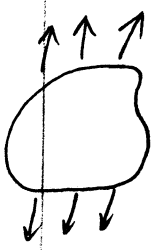
Elasticity Theory



determines stress and displacement in a body as a result of applied (mechanical or thermal) load

—— Barber

A body under elastic deformation reverts to its original state on the removal of loads.

* Elasticity theory is also useful for inelastic deformation, such as fracture and plasticity, by studying the microscopic agents of inelasticity, such as a crack or a dislocation

—— this is called micromechanics

In this course, we will focus on linear, infinitesimal elasticity. Stress and displacements are linear with loads.

Linear superposition can be used to construct solutions.

(* Statics only)

Compared with the undergraduate "Mechanics of Materials" course (ME80 in Stanford), which makes plausible but unsubstantiated assumptions.

Elasticity Theory:

- is a more rigorous treatment
- only makes mathematical assumptions (usually in the last step, to help solve the equation) instead of physical assumptions (hard to justify)
- allows us to assess the quality of assumptions made in "mechanics of materials".
- uses more advanced mathematical tools tensors, partial differential equations, Fourier transform ...

Outline of this course

1. fundamental variables and equations of elasticity:
   variables: stress, strain, displacement, elastic constants
   equations: equilibrium, compatibility
2. Methods to solve these equations:
   2D problems: stress functions
   3D problems: Green functions
3. Applications
   - are going to be interwined with the methods
   - Matlab will be used both for numerical and for symbolic calculations.

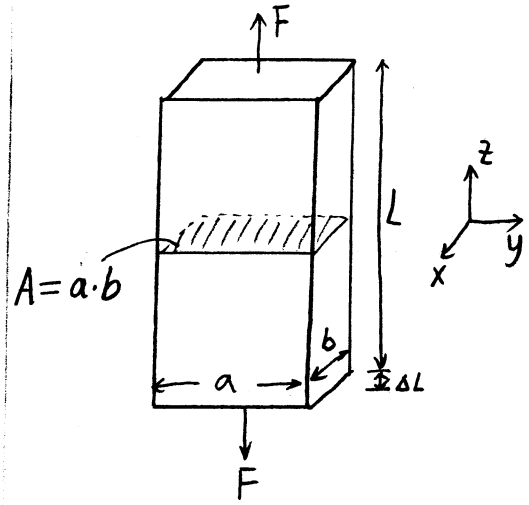
Relationship with other courses.

- undergraduate "Mechanics of materials" as pre-requisite
- students already taken "continuum mechanics" will benefit from the similar notation and assumptions.
  But "continuum mechanics" is not a prerequisite for this class, because we will make a lot of simplifications (linear, infinetesimal elasticity) so that we don't need the full-blown continuum mechanics (finite deformation, etc.)
- Many elasticity problems today are solved by numerical methods such as the "Finite Element Method"

  Elasticity theory provides the fundamental equation to be solved by the numerical methods.

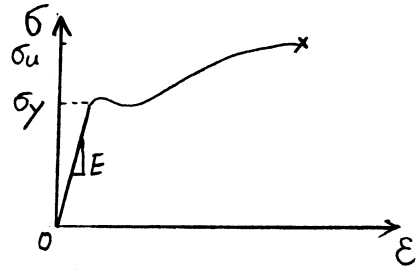
  Analytic solutions also reveal the physics that are difficult to see by numerical methods; they also motivate the

Example 1. (Mechanics of Materials)



(axial) stress: $\sigma_{zz} = \frac{F}{A}$

(axial) strain: $\varepsilon_{zz} = \frac{\Delta L}{L}$



σ_y: yield stress

E: Young's modulus

σ_u: ultimate strength

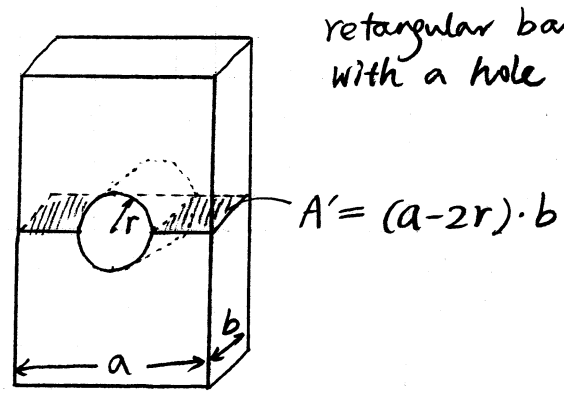
Mechanics of materials:

if $\sigma_{zz} < \sigma_p$ (proportional limit, $\sim \sigma_y$)

then $\sigma_{zz} = E \cdot \varepsilon_{zz}$, $\Delta L = \frac{FL}{EA}$

$\sigma_{zz} < \sigma_y$ is required to prevent <u>yielding</u>

$\sigma_{zz} < \sigma_u$ is required to prevent <u>fracture</u>.



retangular bar with a hole

$$(\sigma_{zz})_{max} = K \cdot \frac{F}{A'}$$

stress concentration factor

Q: where does it come from?

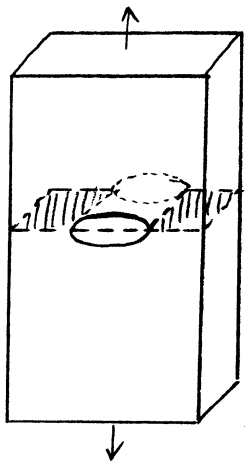
- in "Mechanics of Materials", there is a look-up table.
- in "Elasticity", we compute the stress distribution around the hole.

We will show analytically $K = 3$ when $a \gg r$

* Notice that K is independent of the size (r) of the hole by itself.

It only depends on the ratio $\frac{r}{a}$.

