Fourier integral representation of the Green function for an anisotropic elastic half-space†

By Kevin P. Walker
Engineering Science Software, Inc.,
Smithfield, Rhode Island 02917, U.S.A.

This paper describes the development of a Fourier integral representation of the Green function for an anisotropic elastic half-space. The representation for an isotropic material is integrated in closed form and shown to reduce to Mindlin's solution. An application of the anisotropic representation is made to deduce the exact displacement caused by a two-dimensional periodic vertical force distribution applied to the interior of a half-space with cubic material symmetry.

1. Introduction

Eshelby's (1957, 1959, 1961) nascent treatment of the displacement produced in an infinite isotropic medium by ellipsoidal inclusions undergoing stress-free transformation strains (eigenstrains) has stimulated the production of a voluminous literature on the subject. A keystone ingredient in Eshelby's treatment of inclusion problems is the elastic Green function for an infinite medium, first deduced for an isotropic material by Lord Kelvin in 1848. More recent reviews dealing with eigenstrain distributions in infinite anisotropic media have been given by Bacon *et al.* (1979) and Mura (1987). In these references the displacement $u_i(\boldsymbol{x})$ due to a body force distribution of eigenstrain,

$$f_j(\boldsymbol{x}') = -C_{jlmn}\frac{\partial \varepsilon^*_{mn}(\boldsymbol{x}')}{\partial x'_l}, \tag{1.1}$$

is readily accessible in the form:

$$u_i(\boldsymbol{x}) = -\iiint_V G_{ij}(\boldsymbol{x},\boldsymbol{x}')\,C_{jlmn}\frac{\partial \varepsilon^*_{mn}(\boldsymbol{x}')}{\partial x'_l}\,\mathrm{d}V(\boldsymbol{x}'), \tag{1.2}$$

where repeated tensor suffixes imply summation from 1 to 3.

In this formula $G_{ij}(\boldsymbol{x},\boldsymbol{x}')$ are the components of the Green function tensor which give the displacement in the x_i-direction at the field point $\boldsymbol{x}$ due to the application of a unit point force in the x_j-direction at the source point $\boldsymbol{x}'$; C_{jlmn} are the components of the elasticity tensor with respect to the x_1, x_2, x_3 coordinate system; and V denotes the region over which the body force in equation

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(1.1) is non-zero. For materials which exhibit isotropic or hexagonal symmetry explicit expressions for the Green function tensor components are available (Mura 1987), but the representation for an infinite anisotropic medium is not explicitly available and it is usually given in the form of a triple Fourier integral (see equation (3.8)). Even when explicit algebraic expressions for the Green function are available, it is often convenient to obtain the displacement $u_i(\boldsymbol{x})$ by substituting the Fourier integral representation of $G_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ into equation (1.2). The order of the Fourier wave vector integrations and the spatial integrations over the volume V can then be interchanged in any manner that eases their evaluation.

The treatment of inclusion problems in an isotropic half-space has received some attention in the literature (Mindlin & Cheng 1950; Sneddon 1951; Lin & Tung 1962; Owen & Mura 1967; Owen 1971*a*, *b*; Chiu 1978; Seo & Mura 1979; Mura 1987), but the treatment is complicated by the traction-free boundary condition at the surface. This condition has commonly been treated by evaluating the surface traction on the boundary plane produced by the eigenstrain distribution in an infinite medium and then applying an equal and opposite traction on the boundary plane. Early work on the application of surface forces to isotropic half-spaces was done by Lamé & Clapeyron (1831) and Boussinesq (1885).

The displacement in an isotropic or hexagonal half-space can be obtained by substituting Mindlin's (1953) or Pan & Chou's (1979) expression for $G_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ in equation (1.2). For a general anisotropic half-space it would be convenient to have a Fourier integral representation of $G_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ since this would yield the displacement from equation (1.2) by simple quadrature. Even for an isotropic half-space a Fourier integral representation would prove to be convenient by rendering the Fourier wave vector and spatial integrations as interchangeable items.

In this paper we derive the Fourier integral representation of $G_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ for an anisotropic half-space and show that it is composed of two terms. The first term is the Green function for an infinite anisotropic medium and is represented as a triple Fourier integral, while the second term is represented as a quadruple Fourier integral which accounts for the effect of the traction-free surface. Willis (1966) and later Barnett & Lothe (1975) obtained Fourier integral representations of the surface Green function for an anisotropic half-space, which give the displacement on the surface due to an application of a point force on the surface. It would be interesting in future work to compare these with the Fourier integral representation developed in the present paper when the field and source points lie on the half-space surface.

The present paper is organized on the following lines. A detailed derivation of the Fourier integral representation is first given in § 2. Since a detailed derivation of the anisotropic Green function for an elastic half-space may not be of interest to those researchers who wish only to use the Fourier integral representation, the relevant formulæ have been collected together in § 3. In § 4 we show that the quadruple Fourier integral represents the displacement at the field point $\boldsymbol{x}$ due to the cancellation of the surface traction on the boundary plane of the half-space in the infinite medium when a unit force is applied at the source point $\boldsymbol{x}'$. In § 5 the Fourier integral representation of the Green function for an isotropic half-space is deduced and Mindlin's solution is recovered by integration. Finally, in § 6 we apply the Fourier integral representation to determine the exact displacement caused by a two-dimensional periodic vertical force distribution applied to the interior of an elastic half-space which possesses cubic material symmetry.

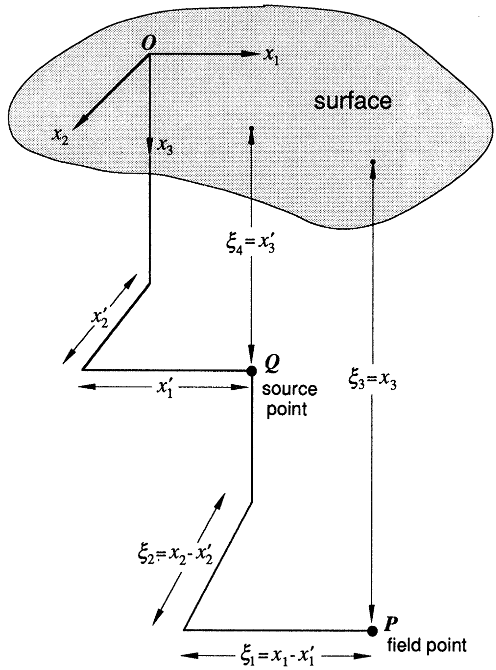

Figure 1. Coordinate systems for half-space.

2. Derivation of Fourier integral representation

We consider the determination of the elastic Green function for a semi-infinite anisotropic medium,

$$-\infty < x_1 < \infty, \quad -\infty < x_2 < \infty, \quad x_3 \geq 0_+,$$

whose coordinate system is shown in figure 1.

If the material is homogeneously anisotropic, then the Green function $G_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ must be invariant with respect to a rigid translation of the field point $\boldsymbol{x}$ and source point $\boldsymbol{x}'$ in any plane parallel to the surface $x_3 = 0$, and must therefore depend on $\boldsymbol{x}$ and $\boldsymbol{x}'$ in terms of the four-vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$, where

$$\xi_1 = x_1 - x_1', \qquad \xi_2 = x_2 - x_2', \qquad \xi_3 = x_3, \qquad \xi_4 = x_3'. \tag{2.1}$$

The displacement at $\boldsymbol{x}$ due to a point force $\boldsymbol{f}\,(\boldsymbol{x}')$ applied at the point $\boldsymbol{x}'$ is given by

$$u_i\,(\boldsymbol{x}) = G_{ij}(\boldsymbol{\xi}) f_j\,(\boldsymbol{x}')\,. \tag{2.2}$$

The strain at $\boldsymbol{x}$ is then

$$\varepsilon_{im}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial G_{ij}(\boldsymbol{\xi})}{\partial x_m} + \frac{\partial G_{mj}(\boldsymbol{\xi})}{\partial x_i}\right) f_j\,(\boldsymbol{x}')\,, \tag{2.3}$$

and the associated stress is

$$\sigma_{pq}(\boldsymbol{x}) = C_{pqim}\varepsilon_{im}(\boldsymbol{x}) = C_{pqim}\frac{\partial G_{ij}(\boldsymbol{\xi})}{\partial x_m} f_j\,(\boldsymbol{x}')\,, \tag{2.4}$$

in which use has been made of the symmetry of C_{pqim} with respect to interchanges of its suffixes.

The traction vector at any point $\boldsymbol{x}$ on the surface S surrounding the source point $\boldsymbol{x}'$ is

$$t_q(\boldsymbol{x}) = n_p(\boldsymbol{x})\,\sigma_{pq}(\boldsymbol{x}) = n_p(\boldsymbol{x})\,C_{pqim}\frac{\partial G_{ij}(\boldsymbol{\xi})}{\partial x_m} f_j\,(\boldsymbol{x}')\,, \tag{2.5}$$

where $n_p(\boldsymbol{x})$ is the outwardly directed unit normal vector at $\boldsymbol{x}$, so that if the volume V enclosed by S is in equilibrium we must require that

$$\iint_S t_q(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) + f_q\,(\boldsymbol{x}') = 0.$$

Using (2.5) and noting that

$$f_q\,(\boldsymbol{x}') = f_j\,(\boldsymbol{x}')\,\delta_{jq} = f_j\,(\boldsymbol{x}') \iiint_V \delta\,(x_1 - x_1')\,\delta\,(x_2 - x_2')\,\delta\,(x_3 - x_3')\,\mathrm{d}V(\boldsymbol{x})\,\delta_{jq},$$

we may use Gauss's divergence theorem to change the surface integral into a volume integral, and obtain

$$f_j\,(\boldsymbol{x}') \iiint_V \left\{ C_{pqim}\frac{\partial^2 G_{ij}(\boldsymbol{\xi})}{\partial x_p \partial x_m} + \delta_{jq}\,\delta\,(x_1 - x_1')\,\delta\,(x_2 - x_2')\,\delta\,(x_3 - x_3') \right\} \mathrm{d}V(\boldsymbol{x}) = 0.$$

Since V and $f_j\,(\boldsymbol{x}')$ are arbitrary, we must require that $G_{ij}(\boldsymbol{\xi})$ satisfy the inhomogeneous equation

$$C_{pqim}\frac{\partial^2 G_{ij}(\boldsymbol{\xi})}{\partial x_p \partial x_m} + \delta_{jq}\,\delta\,(x_1 - x_1')\,\delta\,(x_2 - x_2')\,\delta\,(x_3 - x_3') = 0. \tag{2.6}$$

