<em>{ij} = \sigma</em>{ij}^I + \sigma_{ij}^{\text{im}} )</td>
</tr>
</tbody>
</table>
| finite solid with traction | \( E = E^A + E^F \) | \( H = E - \Delta W_{LM} \)
\( E^A = \frac{1}{2} \int_V \sigma_{ij}^A e_{ij}^A \, dV \) | \( = E^A + E^F - \Delta W_{LM}^A - \Delta W_{LM}^{A-F} \)
\( E^F = -\frac{1}{2} \sigma_{ij}^I e_{ij}^* V_0 \) | \( = E^F - E^A - \sigma_{ij}^A e_{ij}^* V_0 \)

References