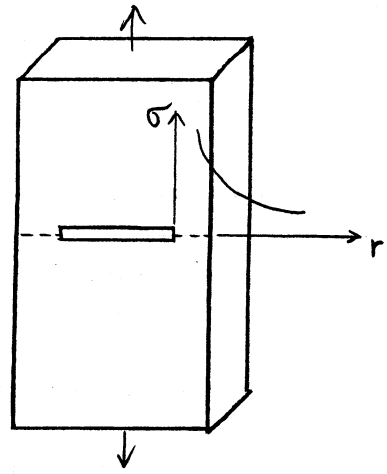
This "scale invariance" is an important feature of Elasticity theory.



elliptic hole

Q: what is the stress concentration factor?

Elasticity theory can answer!

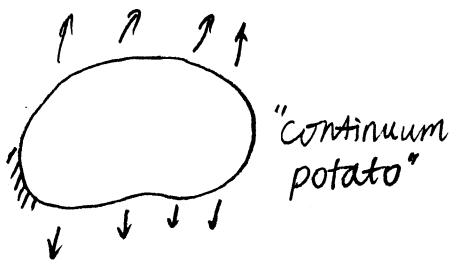


Slit like crack

Stress field becomes singular at crack tip.

Elasticity theory predicts

- $\sigma \sim \frac{1}{\sqrt{r}}$
- stability criteria for crack (advancement)



Given a continuum medium subjected to external loading, we want to find:

1. the displacement field $u_i(\underset{\sim}{x})$ — <u>vector</u>
2. the strain field $\varepsilon_{ij}(\underset{\sim}{x})$
3. the stress field $\sigma_{ij}(\underset{\sim}{x})$

} <u>tensor</u> (rank 2)

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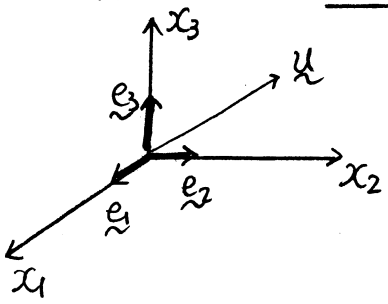
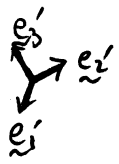
§1. What is a <u>vector</u>?



We can represent a vector by
- an arrow (in a figure)
- a symbol $\sim$
- its components (coordinates)

$$\underset{\sim}{u} = u_1 \underset{\sim}{e}_1 + u_2 \underset{\sim}{e}_2 + u_3 \underset{\sim}{e}_3 = \sum_{i=1}^{3} u_i \underset{\sim}{e}_i$$



<u>index notation</u>: repeated index is summed over from 1 to 3 (dummy index)

$$\underset{\sim}{u} = u_i \underset{\sim}{e}_i$$

Notice here we specify a vector $\underset{\sim}{u}$ by linear combination of three (unit) vectors $\underset{\sim}{e}_i$

$\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ form a <u>coordinate system</u>.

If we choose a different set of (unit) vectors $(\underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3')$ as our coordinate system, then the same vector $\underset{\sim}{u}$ will have different coordinates (u_1', u_2', u_3')

$$\underset{\sim}{u} = u_i \underset{\sim}{e}_i = u_i' \underset{\sim}{e}_i'$$

define: $Q_{ij} = (\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j)$ (dot product)

* Q_{ij} forms an orthogonal matrix

$$Q_{ij} Q_{kj} = \delta_{ik}$$

notice j is a dummy variable. $\delta_{ik} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$ is called <u>Kronecker delta</u>.

In Matrix notation

$$Q \cdot Q^T = I$$

transpose identity matrix

notice $(\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$(\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j') = \delta_{ij}$$

$$u_i' \underset{\sim}{e}_i' = u_i \underset{\sim}{e}_i$$

multiply both sides by $\underset{\sim}{e}_j'$ (dot product)

$$u_j' = u_i (\underset{\sim}{e}_j' \cdot e_i)$$

$$\boxed{u_j' = Q_{ji} u_i}$$

(i is dummy variable, summed over 1, 2, 3
j is a free index, can be 1, 2, 3)

$\Updownarrow$

$$\boxed{\begin{aligned} u_1' &= Q_{11} u_1 + Q_{12} u_2 + Q_{13} u_3 \\ u_2' &= Q_{21} u_1 + Q_{22} u_2 + Q_{23} u_3 \\ u_3' &= Q_{31} u_1 + Q_{32} u_2 + Q_{33} u_3 \end{aligned}}$$

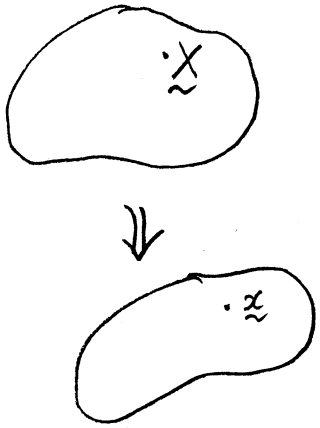
$\Updownarrow$

$$\boxed{u_i' = Q_{ij} u_j}$$

(i is a free index,
j is a dummy index)

The same equation(s) can be expressed using different choices of indices.