In addition, if the surface of the half-space is traction free, we have $t_q\,(x_1, x_2, x_3) = 0$ on the surface $x_3 = 0$, so that from (2.5),

$$n_p(\boldsymbol{x})\,C_{pqim}\frac{\partial G_{ij}(\boldsymbol{\xi})}{\partial x_m} f_j\,(\boldsymbol{x}') = 0 \qquad \text{on} \qquad x_3 = 0.$$

Since $\boldsymbol{n} = (0, 0, -1)$ from figure 1, this condition reduces to

$$C_{3qim}\frac{\partial G_{ij}(\boldsymbol{\xi})}{\partial x_m} f_j\,(\boldsymbol{x}') = 0 \qquad \text{on} \qquad x_3 = 0. \tag{2.7}$$

Equation (2.1) can be used to write (2.6) and (2.7) in the form,

$$C_{ijkl}\frac{\partial^2 G_{km}(\boldsymbol{\xi})}{\partial \xi_l \partial \xi_j} + \delta_{im}\,\delta\,(\xi_1)\,\delta\,(\xi_2)\,\delta\,(\xi_3 - \xi_4) = 0 \tag{2.8}$$

and

$$C_{i3k3}\frac{\partial G_{km}\,(\xi_1, \xi_2, 0, \xi_4)}{\partial \xi_3} = -\left(C_{i3k1}\frac{\partial G_{km}\,(\xi_1, \xi_2, 0, \xi_4)}{\partial \xi_1} + C_{i3k2}\frac{\partial G_{km}\,(\xi_1, \xi_2, 0, \xi_4)}{\partial \xi_2} \right). \tag{2.9}$$

We now define the Fourier wave vectors in three- and four-space by the relations,

$$\boldsymbol{K} = (K_1, K_2, K_3) \qquad \text{and} \qquad \boldsymbol{K}^* = (K_1, K_2, K_3, K_4). \tag{2.10}$$

Equation (2.8) may then be multiplied by

$$\exp(\mathrm{i}\boldsymbol{K}^* \cdot \boldsymbol{\xi})\, \mathrm{d}^4\boldsymbol{\xi} = \mathrm{e}^{\mathrm{i}(K_1\xi_1+K_2\xi_2+K_3\xi_3+K_4\xi_4)}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3\, \mathrm{d}\xi_4$$

and integrated over the four-space to obtain

$$\int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_3=0}^{\infty}\int_{\xi_4=0}^{\infty}\left(C_{ijkl}\frac{\partial^2 G_{km}(\boldsymbol{\xi})}{\partial\xi_l\partial\xi_j} + \delta_{im}\delta(\xi_1)\,\delta(\xi_2)\,\delta(\xi_3-\xi_4)\right) \times \mathrm{e}^{\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3\, \mathrm{d}\xi_4 = 0.$$

Integration by parts severally with respect to ξ_1, ξ_2, ξ_3 and attending to the limiting condition that the derivative, $\partial G_{km}/\partial\xi_j$, vanishes at $(\xi_1, \xi_2) = \pm\infty$ and at $\xi_3 - \infty$ gives

$$\begin{aligned}
&-\int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_4=0}^{\infty} C_{ijk3}\frac{\partial G_{km}(\xi_1,\xi_2,0,\xi_4)}{\partial\xi_j}\,\mathrm{e}^{\mathrm{i}(K_1\xi_1+K_2\xi_2+K_4\xi_4)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4 \\
&-\int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_3=0}^{\infty}\int_{\xi_4=0}^{\infty} C_{ijkl}\frac{\partial G_{km}(\xi_1,\xi_2,\xi_3,\xi_4)}{\partial\xi_j}\,\mathrm{i}K_l \\
&\qquad\qquad \times \mathrm{e}^{\mathrm{i}(K_1\xi_1+K_2\xi_2+K_3\xi_3+K_4\xi_4)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,\mathrm{d}\xi_4 \\
&+\ \delta_{im}\int_{\xi_3=0}^{\infty}\mathrm{e}^{\mathrm{i}(K_3+K_4)\xi_3}\,\mathrm{d}\xi_3 = 0.
\end{aligned} \tag{2.11}$$

If we write the first and second integrals as I_1 and I_2, and note from Appendix A that the last integral is the Heisenberg delta function $\delta_+(K_3+K_4)$, we have

$$I_1 + I_2 + \delta_{im}\,\delta_+(K_3+K_4) = 0. \tag{2.12}$$

Now consider the first integral, which may be written as

$$\begin{aligned}
I_1 = -\int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_4=0}^{\infty}&\left(C_{i1k3}\frac{\partial G_{km}(\xi_1,\xi_2,0,\xi_4)}{\partial\xi_1} + C_{i2k3}\frac{\partial G_{km}(\xi_1,\xi_2,0,\xi_4)}{\partial\xi_2}\right. \\
&\left. + C_{i3k3}\frac{\partial G_{km}(\xi_1,\xi_2,0,\xi_4)}{\partial\xi_3}\right)\mathrm{e}^{\mathrm{i}(K_1\xi_1+K_2\xi_2+K_4\xi_4)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4.
\end{aligned}$$

The last derivative in the integrand can be written in terms of the first two derivatives by means of the traction free surface condition in equation (2.9), so that

$$\begin{aligned}
I_1 = -\int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_4=0}^{\infty}&\left((C_{i1k3}-C_{i3k1})\frac{\partial G_{km}(\xi_1,\xi_2,0,\xi_4)}{\partial\xi_1}\right. \\
&\left. + (C_{i2k3}-C_{i3k2})\frac{\partial G_{km}(\xi_1,\xi_2,0,\xi_4)}{\partial\xi_2}\right)\mathrm{e}^{\mathrm{i}(K_1\xi_1+K_2\xi_2+K_4\xi_4)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4.
\end{aligned}$$

The integrals I_1 and I_2 may now be integrated by parts and substituted into equation (2.12), to yield

$$\int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_4=0}^{\infty} \mathrm{i}\left\{\left(C_{i\alpha k3}-C_{i3k\alpha}\right)K_\alpha + C_{i3kl}K_l\right\} G_{km}\left(\xi_1,\xi_2,0,\xi_4\right)$$

$$\times\, \mathrm{e}^{\mathrm{i}\left(K_1\xi_1+K_2\xi_2+K_4\xi_4\right)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4$$

$$-\int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_3=0}^{\infty}\int_{\xi_4=0}^{\infty} K_lK_jC_{ijkl}G_{km}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)$$

$$\times\, \mathrm{e}^{\mathrm{i}\left(K_1\xi_1+K_2\xi_2+K_3\xi_3+K_4\xi_4\right)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,\mathrm{d}\xi_4 + \delta_{im}\delta_+\left(K_3+K_4\right)=0, \quad (2.13)$$

where the Greek subscript α is summed only over the range $\alpha = 1, 2$.

On summing over α and l the braces in the first integral can be written as $C_{ijk3}K_j$, and if we define $\widehat{G}_{km}\left(\boldsymbol{K}^*\right)$ and $F_{im}\left(\boldsymbol{K}^*\right)$ by the relations

$$\widehat{G}_{km}\left(\boldsymbol{K}^*\right) = \int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_3=0}^{\infty}\int_{\xi_4=0}^{\infty} G_{km}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)$$

$$\times\, \mathrm{e}^{\mathrm{i}\left(K_1\xi_1+K_2\xi_2+K_3\xi_3+K_4\xi_4\right)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,\mathrm{d}\xi_4 \quad (2.14)$$

and

$$F_{im}\left(\boldsymbol{K}^*\right) = \mathrm{i}C_{ijk3}K_j \int_{\xi_1=-\infty}^{\infty}\int_{\xi_2=-\infty}^{\infty}\int_{\xi_4=0}^{\infty} G_{km}\left(\xi_1,\xi_2,0,\xi_4\right)$$

$$\times\, \mathrm{e}^{\mathrm{i}\left(K_1\xi_1+K_2\xi_2+K_4\xi_4\right)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4, \quad (2.15)$$

then (2.13) can be written as

$$K_lK_jC_{ijkl}\,\widehat{G}_{km}\left(\boldsymbol{K}^*\right) = \delta_{im}\,\delta_+\left(K_3+K_4\right) + F_{im}\left(\boldsymbol{K}^*\right). \quad (2.16)$$

We now set $\zeta_i = K_i/K$, where $i = 1, 2, 3$ and $K^2 = K_1^2 + K_2^2 + K_3^2$, so that (2.16) becomes

$$K^2M_{ik}(\boldsymbol{\zeta})\,\widehat{G}_{km}\left(\boldsymbol{K}^*\right) = \delta_{im}\,\delta_+\left(K_3+K_4\right) + F_{im}\left(\boldsymbol{K}^*\right), \quad (2.17)$$

in which

$$M_{ik}(\boldsymbol{\zeta}) = M_{ki}(\boldsymbol{\zeta}) = M_{ik}\left(\zeta_1,\zeta_2,\zeta_3\right) = C_{ijkl}\zeta_j\zeta_l \quad (2.18)$$

is the Christoffel stiffness tensor. In Gibbs's dyadic notation equation (2.17) may be written as

$$K^2\,\boldsymbol{M}\cdot\widehat{\boldsymbol{G}} = \boldsymbol{I}\,\delta_+\left(K_3+K_4\right) + \boldsymbol{F},$$

which may be premultiplied by $K^{-2}\,\boldsymbol{M}^{-1}$ to yield

$$\widehat{\boldsymbol{G}} = K^{-2}\,\boldsymbol{M}^{-1}\cdot\left(\boldsymbol{I}\,\delta_+\left(K_3+K_4\right) + \boldsymbol{F}\right),$$

or, in suffix notation,

$$\widehat{G}_{ij}\left(\boldsymbol{K}^*\right) = \frac{M_{ik}^{-1}(\boldsymbol{\zeta})}{K^2}\left(\delta_{kj}\,\delta_+\left(K_3+K_4\right) + F_{kj}\left(\boldsymbol{K}^*\right)\right). \quad (2.19)$$

