Problem Set 1 Solution
Index Notation and Gauss's Theorem

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Problem 1.1 (10') Index notation.

(a) Show that $\epsilon_{mkq}\epsilon_{nkq} = 2\delta_{mn}$.

$$\begin{aligned}
\epsilon_{mkq}\epsilon_{nkq} &= \delta_{mn}\delta_{kk} - \delta_{mk}\delta_{nk} \\
&= 3\delta_{mn} - \delta_{mn} \\
&= 2\delta_{mn}
\end{aligned}$$

(b) Consider a rank-two tensor $p_{ij} = a\delta_{ij} + bz_iz_j$, where $\mathbf{z}$ is a unit vector ($z_iz_i = 1$). Find the inverse q_{ij} of p_{ij}, which is defined through $q_{ij}p_{jk} = \delta_{ik}$.
[Hint: suppose q_{ij} also has the form of $q_{ij} = c\delta_{ij} + dz_iz_j$.]

$$\begin{aligned}
q_{ij}p_{jk} &= (c\delta_{ij} + dz_iz_j)(a\delta_{jk} + bz_jz_k) \\
&= ac\delta_{ik} + adz_iz_k + cbz_iz_k + bdz_iz_jz_jz_k \\
&= ac\delta_{ik} + (ad + cb + bd)\, z_iz_k = \delta_{ik}
\end{aligned}$$

For this is to be true we require $ac = 1$, which implies $c = \frac{1}{a}$, and $ad + cb + bd = 0$. Thus

$$\begin{aligned}
(a+b)d &= -cb \\
d &= -\frac{cb}{a+b} \\
d &= -\frac{b}{a^2+ab}
\end{aligned}$$

Thus

$$q_{ij} = \frac{1}{a}\delta_{ij} - \frac{b}{a^2+ab}z_iz_j$$

Problem 1.2 (10') Tensor symmetry.

(a) Show that if A_{ij} is a symmetric tensor, and B_{ij} is an arbitrary tensor, then,

$$A_{ij}B_{ij} = A_{ij}B_{(ij)} \tag{1}$$

If A_{ij} is symmetric, then $A_{ij} = A_{ji}$ and

$$\begin{aligned}
(A_{ij} - A_{ji})\, B_{ij} &= 0 \\
A_{ij}B_{ij} - A_{ji}B_{ij} &= 0 \\
A_{ij}B_{ij} - AijB_{ji} &= 0 \\
A_{ij}\,(B_{ij} - B_{ji}) &= 0 \\
A_{ij}B_{[ij]} &= 0
\end{aligned}$$

Now, B can be decomposed into its symmetric and antisymmetric parts

$$A_{ij}\left(B_{(ij)} + B_{[ij]}\right) = A_{ij}B_{(ij)}$$

(b) Show that if A_{ij} is an antisymmetric tensor, then

$$A_{ij}a_i a_j = 0$$

If A_{ij} is antisymmetric then $A_{ij} = -A_{ji}$ and

$$\begin{aligned}
A_{ij} + A_{ji} &= 0 \\
(A_{ij} + A_{ji})\, a_i a_j &= 0 \\
A_{ij}a_i a_j + A_{ji}a_i a_j &= 0 \\
2A_{ij}a_i a_j &= 0 \\
A_{ij}a_i a_j &= 0
\end{aligned}$$

Problem 1.3 (10') Gauss's Theorem.

(a) For a elastic body V with surface S in equilibrium under surface traction T_i and zero body force ($b_i = 0$), show that

$$\int_S T_i u_i \mathrm{d}S = \int_V \sigma_{ij} e_{ij} \mathrm{d}V$$

where u_i, σ_{ij}, e_{ij} are displacement, stress and strain fields.
[Hint: Use the result in Problem 1.2.]

Start with

$$\begin{aligned}
\int_S T_i u_i \mathrm{d}S &= \int_S \sigma_{ij} u_j n_i \mathrm{d}S \\
&= \int_V [\sigma_{ij} u_j]_{,i} \, \mathrm{d}V \\
&= \int_V [\sigma_{ij,i} u_j + \sigma_{ij} u_{j,i}] \, \mathrm{d}V \\
&= \int_V \sigma_{ij} u_{j,i} \mathrm{d}V \\
&= \int_V \sigma_{ij} e_{ij} \mathrm{d}V
\end{aligned}$$

(b) Show that the average stress in the elastic body under zero body force is,

$$\overline{\sigma}_{ij} = \frac{1}{2V} \int_S (T_i \, x_j + T_j \, x_i) \mathrm{d}S$$

$$\begin{aligned}
\int_S T_i \, x_j \mathrm{d}S &= \int_S \sigma_{ik} \, x_j n_k \mathrm{d}S \\
&= \int_V [\sigma_{ik} \, x_j]_{,k} \, \mathrm{d}V \\
&= \int_V [\sigma_{ik,k} \, x_j + \sigma_{ik} \, x_{j,k}] \, \mathrm{d}V \\
&= \int_V \sigma_{ij} \mathrm{d}V
\end{aligned}$$

similarly

$$\int_S T_j \, x_i \mathrm{d}S = \int_V \sigma_{ji} \mathrm{d}V$$

Which gives

$$\int_S (T_i \, x_j + T_j \, x_i) \mathrm{d}S = 2 \int_V \sigma_{ij} \mathrm{d}S$$

The definition of $\overline{\sigma}_{ij}$ is

$$\overline{\sigma}_{ij} = \frac{1}{V} \int_S \sigma_{ij} \mathrm{d}V$$

which gives our final result that

$$\overline{\sigma}_{ij} = \frac{1}{2V} \int_S (T_i \, x_j + T_j \, x_i) \mathrm{d}S$$