§2. <u>displacement (vector) field</u> $u_i(\underset{\sim}{x})$



a material point in the undeformed state is specified by a vector

$$\underset{\sim}{X} = X_i \underset{\sim}{e_i}$$

The same material point in the deformed state is specified by another vector

$$\underset{\sim}{x} = x_i \underset{\sim}{e_i}$$

<u>displacement vector</u> $\underset{\sim}{u} \equiv \underset{\sim}{x} - \underset{\sim}{X}$

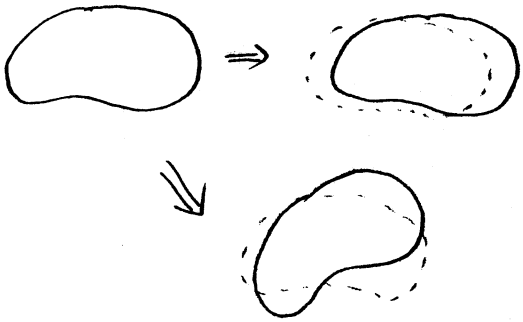
$\underset{\sim}{u}(\underset{\sim}{x})$ is a vector defined for every material point $\underset{\sim}{x}$.
and is called a <u>vector field</u>.
i.e. a vector as a function of another vector.

<u>Infinitesimal elasticity</u>, we assume $|\underset{\sim}{u}| \ll 1$, so that we do not distinguish $\underset{\sim}{u}(\underset{\sim}{x})$ v.s. $\underset{\sim}{u}(\underset{\sim}{X})$

This is a great simplification.
No longer valid for large deformation, which is treated in continuum mechanics.

§3. <u>strain field</u> $\varepsilon_{ij}(\underset{\sim}{x})$

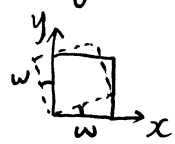
a displacement field $u_i(\underset{\sim}{x})$ does not necessarily lead to "deformation."



rigid-body translation

$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{u}_0$ constant vector

rigid-body rotation



$u_y = +\omega \cdot x$

$u_x = -\omega \cdot y$

(to first order of x, y)

Taylor expand displacement field $u_i(\underset{\sim}{x})$ up to 1st order.

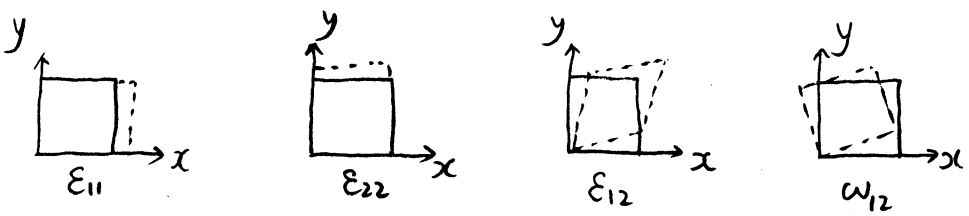
$$u_i(\underset{\sim}{x}) = u_i^0 + \frac{\partial u_i}{\partial x_j} dx_j$$

index notation $u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}$

$$= u_i^0 + u_{i,j}\, dx_j$$

$$= u_i^0 + \frac{1}{2}(u_{i,j} + u_{j,i})\, dx_j + \frac{1}{2}(u_{i,j} - u_{j,i})\, dx_j$$

$$\boxed{\begin{aligned} u_i &= u_i^0 + \varepsilon_{ij}\, dx_j + \omega_{ij}\, dx_j \\ \varepsilon_{ij} &\equiv \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{strain: } \varepsilon_{ij} = \varepsilon_{ji} \text{ symmetric} \\ \omega_{ij} &\equiv \frac{1}{2}(u_{i,j} - u_{j,i}) \qquad \text{rotation: } \omega_{ij} = -\omega_{ji} \text{ anti-symmetric} \end{aligned}}$$



strain strain strain rotation

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

This is the strain component when we express vectors $\underset{\sim}{u}$ and $\underset{\sim}{x}$ in the coordinate system of $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}$

Q: What happens if we choose a different coordinate system?

$$\underset{\sim}{u} = u_i' \underset{\sim}{e_i'}$$

$$\underset{\sim}{x} = x_i' \underset{\sim}{e_i'}$$

$$\varepsilon_{ij}' \equiv \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'}\right)$$

Q: how does ε_{ij} transform with a change of coordinate system?

We can show that

$$\boxed{\varepsilon_{ij}' = Q_{ip} Q_{jq} \varepsilon_{pq}}$$

p, q are dummy indices
i, j are free indices.

* recall $u_i' = Q_{ij} u_j$

The "rule" is that: the first index of Q is the free index
the second index of Q is the dummy.

* The relation in the box can be shown through the "chain rule"

notice that $x_j' = Q_{jq} x_q$ — because $\underset{\sim}{x}$ is a vector

(be careful!) $\rightarrow$ $x_q = Q_{jq} x_j'$ — because $Q \cdot Q^T = I$
Q's inverse is its transpose

we have $\dfrac{\partial x_q}{\partial x_j'} = Q_{jq}$

$$\frac{\partial}{\partial x_j'} f = \left(\frac{\partial}{\partial x_q} f\right) \cdot \left(\frac{\partial x_q}{\partial x_j'}\right) = Q_{jq} \frac{\partial}{\partial x_q} f$$

$f_{,j'} = Q_{jq} f_{,q}$ — spatial derivatives transform as vector

ε_{ij} is a matrix of 9 numbers that transform according to $\varepsilon'_{ij} = Q_{ip} Q_{jq} \varepsilon_{pq}$ — which is the definition of a rank-2 tensor.