We now introduce the Heaviside product $H(\xi_3)H(\xi_4)$ into (2.14), so that the integration in (2.14) runs from $-\infty$ to ∞ for each variable, namely

$$\widehat{G}_{ij}\left(\boldsymbol{K}^*\right) = \iiiint_{-\infty}^{\infty} \left\{G_{ij}(\boldsymbol{\xi})H(\xi_3)H(\xi_4)\right\}\mathrm{e}^{\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}\,\mathrm{d}^4\boldsymbol{\xi}. \quad (2.20)$$

By the Fourier inversion theorem, (2.20) can be inverted to give

$$G_{ij}(\boldsymbol{\xi})H(\xi_3)H(\xi_4) = \iiiint_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}\, \widehat{G}_{ij}(\boldsymbol{K}^*)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}, \tag{2.21}$$

or

$$G_{ij}(\boldsymbol{\xi})H(\xi_3)H(\xi_4)$$
$$= \iiiint_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4} \frac{M_{ik}^{-1}(\boldsymbol{\zeta})}{K^2} \Big(\delta_{kj}\,\delta_+(K_3+K_4) + F_{kj}(\boldsymbol{K}^*)\Big)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}. \tag{2.22}$$

We now have to determine $F_{kj}(\boldsymbol{K}^*)$, which from (2.15), depends on the Green function evaluated on the surface of the half-space. To this end, it is convenient to reduce the quadruple Fourier integral containing the Heisenberg delta function to a triple Fourier integral. By separating out the integral over K_4 containing this delta function, we have

$$G_{ij}(\boldsymbol{\xi})H(\xi_3)H(\xi_4) = \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3} \frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2}\, \mathrm{e}^{-\mathrm{i}(K_1\xi_1+K_2\xi_2+K_3\xi_3)}$$
$$\times \int_{K_4=-\infty}^{\infty} \frac{\mathrm{d}K_4}{2\pi}\, \delta_+(K_3+K_4)\, \mathrm{e}^{-\mathrm{i}K_4\xi_4}$$
$$+ \iiiint_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4} \frac{M_{ik}^{-1}(\boldsymbol{\zeta})}{K^2} F_{kj}(\boldsymbol{K}^*)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}. \tag{2.23}$$

From equation (A 3) of Appendix A, we can write

$$\int_{K_4=-\infty}^{\infty} \frac{\mathrm{d}K_4}{2\pi}\, \delta_+(K_3+K_4)\, \mathrm{e}^{-\mathrm{i}K_4\xi_4} = \mathrm{e}^{\mathrm{i}K_3\xi_4} H(\xi_4) = \mathrm{e}^{\mathrm{i}K_3\xi_4}, \tag{2.24}$$

so that (2.23) may be written as

$$G_{ij}(\boldsymbol{\xi})H(\xi_3)H(\xi_4) = \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3} \frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2}\, \mathrm{e}^{-\mathrm{i}\{K_1\xi_1+K_2\xi_2+K_3(\xi_3-\xi_4)\}}$$
$$+ \iiiint_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4} \frac{M_{ik}^{-1}(\boldsymbol{\zeta})}{K^2}\, F_{kj}(\boldsymbol{K}^*)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}. \tag{2.25}$$

The triple integral which occurs in the definition of $F_{im}(\boldsymbol{K}^*)$ in (2.15) is now written as

$$g_{km}(K_1, K_2, K_4)$$
$$= \int_{\xi_1=-\infty}^{\infty} \int_{\xi_2=-\infty}^{\infty} \int_{\xi_4=-\infty}^{\infty} G_{km}(\xi_1, \xi_2, 0, \xi_4)\, H(\xi_4)\, \mathrm{e}^{\mathrm{i}(K_1\xi_1+K_2\xi_2+K_4\xi_4)}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_4, \tag{2.26}$$

where the Heaviside function $H(\xi_4)$ has been introduced to allow all the variables to run from $-\infty$ to ∞. Then (2.15) can be written as

$$F_{im}(\boldsymbol{K}^*) = \mathrm{i}\, C_{ijk3} K_j\, g_{km}(K_1, K_2, K_4), \tag{2.27}$$

and (2.25) becomes

$$G_{ij}(\xi_1,\xi_2,\xi_3,\xi_4)\,H(\xi_3)H(\xi_4)=\iiint\limits_{-\infty}^{\infty}\frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\,\frac{M_{ij}^{-1}(\zeta)}{K^2}\,\mathrm{e}^{-\mathrm{i}\{K_1\xi_1+K_2\xi_2+K_3(\xi_3-\xi_4)\}}$$

$$+\iiiint\limits_{-\infty}^{\infty}\frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}\,\frac{M_{ik}^{-1}(\zeta)}{K^2}\,\mathrm{i}C_{kpq3}K_p\,g_{qj}(K_1,K_2,K_4)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}. \quad (2.28)$$

If we now let $\xi_3\to 0_+$ and put $H(\xi_3\to 0_+)=1$, then on multiplying both sides of equation (2.28) by $\exp\{\mathrm{i}(L_1\xi_1+L_2\xi_2+L_4\xi_4)\}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4$ and integrating from $-\infty$ to ∞, we obtain

$$\iiint\limits_{-\infty}^{\infty}G_{ij}(\xi_1,\xi_2,0,\xi_4)\,H(\xi_4)\,\mathrm{e}^{\mathrm{i}\{L_1\xi_1+L_2\xi_2+L_4\xi_4\}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4$$

$$=\lim_{\xi_3\to 0}\iiint\limits_{-\infty}^{\infty}\frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\,\frac{M_{ij}^{-1}(\zeta)}{K^2}\,\mathrm{e}^{-\mathrm{i}K_3\xi_3}$$

$$\times\iiint\limits_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\{(K_1-L_1)\xi_1+(K_2-L_2)\xi_2-(K_3+L_4)\xi_4\}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4$$

$$+\lim_{\xi_3\to 0}\iiiint\limits_{-\infty}^{\infty}\frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}\,\frac{M_{ik}^{-1}(\zeta)}{K^2}\,\mathrm{i}C_{kpq3}K_p\,g_{qj}(K_1,K_2,K_4)\,\mathrm{e}^{-\mathrm{i}K_3\xi_3}$$

$$\times\iiint\limits_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\{(K_1-L_1)\xi_1+(K_2-L_2)\xi_2+(K_4-L_4)\xi_4\}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_4. \quad (2.29)$$

The integrations over ξ_1, ξ_2, ξ_4 in the preceding equation are elementary and produce the relations

$$(2\pi)^3\delta(K_1-L_1)\,\delta(K_2-L_2)\,\delta(K_3+L_4)$$

and

$$(2\pi)^3\delta(K_1-L_1)\,\delta(K_2-L_2)\,\delta(K_4-L_4),$$

while from (2.26) the left-hand side is identified as $g_{ij}(L_1,L_2,L_4)$, so that on invoking the sifting properties of the Dirac delta functions, the preceding equation becomes

$$g_{ij}(L_1,L_2,L_4)=f_{ij}(L_1,L_2,L_4)+h_{iq}(L_1,L_2)\,g_{qj}(L_1,L_2,L_4), \quad (2.30)$$

where

$$f_{ij}(L_1,L_2,L_4)=\frac{M_{ij}^{-1}\left(\dfrac{L_1}{\sqrt{L_1^2+L_2^2+L_4^2}},\dfrac{L_2}{\sqrt{L_1^2+L_2^2+L_4^2}},\dfrac{-L_4}{\sqrt{L_1^2+L_2^2+L_4^2}}\right)}{(L_1^2+L_2^2+L_4^2)} \quad (2.31)$$

and

$$h_{iq}(L_1,L_2) = \lim_{\xi_3\to 0} \mathrm{i}\int_{K_3=-\infty}^{\infty} \frac{\mathrm{d}K_3}{2\pi(L_1^2+L_2^2+K_3^2)} \times M_{ik}^{-1}\left(\frac{L_1}{\sqrt{L_1^2+L_2^2+K_3^2}}, \frac{L_2}{\sqrt{L_1^2+L_2^2+K_3^2}}, \frac{K_3}{\sqrt{L_1^2+L_2^2+K_3^2}}\right) \times (C_{k1q3}L_1 + C_{k2q3}L_2 + C_{k3q3}K_3)\,\mathrm{e}^{-\mathrm{i}K_3\xi_3}. \tag{2.32}$$

In taking the limits as $\xi_3 \to 0$ we have used the fact that the sine component in the expression for $h_{iq}(L_1,L_2)$ gives a finite contribution to the integral, whereas in the expression for $f_{ij}(L_1,L_2,L_4)$ the contribution from the sine term vanishes in the limit, and the Dirac delta functions then produce the expression in (2.31).

In Gibbs's dyadic notation, equation (2.30) may be written as

$$(\boldsymbol{I}-\boldsymbol{h})\cdot\boldsymbol{g} = \boldsymbol{f}, \qquad \text{or as} \qquad \boldsymbol{g} = [\boldsymbol{I}-\boldsymbol{h}]^{-1}\cdot\boldsymbol{f},$$

so that on reverting back to suffix notation, we have

$$g_{ij}(L_1,L_2,L_4) = [\delta_{ik} - h_{ik}(L_1,L_2)]^{-1} f_{kj}(L_1,L_2,L_4). \tag{2.33}$$

Observing from (2.1) that

$$\xi_1 = x_1 - x_1', \qquad \xi_2 = x_2 - x_2', \qquad \xi_3 - \xi_4 = x_3 - x_3',$$

we can now put (2.33) into (2.27), and on noting that $(\xi_3,\xi_4) \geq 0_+$ so that $H(\xi_3) = H(\xi_4) = 1$, we finally obtain

$$G_{ij}(\xi_1,\xi_2,\xi_3,\xi_4) = \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{x}')} + \iiiint_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}\frac{M_{ik}^{-1}(\boldsymbol{\zeta})}{K^2} F_{kj}(\boldsymbol{K}^*)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}, \tag{2.34}$$

where

$$F_{kj}(\boldsymbol{K}^*) = \mathrm{i}C_{kpq3}K_p\, g_{qj}(K_1,K_2,K_4). \tag{2.35}$$

Equations (2.18) and (2.31)–(2.35) taken together give the Fourier integral representation of the required Green function. These are collected together in § 3.

3. Collected formulæ for Green's function representation

In this section we gather together the formulæ required to represent the Fourier integral form of the Green function for an anisotropic elastic half-space.