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* An arbitrary ranked tensor $A_{ijklm\cdots n}$ is a set of numbers that transform as

$$A'_{ijklm\cdots n} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} Q_{mt} \cdots Q_{nu} A_{pqrst\cdots u}$$

Always: the first index of Q is free
the second index of Q is dummy
$Q_{ip} \equiv (\underset{\sim}{e}'_i \cdot \underset{\sim}{e}_p)$

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* The differential operator is a vector

$$\underset{\sim}{\nabla} = \frac{\partial}{\partial x_i}(\cdot)\ \underset{\sim}{e}_i \qquad \left(\text{e.g. } \underset{\sim}{\nabla} f = \left(\frac{\partial f}{\partial x_i}\right) \underset{\sim}{e}_i\right)$$

* A tensor can also be expressed in vector notation

$$\underset{\approx}{\varepsilon} = \varepsilon_{ij} (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)$$

$\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ is a tensor product — also called dyadic notation

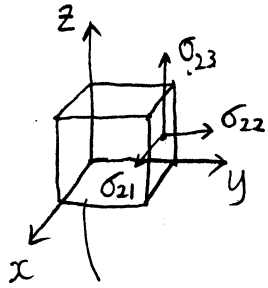
$$= \frac{1}{2}\left[(\underset{\sim}{\nabla} \underset{\sim}{u}) + (\underset{\sim}{\nabla} \underset{\sim}{u})^T\right]$$

Reading assignment:

Sadd Section 1.1 – 1.8

* see section 1.9 for complications that can arise in curvilinear (e.g. cylindrical or polar) coordinate systems.

§4. stress field.



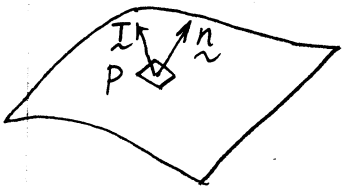
"the stress cube"

σ_{ij} = force per unit area on i-th face in j-th direction (*note correction from previous version!)

* This is the Cauchy stress in continuum mechanics.

Given the stress field, we can obtain the traction force $\underset{\sim}{T}$ per unit area on any surface element with normal vector $\underset{\sim}{n}$:

$$\boxed{T_j = \sigma_{ij}\, n_i}$$



We can show (but not here) that the stress is also a rank-2 (symmetric) tensor, i.e. it transforms as

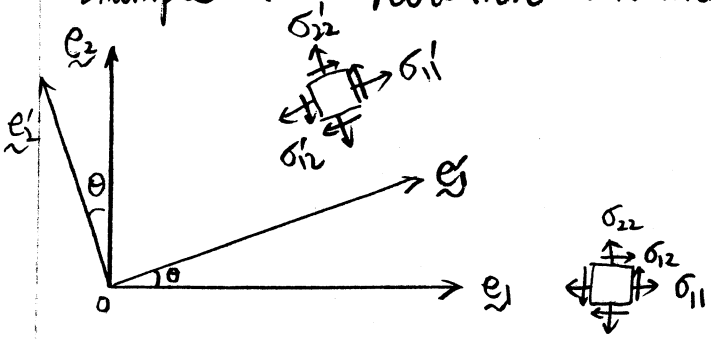
$$\boxed{\sigma'_{ij} = Q_{ip}\, Q_{jq}\, \sigma_{pq}} \qquad \sigma_{ij} = \sigma_{ji}, \qquad \sigma'_{ij} = \sigma'_{ji}$$

* both $\underset{\sim}{T}$ and $\underset{\sim}{n}$ are vectors. i.e. $n'_i = Q_{ip}\, n_p$

$T'_j = Q_{jq}\, T_q$

In fact, we can use these two relations to prove the relation in the box.

Example 1: Rotation around z-axis



$Q_{11} = \underset{\sim}{e}_1' \cdot \underset{\sim}{e}_1 = \cos\theta \quad Q_{12} = \underset{\sim}{e}_1' \cdot \underset{\sim}{e}_2 = \sin\theta$

$Q_{21} = \underset{\sim}{e}_2' \cdot \underset{\sim}{e}_1 = -\sin\theta \quad Q_{22} = \underset{\sim}{e}_2' \cdot \underset{\sim}{e}_2 = \cos\theta$

$Q_{13} = Q_{23} = Q_{31} = Q_{32} = 0$

$Q_{33} = 1$

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

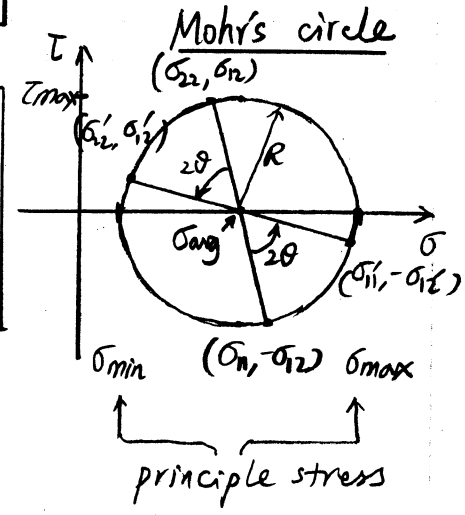
$$\boxed{\begin{aligned} \sigma_{11}' &= \cos^2\theta\, \sigma_{11} + \sin^2\theta\, \sigma_{22} + 2\sin\theta\cos\theta\, \sigma_{12} \\ \sigma_{22}' &= \sin^2\theta\, \sigma_{11} + \cos^2\theta\, \sigma_{22} - 2\sin\theta\cos\theta\, \sigma_{12} \\ \sigma_{12}' &= -\sin\theta\cos\theta\, \sigma_{11} + \sin\theta\cos\theta\, \sigma_{22} + (\cos^2\theta - \sin^2\theta)\, \sigma_{12} \end{aligned}}$$

$\Leftarrow \sigma_{ij}' = Q_{ip} Q_{jq} \sigma_{pq}$

or equivalently

$$\boxed{\begin{aligned} \sigma_{11}' &= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta \\ \sigma_{22}' &= \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta - \sigma_{12}\sin 2\theta \\ \sigma_{12}' &= -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \end{aligned}}$$

Mohr's circle



$$\sigma_{avg} = \frac{\sigma_{11}+\sigma_{22}}{2}$$

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

$$\sigma_{max} = \sigma_{avg} + R$$

$$\sigma_{min} = \sigma_{avg} - R$$

Stress points rotate twice as fast in Mohr's circle as in real space (2θ v.s. θ)

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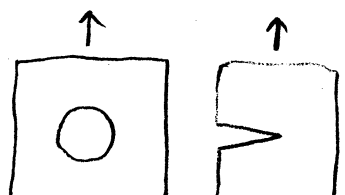
* Strain tensor satisfies a similar transformation rule. Hence there exist a Mohr's circle for strain.

* These relations are useful in <u>polar coordinates</u>, which are convenient for these problems



Both stress and strain are symmetric, rank-2 tensors. that are defined through

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \varepsilon_{ij} = \varepsilon_{ji}$$

$$T_j = \sigma_{ij} n_i \qquad \sigma_{ij} = \sigma_{ji}$$

and they transform as follows

$$\varepsilon'_{ij} = Q_{ip} Q_{jq} \varepsilon_{pq}$$

$$\sigma'_{ij} = Q_{ip} Q_{jq} \sigma_{pq}$$

<u>spherical strain</u>

$$\widetilde{\varepsilon_{ij}} \equiv \frac{1}{3} \varepsilon_{kk} \delta_{ij}$$

<u>deviatoric strain</u>

$$\widehat{\varepsilon_{ij}} \equiv \varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \delta_{ij}$$

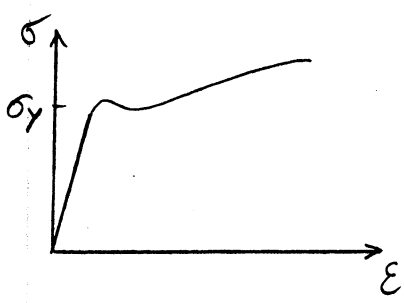
<u>spherical stress</u>

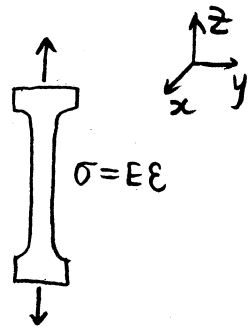
$$\widetilde{\sigma_{ij}} \equiv \frac{1}{3} \sigma_{kk} \delta_{ij}$$

<u>deviatoric strain</u>

$$\widehat{\sigma_{ij}} \equiv \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

§1. Hooke's Law





– Assuming <u>isotropic material</u>

<u>tensile test</u>

$$\sigma_{zz} = E\,\varepsilon_{zz}$$

E: Young's modulus

$$\sigma_{xx} = \sigma_{yy} = 0$$

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$$

i.e. σ_{zz} is the only non-zero stress component ($\sigma_{zz} > 0 \Leftrightarrow$ tension)

Q: Is ε_{zz} the only non-zero strain component?

$$\varepsilon_{xx} = \varepsilon_{yy} = -\nu\,\varepsilon_{zz} \qquad (\varepsilon_{xx} < 0 \Leftrightarrow \text{contraction})$$

Poisson effect. ν — Poisson's ratio ~0.2–0.3 for most mater.

$$\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

If we put an isotropic material under pure shear,

i.e. $\varepsilon_{xy} \neq 0$, $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0$.

then $\sigma_{xy} = 2\mu\, \varepsilon_{xy}$ $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = \sigma_{31} = 0$

(we will see why the factor of "2" later)

μ: shear modulus.

But an isotropic material only has <u>2 independent</u> elastic constants. (we will see why later)

$$\boxed{E = 2\mu\,(1+\nu)}$$

— We need to remember this one.

§2. Generalized Hooke's Law.

$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$ — k, l dummy indices

C_{ijkl} — elastic stiffness tensor. (rank-4)
usually called elastic constants

As a tensor, C_{ijkl} transform as follows

$$\boxed{C'_{ijkl} = \qquad\qquad\qquad C_{pqrs}}$$

← fill in the blanks

Because $\sigma_{ij} = \sigma_{ji}$, $\varepsilon_{ij} = \varepsilon_{ji}$, C_{ijkl} has symmetries as well

$C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$ ← minor symmetries

We also have

$C_{ijkl} = C_{klij}$ ← major symmetry

(We will see later that it is a consequence of)

σ_{ij} or ε_{ij} 9 components

$(\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}, \cdots)$

$\downarrow$ due to symmetry $\sigma_{ij} = \sigma_{ji}$

6 <u>independent components</u>

$(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12})$

C_{ijkl} 81 components

$(C_{1111}, C_{1112}, C_{1113}, C_{1121}, \cdots)$

$\downarrow$ due to major and minor symmetries

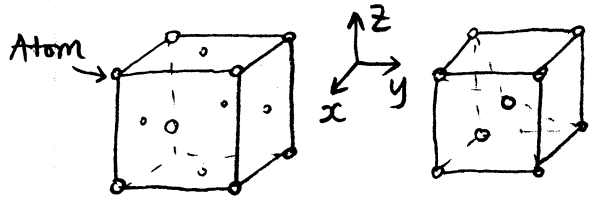
21 <u>independent components</u>

(can you enumerate them?)

for a general anisotropic material.

But real materials usually has more symmetries, which further reduces the number of independent elastic constants.

Most engineering materials are made of crystals having a cubic symmetry.



face-centered cubic crystal structure (FCC) e.g. Al, Cu, Au

body-centered cubic crystal structure (BCC) e.g. Fe, W, Nb

The cubic symmetry reduces the number of independent elastic constants to <u>3</u>

$$\boxed{\begin{aligned} C_{1111} = C_{2222} = C_{3333} &\equiv C_{11} \\ C_{1122} = C_{2233} = C_{3311} &\equiv C_{12} \\ C_{1212} = C_{1313} = C_{2323} &\equiv C_{44} \end{aligned}}$$

other components are obtained either by symmetry or are zero.