The components of the elasticity tensor C_{ijkl} are first evaluated with respect to the global coordinate system x_1, x_2, x_3 shown in figure 1. With the definitions

$$\boldsymbol{\zeta} = (\zeta_1,\zeta_2,\zeta_3) = \left(\frac{K_1}{\sqrt{K_1^2+K_2^2+K_3^2}}, \frac{K_2}{\sqrt{K_1^2+K_2^2+K_3^2}}, \frac{K_3}{\sqrt{K_1^2+K_2^2+K_3^2}}\right) \tag{3.1}$$

and

$$\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) = \left(\frac{K_1}{\sqrt{K_1^2 + K_2^2 + K_4^2}}, \frac{K_2}{\sqrt{K_1^2 + K_2^2 + K_4^2}}, \frac{-K_4}{\sqrt{K_1^2 + K_2^2 + K_4^2}} \right), \tag{3.2}$$

the inverse of the Christoffel stiffness tensor is computed in component form as

$$M_{ik}^{-1}(\boldsymbol{\zeta}) = M_{ik}^{-1}(\zeta_1, \zeta_2, \zeta_3) = (C_{ijkl}\zeta_j\zeta_l)^{-1}. \tag{3.3}$$

We then form the tensor components

$$f_{lj}(K_1, K_2, K_4) = \frac{M_{lj}^{-1}(\eta_1, \eta_2, \eta_3)}{K_1^2 + K_2^2 + K_4^2} \tag{3.4}$$

and

$$h_{ql}(K_1, K_2) = \lim_{\xi_3 \to 0} \mathrm{i} \int_{K_3=-\infty}^{\infty} \frac{\mathrm{d}K_3}{2\pi} \frac{M_{qk}^{-1}(\zeta_1, \zeta_2, \zeta_3)}{K_1^2 + K_2^2 + K_3^2} C_{kpl3} K_p \mathrm{e}^{-\mathrm{i}K_3\xi_3}, \tag{3.5}$$

and compute the inverse tensor components

$$g_{qj}(K_1, K_2, K_4) = [\delta_{ql} - h_{ql}(K_1, K_2)]^{-1} f_{lj}(K_1, K_2, K_4). \tag{3.6}$$

The tensor components

$$F_{kj}(\boldsymbol{K}^*) = \mathrm{i}C_{kpq3}K_p\, g_{qj}(K_1, K_2, K_4) \tag{3.7}$$

are then evaluated and substituted into the Fourier integral representation of the Green function tensor to give the component form:

$$G_{ij}(\boldsymbol{\xi}) = \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3} \frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{x}')} + \iiiint_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4} \frac{M_{ik}^{-1}(\boldsymbol{\zeta})}{K^2} F_{kj}(\boldsymbol{K}^*) \mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}, \tag{3.8}$$

where

$$K^2 = K_1^2 + K_2^2 + K_3^2, \quad \boldsymbol{K} = (K_1, K_2, K_3),$$
$$\boldsymbol{K}^* = (K_1, K_2, K_3, K_4), \quad \boldsymbol{\xi} = (x_1 - x_1', x_2 - x_2', x_3, x_3').$$

The first triple Fourier integral in this representation is the Green function for the infinite medium, while the quadruple Fourier integral represents the effect due to the surface of the half-space, and where it is understood that since the Green function is real, only the real part of equation (3.8) is to be taken. The triple integral representing the Green function for an infinite medium has a singularity at $\boldsymbol{x} = \boldsymbol{x}'$, but the quadruple integral, which represents the effect of the surface, has no singularities within the region occupied by the elastic half-space.

The integrands in the Fourier integral representation are related to those used in Mura's book, *Micromechanics of defects in solids* (1987), by the relationship

$$\frac{M_{ij}^{-1}(\boldsymbol{\zeta})}{K^2} = \frac{N_{ij}(\boldsymbol{K})}{D(\boldsymbol{K})}, \tag{3.9}$$

where

$$\left.\begin{aligned} N_{ij}(\boldsymbol{K}) &= \tfrac{1}{2}\epsilon_{ikl}\,\epsilon_{jmn}M_{km}(\boldsymbol{K})\,M_{ln}(\boldsymbol{K}), \\ D(\boldsymbol{K}) &= \tfrac{1}{6}\epsilon_{ikl}\,\epsilon_{jmn}M_{ij}(\boldsymbol{K})\,M_{km}(\boldsymbol{K})\,M_{ln}(\boldsymbol{K}) \end{aligned}\right\} \tag{3.10}$$

are the cofactors and determinant, respectively, of the matrix components

$$M_{ik}(\boldsymbol{K}) = C_{ijkl}K_jK_l,$$

with ϵ_{mnl} being the components of the permutation tensor.

4. Contribution to Green's function from the surface

We may now examine the quadruple Fourier integral. Since this represents the effect of the surface, we expect that it vanishes when $x_3' = \xi_4 \to \infty$, but $x_3 - x_3' = \xi_3 - \xi_4$ remains finite (see figure 1). To show this, we write the effect of the surface as

$$G^S_{km}(\boldsymbol{\xi}) = \iiiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}\,\frac{M^{-1}_{ki}(\boldsymbol{\zeta})}{K^2}\,F_{im}(\boldsymbol{K}^*)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}. \tag{4.1}$$

Inserting (2.26) and (2.27) into this integral gives

$$G^S_{km}(\boldsymbol{\xi}) = \iiiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}\,\frac{M^{-1}_{ki}(\boldsymbol{\zeta})}{K^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}\,\mathrm{i}C_{iqp3}K_q$$

$$\times \iiint\limits_{-\infty}^{\infty} G_{pm}(\alpha_1,\alpha_2,0,\alpha_4)\,H(\alpha_4)\,\mathrm{e}^{\mathrm{i}(K_1\alpha_1+K_2\alpha_2+K_4\alpha_4)}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_4. \tag{4.2}$$

On separating the integration over K_4, this can be written as

$$G^S_{km}(\boldsymbol{\xi}) = \iiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\,\frac{M^{-1}_{ki}(\boldsymbol{\zeta})}{K^2}\,\mathrm{e}^{-\mathrm{i}(K_1\xi_1+K_2\xi_2+K_3\xi_3)}\,\mathrm{i}C_{iqp3}K_q$$

$$\times \iiint\limits_{-\infty}^{\infty} G_{pm}(\alpha_1,\alpha_2,0,\alpha_4)\,H(\alpha_4)\,\mathrm{e}^{\mathrm{i}(K_1\alpha_1+K_2\alpha_2)}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_4$$

$$\times \int_{K_4=-\infty}^{\infty} \frac{\mathrm{d}K_4}{2\pi}\,\mathrm{e}^{\mathrm{i}K_4(\alpha_4-\xi_4)}. \tag{4.3}$$

The integral over K_4 is the Dirac delta function $\delta(\alpha_4 - \xi_4)$, and invoking its sifting properties gives

$$G^S_{km}(\boldsymbol{\xi}) = \iiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\,\frac{M^{-1}_{ki}(\boldsymbol{\zeta})}{K^2}\,\mathrm{e}^{-\mathrm{i}(K_1\xi_1+K_2\xi_2+K_3\xi_3)}\,\mathrm{i}C_{iqp3}K_q$$

$$\times \iint\limits_{-\infty}^{\infty} G_{pm}(\alpha_1,\alpha_2,0,\xi_4)\,H(\xi_4)\,\mathrm{e}^{\mathrm{i}(K_1\alpha_1+K_2\alpha_2)}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2. \tag{4.4}$$

The integral in (4.4) can be rearranged into a physically more revealing form as follows. First, make the change of variables $\alpha_1 = \xi_1 - \beta_1$, $\alpha_2 = \xi_2 - \beta_2$, and set $H(\xi_4) = 1$, so that

$$G^S_{km}(\boldsymbol{\xi}) = \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3} \frac{M^{-1}_{ki}(\boldsymbol{\zeta})}{K^2} \mathrm{i}C_{iqp3}K_q \iint_{-\infty}^{\infty} G_{pm}(\xi_1 - \beta_1, \xi_2 - \beta_2, 0, \xi_4) \times \mathrm{e}^{-\mathrm{i}(K_1\beta_1 + K_2\beta_2 + K_3\xi_3)}\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2. \quad (4.5)$$

By the reciprocity theorem in Appendix B, the position vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$ of the field and source points can be interchanged in the Green function provided the suffixes are transposed. Thus, from (2.1), we may write

$$\left.\begin{aligned} G^S_{km}(\xi_1, \xi_2, \xi_3, \xi_4) &= G^S_{mk}(-\xi_1, -\xi_2, \xi_4, \xi_3), \\ G_{km}(\xi_1, \xi_2, \xi_3, \xi_4) &= G_{mk}(-\xi_1, -\xi_2, \xi_4, \xi_3), \end{aligned}\right\} \quad (4.6)$$

so that on performing the required interchanges on $\boldsymbol{\xi}$ in the integral on the right-hand side of (4.5) and transposing the suffixes we obtain

$$G^S_{km}(\boldsymbol{\xi}) = \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3} \frac{M^{-1}_{mi}(\boldsymbol{\zeta})}{K^2} \mathrm{i}\,C_{iqp3}K_q \iint_{-\infty}^{\infty} G_{pk}(-\xi_1 - \beta_1, -\xi_2 - \beta_2, 0, \xi_3) \times \mathrm{e}^{-\mathrm{i}(K_1\beta_1 + K_2\beta_2 + K_3\xi_4)}\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2. \quad (4.7)$$

Using the preceding reciprocity theorem again on the Green function inside the integral, namely

$$G_{pk}(-\xi_1 - \beta_1, -\xi_2 - \beta_2, 0, \xi_3) = G_{kp}(\xi_1 + \beta_1, \xi_2 + \beta_2, \xi_3, 0), \quad (4.8)$$

and introducing the change in variables,

$$x''_1 = x'_1 - \beta_1, \qquad x''_2 = x'_2 - \beta_2,$$

together with the use of equation (2.1), allows the surface Green function to be expressed in the form

$$G^S_{km}(\boldsymbol{\xi}) = \iint_{-\infty}^{\infty} \mathrm{d}x''_1\,\mathrm{d}x''_2\, G_{kp}(x_1 - x''_1, x_2 - x''_2, x_3, 0) \times \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3} \frac{M^{-1}_{mi}(\boldsymbol{\zeta})}{K^2} \mathrm{i}C_{iqp3}K_q\, \mathrm{e}^{\mathrm{i}\{K_1(x''_1 - x'_1) + K_2(x''_2 - x'_2) - K_3x'_3\}}, \quad (4.9)$$

where the order of the space and wave vector integrations has been changed.