* This is the case only when the x, y, z axes of the coordinate system are aligned with the cubic axes of the crystal.

Otherwise we get C'_{ijkl} which follows the tensor's transformation rule and is in general non-zero. (Homework)

---

cubic material

Generalized Hooke's Law

$$\boxed{\begin{aligned}
\sigma_{11} &= C_{11}\,\varepsilon_{11} + C_{12}\,\varepsilon_{22} + C_{12}\,\varepsilon_{33} \\
\sigma_{22} &= C_{12}\,\varepsilon_{11} + C_{11}\,\varepsilon_{22} + C_{12}\,\varepsilon_{33} \\
\sigma_{33} &= C_{12}\,\varepsilon_{11} + C_{12}\,\varepsilon_{22} + C_{11}\,\varepsilon_{33} \\
\sigma_{12} &= 2\,C_{44}\,\varepsilon_{12} \\
\sigma_{23} &= 2\,C_{44}\,\varepsilon_{23} \\
\sigma_{31} &= 2\,C_{44}\,\varepsilon_{31}
\end{aligned}}$$

Q: what is the physical meaning of C_{11}, C_{12}, C_{44}?

Isotropic material

can be regarded as a special case of a cubic material which satisfies $\boxed{C_{11} = C_{12} + 2\,C_{44}}$

* We can define <u>anisotropic factor</u> $A \equiv \dfrac{2\,C_{44}}{C_{11} - C_{12}}$

$$A = 1 \iff \text{isotropic material}$$

Many engineering materials, e.g. metals, are <u>isotropic</u> because they are <u>polycrystals</u>, i.e. made of many small randomly oriented grains, even though each crystal grain is elastically anisotropic

For isotropic materials, the Lamé constants (λ, μ) are usually used

$$\boxed{\begin{aligned} C_{12} &= \lambda \\ C_{44} &= \mu \\ C_{11} &= \lambda + 2\mu \end{aligned}}$$

In index notation

$$\boxed{C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

← can be derived by plugging the above expr. to $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$. a good exercise for index notation.

(* notice the factor of "2")

The inverse of expression $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ is $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$, S_{ijkl} is the elastic compliance tensor.

For isotropic material

$S_{ijkl} =$ _____ (Homework)

$\varepsilon_{ij} =$ _____

$$\boxed{\begin{aligned} \varepsilon_{11} &= \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} \\ \varepsilon_{22} &= -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} \\ \varepsilon_{33} &= -\frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{33} \\ \varepsilon_{12} &= \frac{1}{2\mu}\sigma_{12} \\ \varepsilon_{23} &= \frac{1}{2\mu}\sigma_{23} \\ \varepsilon_{31} &= \frac{1}{2\mu}\sigma_{31} \end{aligned}}$$

(isotropic material)

§3. Thermo-elasticity

The Generalized Hooke's Law describes the relationship between <u>stress</u> and <u>elastic strain</u>, more precisely,

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}^{el}$$

(* don't ever say "elastic stress")

Temperature change can also cause shape change (thermal expansion)

Thermal strain is the strain under zero stress

$$\varepsilon_{ij}^{T} = \alpha_{ij}(T - T_0)$$

T_0 : reference temperature

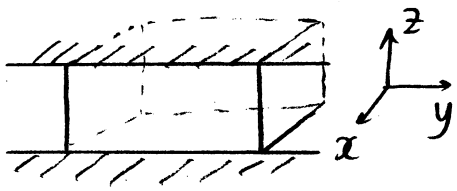
α_{ij} : linear thermal expansion coefficient

isotropic material $\alpha_{ij} = \alpha\,\delta_{ij}$

$$\varepsilon_{ij}^{T} = \alpha(T - T_0)\cdot\delta_{ij}$$

$$\varepsilon_{ij}^{tot} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{T}$$

Example 1: Stress can be developed if material is not allowed to expand freely.



A material constrained between two rigid plates

rigid plates $\Longleftrightarrow$ $\varepsilon_{zz}^{tot} = 0$

free expansion in x, y $\Leftrightarrow$ $\sigma_{xx} = \sigma_{yy} = 0$

our task is to find the remaining stress, strain components, e.g. σ_{zz}, ε_{xx}, ε_{yy}

$$0 = \varepsilon_{zz}^{tot} = \varepsilon_{zz}^{el} + \varepsilon_{zz}^{T} \qquad \varepsilon_{zz}^{el} = -\varepsilon_{zz}^{T} = -\alpha(T - T_0)$$

$$\varepsilon_{zz}^{el} = \frac{1}{E}(-\nu\cancel{\sigma_{xx}} - \nu\cancel{\sigma_{yy}} + \sigma_{zz}) \qquad \sigma_{zz} = E\,\varepsilon_{zz}^{el} = -\alpha E(T - T_0)$$

$$\varepsilon_{xx}^{el} = \frac{1}{E}(\cancel{\sigma_{xx}} - \nu\cancel{\sigma_{yy}} - \nu\sigma_{zz}) = -\frac{\nu}{E}\sigma_{zz} = \alpha\nu(T - T_0) = \varepsilon_{yy}^{el}$$

Fundamental Equations of Elasticity:

compatibility. Equilibrium.