The displacement at the field point $\boldsymbol{x}$ due to the applied force $\boldsymbol{f}$ at $\boldsymbol{x}'$ arising from the surface effect is then expressed as a Faltung representation (Sneddon 1951) of equation (4.1) in the form

$$u^S_k(\boldsymbol{x}) = G^S_{km}(\boldsymbol{\xi})f_m = \iint_{-\infty}^{\infty} \mathrm{d}x''_1\,\mathrm{d}x''_2\, G_{kp}(x_1 - x''_1, x_2 - x''_2, x_3, 0)\, t_p(x''_1 - x'_1, x''_2 - x'_2, x'_3), \quad (4.10)$$

where

$$t_p\,(x_1''-x_1',x_2''-x_2',x_3')$$

$$=\iiint_{-\infty}^{\infty}\frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\,\frac{M_{mi}^{-1}(\boldsymbol{\zeta})}{K^2}\,\mathrm{i}C_{iqp3}K_q\,\mathrm{e}^{\mathrm{i}\{K_1(x_1''-x_1')+K_2(x_2''-x_2')-K_3x_3'\}}f_m$$

$$=C_{iqp3}\left[\frac{\partial}{\partial x_q''}\iiint_{-\infty}^{\infty}\frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\,\frac{M_{mi}^{-1}(\boldsymbol{\zeta})}{K^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{x}''-\boldsymbol{x}')}\right]_{x_3''=0}f_m$$

$$=C_{iqp3}\left[\frac{\partial G_{mi}^{\infty}(\boldsymbol{x}''-\boldsymbol{x}')}{\partial x_q''}\right]_{x_3''=0}f_m, \tag{4.11}$$

in which $G_{mi}^{\infty}(\boldsymbol{x}''-\boldsymbol{x}')$ is the Green function for the infinite medium. But

$$\left[\frac{\partial G_{mi}^{\infty}(\boldsymbol{x}''-\boldsymbol{x}')}{\partial x_q''}\right]_{x_3''=0}\to 0 \qquad \text{when} \qquad x_3'=\xi_4\to\infty \tag{4.12}$$

so that $G_{km}^{S}(\boldsymbol{\xi})\to 0$ when $\xi_4\to\infty$ as expected.

From (2.5) it is evident that the expression for $t_p\,(x_1''-x_1',x_2''-x_2',x_3')$ in (4.11) therefore represents the surface traction at the point (x_1'',x_2'') on the surface $x_3''=0$ due to the applied point force at $\boldsymbol{x}'$ in the infinite medium, but with its sign reversed.

Thus, the displacement at $\boldsymbol{x}$ due to a point force at $\boldsymbol{x}'$ is determined by the superposition of two deformations. The first is that due to the application of the point force in an infinite medium. On the surface $x_3''=0$, the surface traction in the infinite medium due to the applied point force at $\boldsymbol{x}'$ is given by $-t_p\,(x_1''-x_1',x_2''-x_2',x_3')$. If we cut the infinite medium at $x_3''=0$ and reverse the surface traction (i.e. we apply $+t_p\,(x_1''-x_1',x_2''-x_2',x_3')$ to the surface), the second additional displacement due to the reversed surface traction is given by equation (4.10), in which the office of the half-space Green function $G_{kp}\,(x_1-x_1'',x_2-x_2'',x_3,0)$ is to give the displacement at the field point $\boldsymbol{x}$ due to distributed tractions at $\boldsymbol{x}''$ on the surface $x_3''=0$. This method of generating the Green function for finite media was suggested by Eshelby (1961), and was used by Owen & Mura (1967), Owen (1971 *a*, *b*), and Seo & Mura (1979) to obtain the displacement due to eigenstrain distributions in an isotropic half-space using Mindlin's explicit representation of the half-space Green function, while Chiu (1978) essentially used the Fourier integral representation of the half-space surface Green function.

5. Mindlin's solution for an isotropic half-space

In this section we give the Fourier integral representation of the Green function for an isotropic half-space and demonstrate that on integration Mindlin's explicit algebraic expression is recovered.

The triple Fourier integral in the half-space representation corresponds to the Green function for an infinite medium. For an isotropic medium we then recover

Kelvin's solution (Mura 1987),

$$G_{ij}^{\infty}\left(\boldsymbol{x}-\boldsymbol{x}'\right)=\frac{1}{16\pi\mu(1-\nu)R_1}\left\{(3-4\nu)\delta_{ij}+\frac{\left(x_i-x_i'\right)\left(x_j-x_j'\right)}{R_1^2}\right\}, \tag{5.1}$$

where $R_1^2=(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2$.

The quadruple Fourier integral in (3.8) requires the evaluation of the components $F_{kj}\left(\boldsymbol{K}^*\right)$. The elasticity tensor components are given by

$$C_{ijkl}=\frac{2\mu\nu}{1-2\nu}\,\delta_{ij}\delta_{kl}+\mu\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right), \tag{5.2}$$

where μ is the Lamé shear modulus and ν is Poisson's ratio. From this relation the inverse Christoffel tensor components are

$$M_{ij}^{-1}\left(\boldsymbol{\zeta}\right)=\frac{\delta_{ij}}{\mu}-\frac{\zeta_i\zeta_j}{2\mu\left(1-\nu\right)}. \tag{5.3}$$

When these relations are substituted into (3.5) we obtain, on integration,

$$h_{pq}\left(K_1,K_2\right)=\tfrac{1}{2}\delta_{pq}+\mathrm{i}\,\frac{1-2\nu}{4\left(1-\nu\right)}\left(\delta_{3p}\beta_q-\delta_{3q}\beta_p\right) \tag{5.4}$$

where

$$\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)=\left(\frac{K_1}{\sqrt{K_1^2+K_2^2}},\frac{K_2}{\sqrt{K_1^2+K_2^2}},0\right). \tag{5.5}$$

The components $\left[\delta_{pq}-h_{pq}\right]^{-1}$ are given by

$$\begin{aligned}a_{pq}\left(K_1,K_2\right)&=\left[\delta_{pq}-h_{pq}\left(K_1,K_2\right)\right]^{-1}\\&=2\delta_{pq}+\frac{2\left(1-2\nu\right)^2}{3-4\nu}\left(\beta_p\beta_q+\delta_{3p}\delta_{3q}\right)\\&\qquad+\mathrm{i}\,\frac{4\left(1-\nu\right)\left(1-2\nu\right)}{3-4\nu}\left(\delta_{3p}\beta_q-\delta_{3q}\beta_p\right).\end{aligned} \tag{5.6}$$

It is of interest to note that for an isotropic material the real parts of h_{pq} and a_{pq} are symmetric, while the imaginary parts are antisymmetric.

The tensor components $F_{kj}\left(\boldsymbol{K}^*\right)$ are then obtained by substituting (5.2), (5.3) and (5.6) into the relation

$$F_{kj}\left(\boldsymbol{K}^*\right)=\mathrm{i}C_{kpq3}\,K_p\,a_{ql}\left(K_1,K_2\right)\,\frac{M_{lj}^{-1}\left(\eta_1,\eta_2,\eta_3\right)}{K_1^2+K_2^2+K_4^2}, \tag{5.7}$$

and the quadruple Fourier integral to be evaluated for an isotropic material is

$$G_{ij}^{S}\left(\boldsymbol{\xi}\right)=\mathrm{i}C_{kpq3}\iiiint_{-\infty}^{\infty}\frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}\,\frac{K_p\,a_{ql}\left(K_1,K_2\right)M_{ik}^{-1}\left(\boldsymbol{\zeta}\right)M_{lj}^{-1}\left(\boldsymbol{\eta}\right)}{\left(K_1^2+K_2^2+K_3^2\right)\left(K_1^2+K_2^2+K_4^2\right)}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}. \tag{5.8}$$

The integration of this relation may now be demonstrated for the component

Table 1. *Components f_{ij}, h_{ij} and a_{ij} for an isotropic material*

$f_{11}=\frac{(1-2\nu)K_1^2+2(1-\nu)(K_2^2+K_4^2)}{2\mu(1-\nu)(K_1^2+K_2^2+K_4^2)^2}$	$h_{11}=\frac{1}{2}$	$a_{11}=\frac{2\{4K_1^2(1-\nu)^2+K_2^2(3-4\nu)\}}{(3-4\nu)(K_1^2+K_2^2)}$
$f_{12}=-\frac{K_1K_2}{2\mu(1-\nu)(K_1^2+K_2^2+K_4^2)^2}$	$h_{12}=0$	$a_{12}=\frac{2K_1K_2(1-2\nu)^2}{(3-4\nu)(K_1^2+K_2^2)}$
$f_{13}=\frac{K_1K_4}{2\mu(1-\nu)(K_1^2+K_2^2+K_4^2)^2}$	$h_{13}=-\mathrm{i}\frac{K_1(1-2\nu)}{4(1-\nu)(K_1^2+K_2^2)^{1/2}}$	$a_{13}=-\mathrm{i}\frac{4K_1(1-\nu)(1-2\nu)}{(3-4\nu)(K_1^2+K_2^2)^{1/2}}$
$f_{21}=f_{12}$	$h_{21}=0$	$a_{21}=\frac{2K_1K_2(1-2\nu)^2}{(3-4\nu)(K_1^2+K_2^2)}$
$f_{22}=\frac{(1-2\nu)K_2^2+2(1-\nu)(K_1^2+K_4^2)}{2\mu(1-\nu)(K_1^2+K_2^2+K_4^2)^2}$	$h_{22}=\frac{1}{2}$	$a_{22}=\frac{2\{4K_2^2(1-\nu)^2+K_1^2(3-4\nu)\}}{(3-4\nu)(K_1^2+K_2^2)}$
$f_{23}=\frac{K_2K_4}{2\mu(1-\nu)(K_1^2+K_2^2+K_4^2)^2}$	$h_{23}=-\mathrm{i}\frac{K_2(1-2\nu)}{4(1-\nu)(K_1^2+K_2^2)^{1/2}}$	$a_{23}=-\mathrm{i}\frac{4K_2(1-\nu)(1-2\nu)}{(3-4\nu)(K_1^2+K_2^2)^{1/2}}$
$f_{31}=f_{13}$	$h_{31}=\mathrm{i}\frac{K_1(1-2\nu)}{4(1-\nu)(K_1^2+K_2^2)^{1/2}}$	$a_{31}=\mathrm{i}\frac{4K_1(1-\nu)(1-2\nu)}{(3-4\nu)(K_1^2+K_2^2)^{1/2}}$
$f_{32}=f_{23}$	$h_{32}=\mathrm{i}\frac{K_2(1-2\nu)}{4(1-\nu)(K_1^2+K_2^2)^{1/2}}$	$a_{32}=\mathrm{i}\frac{4K_2(1-\nu)(1-2\nu)}{(3-4\nu)(K_1^2+K_2^2)^{1/2}}$
$f_{33}=\frac{(1-2\nu)K_4^2+2(1-\nu)(K_1^2+K_2^2)}{2\mu(1-\nu)(K_1^2+K_2^2+K_4^2)^2}$	$h_{33}=\frac{1}{2}$	$a_{33}=\frac{8(1-\nu)^2}{3-4\nu}$

$G_{33}^S(\boldsymbol{\xi})$. We first integrate with respect to K_1 and K_2 by introducing the cylindrical coordinates

$$K_1=r\cos\theta,\quad K_2=r\sin\theta,\quad \mathrm{d}K_1\,\mathrm{d}K_2=r\,\mathrm{d}r\,\mathrm{d}\theta,$$

which reduces equation (5.8) to the form,

$$G_{33}^S(\boldsymbol{\xi})=\iint_{-\infty}^{\infty}\frac{\mathrm{d}K_3\,\mathrm{d}K_4}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}(K_3\xi_3+K_4\xi_4)} \times\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{-\mathrm{i}r(\xi_1\cos\theta+\xi_2\sin\theta)}\,Q(r,K_3,K_4), \qquad (5.9)$$

where

$$\begin{aligned}Q(r,K_3,K_4)&=\frac{1}{(r^2+K_3^2)^2(r^2+K_4^2)^2}\\ &\times\left\{\left[-\frac{2rK_3^3K_4(1-2\nu)}{\mu(3-4\nu)}-\frac{2rK_3^2K_4^2\,\nu(1-2\nu)^2}{\mu(1-\nu)(3-4\nu)}-\frac{4r^3K_3^2\,\nu(1-2\nu)}{\mu(3-4\nu)}\right.\right.\\ &-\left.\frac{2r^3K_3K_4(2-\nu)(1-2\nu)}{\mu(1-\nu)(3-4\nu)}+\frac{2r^3K_4^2(1-2\nu)^2}{\mu(3-4\nu)}+\frac{4r^5(1-\nu)(1-2\nu)}{\mu(3-4\nu)}\right]\\ &+\mathrm{i}\left[\frac{4K_3^3K_4^2(1-\nu)(1-2\nu)}{\mu(3-4\nu)}+\frac{8r^2K_3^3(1-\nu)^2}{\mu(3-4\nu)}-\frac{4r^2K_3^2K_4\,\nu}{\mu(3-4\nu)}\right.\\ &+\left.\left.\frac{4r^2K_3K_4^2(2-\nu)(1-2\nu)}{\mu(3-4\nu)}+\frac{8r^4K_3(2-\nu)(1-\nu)}{\mu(3-4\nu)}+\frac{4r^4K_4(1-\nu)}{\mu(3-4\nu)}\right]\right\}. \qquad (5.10)\end{aligned}$$

Now put $\xi_1 = \varrho\cos\phi$, $\xi_2 = \varrho\sin\phi$. Then

$$\varrho^2 = \xi_1^2 + \xi_2^2 \quad \text{and} \quad \exp\{-\mathrm{i}r(\xi_1\cos\theta + \xi_2\sin\theta)\} = \exp\{-\mathrm{i}r\varrho\cos(\theta-\phi)\}. \tag{5.11}$$

The integration over θ produces the result $2\pi J_0(\varrho r)$, so that $G_{33}^S(\boldsymbol{\xi})$ is obtained in the form of a zero-order Hankel transform (Sneddon 1951) of the Fourier transform of the function $Q(r, K_3, K_4)$, namely

$$G_{33}^S(\boldsymbol{\xi}) = \iint_{-\infty}^{\infty} \frac{\mathrm{d}K_3\,\mathrm{d}K_4}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}(K_3\xi_3+K_4\xi_4)} \int_{r=0}^{\infty} rJ_0(\varrho r)\,Q(r, K_3, K_4)\,\mathrm{d}r. \tag{5.12}$$

The integrations over K_3 and K_4 may now be performed by means of the relation

$$\int_{K=-\infty}^{\infty} \frac{K^p\,\mathrm{e}^{-\mathrm{i}K\xi}}{(r^2+K^2)^2}\,\mathrm{d}K = \left(\mathrm{i}\frac{\partial}{\partial\xi}\right)^p \left\{\frac{\pi(r\xi+1)\,\mathrm{e}^{-r\xi}}{2r^3}\right\}, \tag{5.13}$$

and we obtain

$$G_{33}^S(\boldsymbol{\xi}) = \frac{1}{16\pi\mu(1-\nu)} \int_{r=0}^{\infty} \mathrm{d}r\,J_0(\varrho r)\,\mathrm{e}^{-r(\xi_3+\xi_4)} \times \left[\left(8\nu^2 - 12\nu + 5\right) + (3-4\nu)(\xi_3+\xi_4)\,r + 2\xi_3\xi_4 r^2\right]. \tag{5.14}$$

On integration this relation yields the surface component of the Green function in the form

$$G_{33}^S(\boldsymbol{\xi}) = 1/[16\pi\mu(1-\nu)] \times \left[\frac{8\nu^2-12\nu+5}{R_2} + \frac{(3-4\nu)(\xi_3+\xi_4)^2 - 2\xi_3\xi_4}{R_2^3} + \frac{6\xi_3\xi_4(\xi_3+\xi_4)^2}{R_2^5}\right], \tag{5.15}$$

where $R_2^2 = \varrho^2 + (\xi_3+\xi_4)^2 = \xi_1^2+\xi_2^2+(\xi_3+\xi_4)^2$. When $G_{33}^\infty(\boldsymbol{x}-\boldsymbol{x}')$ and $G_{33}^S(\boldsymbol{\xi})$ are added we obtain Mindlin's solution (Mura 1987) for the component $G_{33}(\boldsymbol{\xi})$.

On the surface of the half-space the Green function component $G_{33}(\boldsymbol{\xi})$, obtained by adding the 33-component of (5.1) and (5.15) and setting $\xi_3 = 0$, is obtained in the form

$$G_{33}(\xi_1, \xi_2, 0, \xi_4) = (1-\nu)/2\pi\mu R + \xi_4^2/4\pi\mu R^3, \tag{5.16}$$

where $R^2 = \xi_1^2 + \xi_2^2 + \xi_4^2$. It is interesting to note that this expression can also be obtained from the inverse Fourier transform of equation (2.26). The required inverse transform is

$$G_{ij}(\xi_1, \xi_2, 0, \xi_4) = \iiint_{-\infty}^{\infty} \frac{\mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K_4}{(2\pi)^3}\,g_{ij}(K_1, K_2, K_4)\,\mathrm{e}^{-\mathrm{i}(K_1\xi_1+K_2\xi_2+K_4\xi_4)}. \tag{5.17}$$

From table 1 the component $g_{33} = a_{3k}f_{k3}$ can be inserted into the preceding triple integral and integrated using the method indicated earlier in this section to yield equation (5.16) directly.

The tensor components f_{ij}, h_{ij} and a_{ij} required for the computation of $F_{ij}(\boldsymbol{K}^*)$ in equation (5.7) and defined in equations (3.4), (3.5) and (5.6), are given for an isotropic medium in table 1.

6. Periodic force distribution in a cubic half-space

In the final section we now use the Fourier integral representation to determine the displacement produced in a half-space when a periodic vertical force distribution is applied at a distance $\xi_4 = x_3'$ below its surface. Results are obtained for a material which possesses cubic material symmetry. It is assumed that the cubic crystallographic axes coincide with the global axes x_1, x_2, x_3, so that the components of the elasticity and Christoffel stiffness tensors may be evaluated in their simplest form.

The vertical force distribution applied at a distance x_3' below the surface is assumed to be given by

$$p\left(x_1, x_2\right) = p_0 \cos\left(2\pi/\lambda\right)\left(x_1 + x_2\right), \tag{6.1}$$

where λ is the wavelength of the periodic distribution. This generates a vertical displacement in the half-space given by the real part of the expression,

$$u_3\left(\boldsymbol{x}\right) = p_0 \iint\limits_{-\infty}^{\infty} G_{33}\left(\xi_1, \xi_2, \xi_3, \xi_4\right) \mathrm{e}^{-\mathrm{i}L\left(x_1'+x_2'\right)}\, \mathrm{d}x_1'\, \mathrm{d}x_2', \tag{6.2}$$

where $L = 2\pi/\lambda$ is the wavenumber.

When the relation for $G_{ij}\left(\boldsymbol{\xi}\right)$ from (3.8) is substituted into (6.2), the vertical displacement can be written as the sum

$$u_3\left(\boldsymbol{x}\right) = u_3^{\infty}\left(\boldsymbol{x}\right) + u_3^{S}\left(\boldsymbol{x}\right), \tag{6.3}$$

where the term $u_3^{\infty}\left(\boldsymbol{x}\right)$ represents the displacement caused by the application of the force distribution $p\left(x_1', x_2'\right)$ in the infinite medium and $u_3^{S}\left(\boldsymbol{x}\right)$ is the displacement due to the presence of the surface.