§1. Compatibility condition for strain

u_i has 3 degrees of freedom (at every point $\underset{\sim}{x}$)

ε_{ij} has 6 degrees of freedom

given $u_i(\underset{\sim}{x})$, we can always find $\varepsilon_{ij}(\underset{\sim}{x})$ by differentiation: $\varepsilon_{ij}(\underset{\sim}{x}) = \frac{1}{2}(u_{i,j} + u_{j,i})$

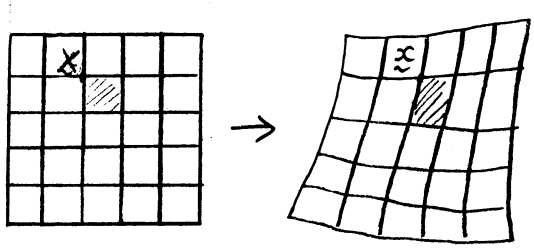
on the other hand, if we are given an arbitrary $\varepsilon_{ij}(\underset{\sim}{x})$, we may not always be able to find a single valued, continuous u_j

In order to be able to find a corresponding $u_j(\underset{\sim}{x})$ $\varepsilon_{ij}(\underset{\sim}{x})$ must satify some constraints.

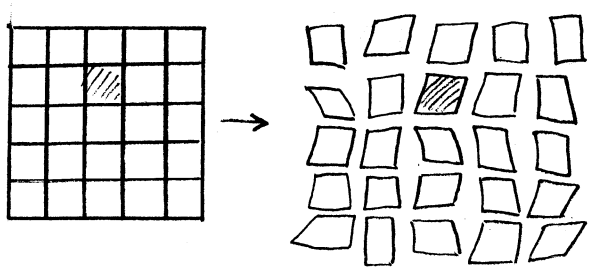
$$\boxed{\text{compatibility condition: } \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0}$$

— This can be verified by plugging it into $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

* What is more difficult is to show this is a sufficient condition for us to find u_j corresponding to ε_{ij}. (omitted here)



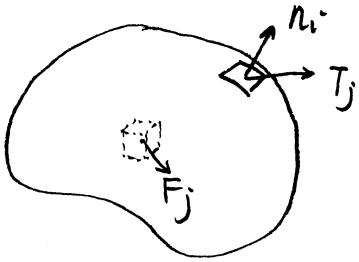
Given a displacement field $u_i(\underset{\sim}{x})$, it is straightforward to obtain the strain field.



Given a strain field, $\varepsilon_{ij}(\underset{\sim}{x})$, we can imagine breaking the original medium into many pieces, each is deformed according to the local strain.

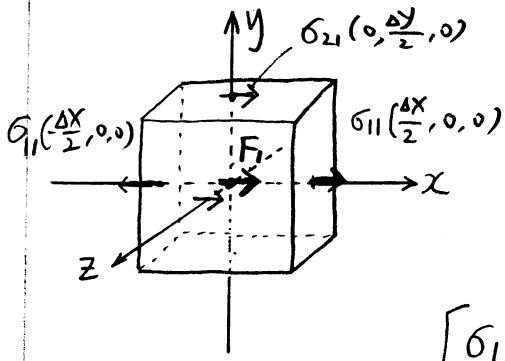
— but the deformed pieces may not fit together (incompatible).

§2. Equilibrium condition for Stress



consider a continuum body subjected to traction force T_j per unit area on the surface (normal vector n_i) and body force F_j per unit volume

$$\boxed{\text{equilibrium condition: } \sigma_{ij,i} + F_j = 0}$$



cube size $\Delta x \cdot \Delta y \cdot \Delta z$

recall: σ_{ij} is force per unit area on i-th face of the stress cube in j-th direction

Total force balance in x-direction:

$$\left[\sigma_{11}\left(\frac{\Delta x}{2},0,0\right) - \sigma_{11}\left(-\frac{\Delta x}{2},0,0\right)\right]\cdot \underbrace{\Delta y \Delta z}_{\text{area}}$$

$$+ \left[\sigma_{21}\left(0,\frac{\Delta y}{2},0\right) - \sigma_{21}\left(0,-\frac{\Delta y}{2},0\right)\right]\cdot \Delta x \Delta z$$

$$+ \left[\sigma_{31}\left(0,0,\frac{\Delta z}{2}\right) - \sigma_{31}\left(0,0,-\frac{\Delta z}{2}\right)\right]\cdot \Delta x \Delta y$$

$$+ F_1(0,0,0)\cdot \Delta x \Delta y \cdot \Delta z = 0$$

In the limit of $\Delta x, \Delta y, \Delta z \to 0$

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + F_1 = 0$$

$\sigma_{i1,i} + F_1 = 0$ — total force balance in x-direction.

Similarly: $\sigma_{i2,i} + F_2 = 0$ — total force balance in y-direction

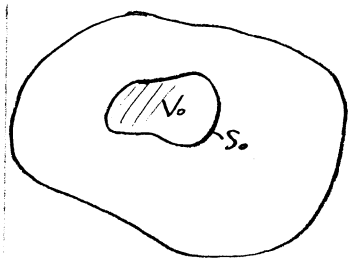
$\sigma_{i3,i} + F_3 = 0$ — total force balance in z-direction.

$\therefore$ $\sigma_{ij,i} + F_j = 0.$ j is free index

A more formal proof can be obtained using <u>Gauss's Theorem</u> (divergence theorem)

$$\int_S f_i n_i \, dS = \int_V f_{i,i} \, dV \quad \text{(Index notation)}$$

$$\int_S \underset{\sim}{f} \cdot \underset{\sim}{n} \, dS = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{f} \, dV \quad \text{(vector notation)}$$



consider an arbitrary volume V_0 inside the elastic medium.

Force equilibrium requires

$$\underbrace{\int_{S_0} T_j \, dS}_{\text{traction force on the surface } S_0} + \underbrace{\int_{V_0} F_j \, dV}_{\text{body force inside } V_0} = 0$$

$$\int_{S_0} \sigma_{ij} n_i \, dS + \int_{V_0} F_j \, dV = 0$$

Apply Gauss's theorem, considering σ_{ij} as f_i

$$\int_{V_0} (\sigma_{ij,i} + F_j) \, dV = 0$$

Because this is true for any volume V_0,

$$\sigma_{ij,i} + F_j = 0$$

at every point inside the continuum body.