The term involving K_p in the quadruple Fourier integral in (5.8) can be removed from the integrand by operating on the Fourier integral with $\partial/\partial x_p$, namely

$$G_{ij}^{S}\left(\boldsymbol{\xi}\right) = -\, C_{kpq3} \frac{\partial}{\partial x_p} \iiiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4} \frac{a_{ql}\left(K_1, K_2\right) M_{ik}^{-1}\left(\boldsymbol{\zeta}\right) M_{lj}^{-1}\left(\boldsymbol{\eta}\right)}{\left(K_1^2 + K_2^2 + K_3^2\right)\left(K_1^2 + K_2^2 + K_4^2\right)} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}. \tag{6.4}$$

The use of (3.9) then allows the displacement components to be written in the respective forms:

$$u_3^{\infty}\left(\boldsymbol{x}\right) = p_0 \iint\limits_{-\infty}^{\infty} \mathrm{d}x_1'\, \mathrm{d}x_2' \iiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3} \frac{N_{33}\left(\boldsymbol{K}\right)}{D\left(\boldsymbol{K}\right)} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\left(\boldsymbol{x}-\boldsymbol{x}'\right)} \mathrm{e}^{-\mathrm{i}L\left(x_1'+x_2'\right)} \tag{6.5}$$

and

$$u_3^{S}\left(\boldsymbol{x}\right) = -\, p_0\, C_{kpq3} \frac{\partial}{\partial x_p} \iint\limits_{-\infty}^{\infty} \mathrm{d}x_1'\, \mathrm{d}x_2' \iiiint\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4\boldsymbol{K}^*}{(2\pi)^4}$$

$$\times \frac{a_{ql}\left(K_1, K_2\right) N_{3k}\left(K_1, K_2, K_3\right) N_{l3}\left(K_1, K_2, -K_4\right)}{D\left(K_1, K_2, K_3\right) D\left(K_1, K_2, -K_4\right)} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}^*\cdot\boldsymbol{\xi}}\, \mathrm{e}^{-\mathrm{i}L\left(x_1'+x_2'\right)}. \tag{6.6}$$

The integrations over x_1' and x_2' are elementary and give

$$(2\pi)^2 \delta(K_1 - L)\, \delta(K_2 - L),$$

and the sifting properties of the Dirac delta functions then yield

$$u_3^\infty(\boldsymbol{x}) = p_0\, \mathrm{e}^{-\mathrm{i}L(x_1+x_2)}\, \Phi_{33}\left(|x_3 - x_3'|\right) \tag{6.7}$$

and

$$u_3^S(\boldsymbol{x}) = -p_0\, C_{kpq3}\, a_{ql}(L, L)\, \Phi_{l3}^*(x_3') \frac{\partial}{\partial x_p} \left\{ \mathrm{e}^{-\mathrm{i}L(x_1+x_2)}\, \Phi_{3k}(x_3) \right\}, \tag{6.8}$$

where

$$\Phi_{ij}(x) = \int_{K=-\infty}^{\infty} \frac{\mathrm{d}K}{2\pi} \frac{N_{ij}(L, L, K)}{D(L, L, K)}\, \mathrm{e}^{-\mathrm{i}Kx} \tag{6.9}$$

and $\Phi_{ij}^*(x)$ denotes the complex conjugate of $\Phi_{ij}(x)$.

The cofactor components $N_{ij}(\boldsymbol{K})$ and the determinant $D(\boldsymbol{K})$ are required in (6.9) for the particular case in which $K_1 = K_2 = L$, and when they are expressed in ascending powers of K, we obtain (cf. Mura 1987)

$$\left.\begin{aligned}
D(L, L, K) &= \tau + \theta K^2 + \rho K^4 + \alpha K^6, \\
N_{11}(L, L, K) &= 2(2\mu^2 + \beta)L^4 + (4\mu^2 + 3\beta + \gamma)L^2K^2 + (\mu^2 + \beta)K^4, \\
N_{12}(L, L, K) &= -2\mu(\lambda + \mu)L^4 - (\lambda + \mu)(\mu + \mu^*)L^2K^2, \\
N_{13}(L, L, K) &= -(\lambda + \mu)(2\mu + \mu^*)L^3K - \mu(\lambda + \mu)LK^3, \\
N_{21}(L, L, K) &= N_{12}(L, L, K), \\
N_{22}(L, L, K) &= N_{11}(L, L, K), \\
N_{23}(L, L, K) &= N_{13}(L, L, K), \\
N_{31}(L, L, K) &= N_{13}(L, L, K), \\
N_{32}(L, L, K) &= N_{13}(L, L, K), \\
N_{33}(L, L, K) &= (4\mu^2 + 4\beta + \gamma)L^4 + 2(2\mu^2 + \beta)L^2K^2 + \mu^2K^4,
\end{aligned}\right\} \tag{6.10}$$

where

$$\left.\begin{aligned}
&\tau = 2L^6\mu(2\mu + \mu^*)(2\lambda + 4\mu + \mu^*), \\
&\theta = L^4\left\{\lambda(12\mu^2 + 10\mu\mu^* + 3\mu^{*2}) + 24\mu^3 + 22\mu^2\mu^* + 8\mu\mu^{*2} + \mu^{*3}\right\}, \\
&\rho = 2L^2\mu\left\{\lambda(3\mu + 2\mu^*) + (2\mu + \mu^*)(3\mu + \mu^*)\right\}, \\
&\alpha = \mu^2(\lambda + 2\mu + \mu^*), \\
&\beta = \mu(\lambda + \mu + \mu^*), \quad \gamma = \mu^*(2\lambda + 2\mu + \mu^*), \\
&\lambda = c_{12}, \quad \mu = c_{44}, \quad \mu^* = c_{11} - c_{12} - 2c_{44},
\end{aligned}\right\} \tag{6.11}$$

with the two-dimensional array of elastic constants c_{pq} defined by the Voigt convention $c_{11} = C_{1111}$, $c_{12} = C_{1122}$, $c_{44} = C_{1212}$. The parameter λ now represents an elastic modulus component.

The determinant $D(L, L, K)$ in (6.10) can be factored into the product

$$D(L, L, K) = \alpha\left(K^2 + a^2L^2\right)\left(K^2 + b^2L^2\right)\left(K^2 + c^2L^2\right), \tag{6.12}$$

where

$$a = \left(\frac{2\mu + \mu^*}{\mu}\right)^{1/2}, \tag{6.13}$$

$$b = \left[\frac{\lambda(4\mu + 3\mu^*) + (2\mu + \mu^*)(4\mu + \mu^*) + \omega}{2\mu(\lambda + 2\mu + \mu^*)}\right]^{1/2}, \tag{6.14}$$

$$c = \left[\frac{\lambda(4\mu + 3\mu^*) + (2\mu + \mu^*)(4\mu + \mu^*) - \omega}{2\mu(\lambda + 2\mu + \mu^*)}\right]^{1/2}, \tag{6.15}$$

$$\omega = \left[\mu^*\left\{\lambda(8\mu + 3\mu^*) + 8\mu^2 + 6\mu\mu^* + \mu^{*2}\right\}\left\{3\lambda + 6\mu + \mu^*\right\}\right]^{1/2}. \tag{6.16}$$

The tensor components $\Phi_{ij}(x)$ in (6.9) then involve integrals of the type

$$\begin{aligned} I_p(x) &= \int_{K=-\infty}^{\infty} \frac{K^p \, \mathrm{e}^{-\mathrm{i}Kx}}{(K^2 + a^2L^2)(K^2 + b^2L^2)(K^2 + c^2L^2)} \frac{\mathrm{d}K}{2\pi} \\ &= \frac{1}{2L^5}\left(\mathrm{i}\frac{\partial}{\partial x}\right)^p \left\{\frac{\mathrm{e}^{-aLx}}{a(a^2 - b^2)(a^2 - c^2)}\right. \\ &\qquad \left. + \frac{\mathrm{e}^{-bLx}}{b(b^2 - a^2)(b^2 - c^2)} + \frac{\mathrm{e}^{-cLx}}{c(c^2 - a^2)(c^2 - b^2)}\right\}, \end{aligned} \tag{6.17}$$

and if we write the explicit dependence of the cofactor components on the powers of K in (6.10) in the form

$$N_{ij}\left(L, L, K^0, K^1, K^2, K^3, K^4\right) \equiv N_{ij}(L, L, K), \tag{6.18}$$

it follows that

$$\Phi_{ij}(x) = N_{ij}\left[L, L, I_0(x), I_1(x), I_2(x), I_3(x), I_4(x)\right]/\alpha. \tag{6.19}$$

The tensor components in (6.10) may be substituted into (3.5) to give

$$\begin{aligned} h_{ql}(L, L) &= \lim_{\xi_3 \to 0} \mathrm{i} \int_{K_3=-\infty}^{\infty} \frac{\mathrm{d}K_3}{2\pi} \frac{N_{qk}(L, L, K_3)}{D(L, L, K_3)} C_{kpl3} \, z_p(K_3) \, \mathrm{e}^{-\mathrm{i}K_3\xi_3} \\ &= \lim_{\xi_3 \to 0} \left(\mathrm{i}\,(C_{k1l3} + C_{k2l3})\, L - C_{k3l3} \frac{\partial}{\partial \xi_3}\right) \Phi_{qk}(\xi_3), \end{aligned} \tag{6.20}$$

where the vector $\boldsymbol{z} = (z_1, z_2, z_3) = (L, L, K_3)$. For cubic anisotropy the elasticity tensor components are given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \mu^*\delta_{iJ}\delta_{JL}\delta_{Lk}, \tag{6.21}$$

in which the capital letters imply that the summation convention for repeated suffixes is temporarily suspended. The tensor components

$$a_{ql}(L, L) = \left[\delta_{ql} - h_{ql}(L, L)\right]^{-1}$$

are then obtained by inversion of the 3×3 matrix.

The matrix representation of equation (6.20) gives

$$\boldsymbol{h}(L,L)=\begin{pmatrix} \frac{1}{2} & 0 & h_{13} \\ 0 & \frac{1}{2} & h_{13} \\ h_{31} & h_{31} & \frac{1}{2} \end{pmatrix}, \tag{6.22}$$

where

$$h_{13}=-\mathrm{i}\left\{\frac{\Gamma(a+b+c)+\Lambda abc+\Sigma abc(ab+bc+ac)}{2\mu(\lambda+2\mu+\mu^*)abc(a+b)(b+c)(a+c)}\right\}, \tag{6.23}$$

$$h_{31}=-\mathrm{i}\left\{\frac{\Xi(a+b+c)+\Upsilon abc+\Omega abc(ab+bc+ac)}{2\mu(\lambda+2\mu+\mu^*)abc(a+b)(b+c)(a+c)}\right\}, \tag{6.24}$$

and

$$\left.\begin{array}{ll} \Gamma=-2\lambda(2\mu+\mu^*), & \Lambda=\lambda\mu^*+(2\mu+\mu^*)^2, \\ \Xi=-(2\mu+\mu^*)(2\lambda+4\mu+\mu^*), & \Upsilon=\lambda\mu^*-\mu(4\mu+\mu^*), \\ \Sigma=\mu(\lambda+2\mu+\mu^*), & \Omega=\lambda\mu. \end{array}\right\} \tag{6.25}$$

The matrix $[\boldsymbol{I}-\boldsymbol{h}]$ is now easily inverted and we obtain

$$\boldsymbol{a}(L,L)=\frac{1}{1-8h_{13}h_{31}}\begin{pmatrix} 2-8h_{13}h_{31} & 8h_{13}h_{31} & 4h_{13} \\ 8h_{13}h_{31} & 2-8h_{13}h_{31} & 4h_{13} \\ 4h_{31} & 4h_{31} & 2 \end{pmatrix}. \tag{6.26}$$

The real part of equation (6.3) together with equations (6.7)–(6.26) then gives the exact displacement produced in the cubic half-space by the two-dimensional periodic force distribution. It is of interest to note that for an isotropic material the tensors $\boldsymbol{h}$ and $\boldsymbol{a}$ are hermitian, with the properties

$$h_{pq}(K_1,K_2)=h^*_{qp}(K_1,K_2)=h_{qp}(-K_1,-K_2) \tag{6.27}$$

and

$$a_{pq}(K_1,K_2)=a^*_{qp}(K_1,K_2)=a_{qp}(-K_1,-K_2), \tag{6.28}$$

where the asterisks denote the complex conjugate function. The hermitian properties do not hold for a general anisotropic material, as may be observed from equations (6.22)–(6.26); the real parts of $\boldsymbol{h}$ and $\boldsymbol{a}$ are still symmetric, but the imaginary parts are no longer antisymmetric.

A consistency check on the preceding equations can be achieved by computing the vertical displacement on the surface of an isotropic half-space when the periodic force distribution is also applied on the surface. For an isotropic half-space we may put $\mu^*=0$ and $\lambda=2\mu\nu/(1-2\nu)$ into the preceding equations. The tensors $\boldsymbol{h}$ and $\boldsymbol{a}$ then reduce to the forms given in table 1 with $K_1=K_2=L$, and the constants in (6.13)–(6.15) reduce to $a=b=c=\sqrt{2}$. On the surface of the half-space the vertical displacement due to the force distribution in (6.1) applied on the surface, when calculated from (6.3), (6.7) and (6.8) with $x_3=x_3'=0$, then reduces to

$$u_3(x_1,x_2)=\frac{1-\nu}{2\sqrt{2}\pi\mu}\lambda p_0\cos\frac{2\pi}{\lambda}(x_1+x_2), \tag{6.29}$$

where $\lambda=2\pi/L$ now refers to the wavelength of the force distribution. This result is in conformity with that obtained by substituting (5.16) into (6.2) and putting $\xi_3 = \xi_4 = 0$.

7. Summary

A Fourier integral representation of the Green function for an anisotropic elastic half-space has been developed. For an isotropic material the Fourier integrals can be evaluated in closed form and are shown to yield Mindlin's expression for the Green function. The anisotropic Green function is then used to deduce an exact representation for the vertical displacement in a half-space possessing cubic material symmetry when a two-dimensional periodic vertical force distribution is applied in its interior.

A quadruple Fourier integral is the natural representation for a Green function which depends on the four-vector $\boldsymbol{\xi}$, and is useful when interchanges of the integration order simplifies the algebraic result, as demonstrated in § 6. It is not useful, however, for extracting numerical results. In the case of an infinite medium, the triple Fourier integral representation (the first integral in (3.8)) can be reduced to a single contour integral (Mura 1987) which can be evaluated either by numerical quadrature, or by means of Cauchy's residue theorem, as the sum of Stroh matrices (Malén 1971).

A referee has pointed out that by expressing the half-space Green function in equation (3.8) as $G_{ij} = G_{ij}^{\infty} + G_{ij}^{S}$ and then representing G_{ij}^{S} as the Faltung integral in (4.10), that a double Fourier transform of the surface Green function $G_{kp}(x_1 - x_1', x_2 - x_2', x_3, 0)$ enables it to be expressed in terms of the infinite medium Green function and its first derivatives. An inverse double Fourier transform then yields the surface Green function which can be substituted into the Faltung representation to yield G_{ij}^{S} as a double integral.

Other methods of reduction are also possible. If we write

$$\Psi_{ij}(K_1, K_2, x) = \int_{K=-\infty}^{\infty} \frac{\mathrm{d}K}{2\pi} \frac{N_{ij}(K_1, K_2, K)}{D(K_1, K_2, K)} \mathrm{e}^{-\mathrm{i}Kx}, \tag{7.1}$$

then G_{ij}^{S} in equation (6.4) may be expressed as

$$G_{ij}^{S}(\boldsymbol{\xi}) = -C_{kpq3} \frac{\partial}{\partial x_p} \iint_{-\infty}^{\infty} \frac{\mathrm{d}K_1\,\mathrm{d}K_2}{(2\pi)^2} a_{ql}(K_1, K_2) \times \Psi_{ik}(K_1, K_2, \xi_3)\, \Psi_{lj}^{*}(K_1, K_2, \xi_4)\, \mathrm{e}^{-\mathrm{i}(K_1\xi_1 + K_2\xi_2)}, \tag{7.2}$$

where $\Psi_{ik}(K_1, K_2, x)$ will be similar in form to equation (6.19), but the integral in (6.17) will now contain parameters a, b, c which depend on K_1 and K_2.

Other alternative reductions of the quadruple Fourier integral to lower order integral representations can also be effected by expressing the integration over K_1, K_2, K_3 in spherical polar coordinates.

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Appendix A. Heisenberg delta transformation

The Heisenberg delta function (cf. Sneddon (1951) & Schwinger (1958)) is the Fourier transform of the Heaviside step function, namely

$$\delta_+(K_3+\alpha) = \int_{-\infty}^{\infty} \left\{ e^{i\alpha x_3} H(x_3) \right\} e^{iK_3 x_3} \, dx_3. \tag{A 1}$$

This definition is 2π times the quantity defined in the references, since we have associated the factor 2π with the wave vector integrations rather than with the spatial integrations. From the references cited,

$$\delta_+(K_3) = \pi\delta(K_3) - 1/(iK_3), \tag{A 2}$$

and by the Fourier inversion theorem, the inverse of (A 1) is

$$e^{i\alpha x_3} H(x_3) = \int_{-\infty}^{\infty} \frac{dK_3}{2\pi} \, \delta_+(K_3+\alpha) \, e^{-iK_3 x_3}. \tag{A 3}$$

Appendix B. Green function reciprocity

From the definition of the Green function the displacement at the field point $\boldsymbol{x}$ due to a point force $\boldsymbol{f}(\boldsymbol{x}')$ at the source point $\boldsymbol{x}'$ can be written in the equivalent forms,

$$u_i(\boldsymbol{x}) = G_{ij}(\boldsymbol{x},\boldsymbol{x}') f_j(\boldsymbol{x}') \qquad \text{and} \qquad u_j(\boldsymbol{x}) = G_{ji}(\boldsymbol{x},\boldsymbol{x}') f_i(\boldsymbol{x}'). \tag{B 1}$$

On multiplying these relations by $f_i(\boldsymbol{x})$ and $f_j(\boldsymbol{x})$ respectively, we obtain the equivalent scalar representations,

$$S(\boldsymbol{x},\boldsymbol{x}') = G_{ij}(\boldsymbol{x},\boldsymbol{x}') f_i(\boldsymbol{x}) f_j(\boldsymbol{x}') = G_{ji}(\boldsymbol{x},\boldsymbol{x}') f_i(\boldsymbol{x}') f_j(\boldsymbol{x}). \tag{B 2}$$

We now integrate with respect to $\boldsymbol{x}$ and $\boldsymbol{x}'$ over a volume V which contains both the field and source points to arrive at

$$\iiint_V dV(\boldsymbol{x}) \iiint_V dV(\boldsymbol{x}') \, G_{ij}(\boldsymbol{x},\boldsymbol{x}') f_i(\boldsymbol{x}) f_j(\boldsymbol{x}')$$
$$= \iiint_V dV(\boldsymbol{x}) \iiint_V dV(\boldsymbol{x}') \, G_{ji}(\boldsymbol{x},\boldsymbol{x}') f_i(\boldsymbol{x}') f_j(\boldsymbol{x}). \tag{B 3}$$

Since $\boldsymbol{x}$ and $\boldsymbol{x}'$ are dummy integration variables we can interchange them in the double volume integral on the right-hand side and rearrange the equation to obtain

$$\iiint_V dV(\boldsymbol{x}) \iiint_V dV(\boldsymbol{x}') \left\{ G_{ij}(\boldsymbol{x},\boldsymbol{x}') - G_{ji}(\boldsymbol{x}',\boldsymbol{x}) \right\} f_i(\boldsymbol{x}) f_j(\boldsymbol{x}') = 0. \tag{B 4}$$

Because V and $\boldsymbol{f}(\boldsymbol{x}')$ can be arbitrarily chosen we see that the Green function is subject to the restriction that

$$G_{ij}(\boldsymbol{x},\boldsymbol{x}') = G_{ji}(\boldsymbol{x}',\boldsymbol{x}). \tag{B 5}$$

Moreover, since

$$G_{ij}(\boldsymbol{x},\boldsymbol{x}') = G_{ij}^{\infty}(\boldsymbol{x}-\boldsymbol{x}') + G_{ij}^{S}(\boldsymbol{x},\boldsymbol{x}') \quad \text{and} \quad G_{ij}^{\infty}(\boldsymbol{x}-\boldsymbol{x}') = G_{ji}^{\infty}(\boldsymbol{x}'-\boldsymbol{x}),$$

we also find that

$$G^S_{ij}(\boldsymbol{x}, \boldsymbol{x}') = G^S_{ji}(\boldsymbol{x}', \boldsymbol{x}), \tag{B 6}$$

so that the field and source points can be exchanged in both the full half-space Green function $G_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ and in $G^S_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ provided the tensor suffixes are transposed.

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