In Summary, here are all the fundamental Equations of elasticity relating the following variables

displacement		strain		stress		traction force
u_i	——	ε_{ij}	——	σ_{ij}	——	T_j

definition of strain, stress: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$T_j = \sigma_{ij} n_i$

compatibility: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$

equilibrium: $\sigma_{ij,i} + F_j = 0$

elastic constitutive law:
- Generalized Hooke's Law for isotropic medium

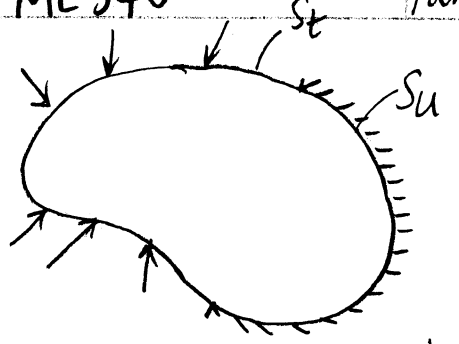
$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$

or $\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$

All elasticity problems in this class satisfy the same equations as above.

Different applications correspond to different boundary conditions, which will lead to different solutions.

Boundary Value Problem (B.V.P.): Find the u_i, ε_{ij}, σ_{ij} fields that satisfy compatibility, equilibrium, and constitutive equations and the specified boundary conditions.



In general, two types of boundary conditions can be applied on the surface.

traction boundary condition: $T_j(\underset{\sim}{x}) = g_j(\underset{\sim}{x})$ on S_t

i.e. $\sigma_{ij}(\underset{\sim}{x}) \cdot n_i(\underset{\sim}{x}) = g_j(x)$ on S_t

or

displacement boundary condition $u_i(\underset{\sim}{x}) = h_i(x)$ on S_u.

(or some combination of the two)

In the remaining of this class, we will

1. practice how to <u>formulate</u> the B.V.P.
2. learn the tools to <u>solve</u> the B.V.P.

General strategies to solve B.V.P.

1. Avoid compatibility eq. by working with u_i directly (<u>displacement formulation</u>)

$\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$

$\downarrow$ plug into equilibrium cond.

$\mu u_{i,kk} + (\lambda+\mu) u_{k,ki} + F_i = 0$

$\mu \nabla^2 \underset{\sim}{u} + (\lambda+\mu) \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{F} = 0$

Boundary Condition:

S_u: is easy

S_t: is a little more complicated.

mostly for <u>3D</u> problems

2. <u>Stress formulation</u>

need to work with compatibility condition.

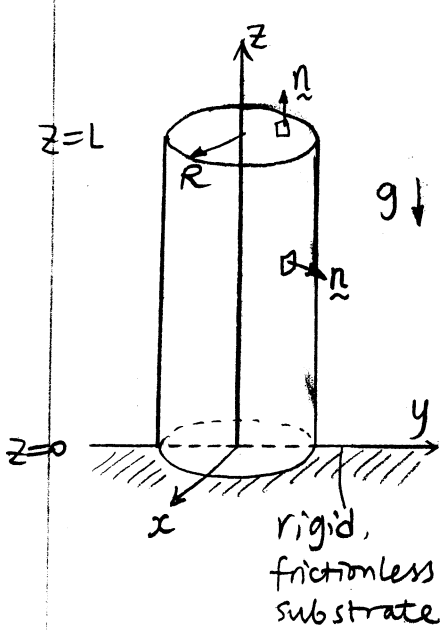
rewrite compatibility cond. in terms of stress

+ equilibrium cond. $\sigma_{ij,i} + F_j = 0$

$\Rightarrow$ Beltrami-Michel compatibility eq.

$$\sigma_{ij,kk} + \frac{1}{1+\nu} \sigma_{kk,ij} = -\frac{\nu}{1-\nu} \delta_{ij} F_{k,k} - F_{i,j} - F_{j,i}$$

mostly for <u>2D</u> problems



Example: Elastic rod standing vertically in gravitational field on a rigid, frictionless substrate

Formulation of the B.V.P.

body force $F_z = -\rho g$

equilibrium: $\sigma_{ij,i} + F_j = 0$

i.e. $\begin{cases} \sigma_{ix,i} = 0 \\ \sigma_{iy,i} = 0 \\ \sigma_{iz,i} = \rho g \end{cases}$

Boundary Condition:

top surface: $(z=L)$	zero traction force $(\underset{\sim}{n} = \underset{\sim}{e}_z)$ $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$	S_t
side surface: $\sqrt{x^2+y^2} = R$	zero traction force $\sigma_{xi} n_i = \sigma_{yi} n_i = \sigma_{zi} n_i = 0$	S_t
bottom surface: $z=0$	$u_z = 0$, $\sigma_{xz} = \sigma_{yz} = 0$	mixed B.C.

How to solve this B.V.P.?

By trial and error.

Step ①

Let's <u>try</u> a solution with

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0$$

the only non-zero stress component is σ_{zz}

equilibrium condition reduces to: $\frac{\partial}{\partial z}\sigma_{zz} = \rho g$

$\therefore \quad \sigma_{zz} = \rho g z + C$ (C is a constant)

Boundary Condition: $\sigma_{zz}=0$ at $z=L$

$$\therefore \quad C=-\rho g L$$

$$\sigma_{zz} = \rho g (z-L)$$

(This is a solution that satisfies the equilibrium condition and traction boundary conditions.)

Step ② Let's find all strain components from the generalized Hooke's Law

$$\varepsilon_{xx} = -\frac{\nu}{E}\sigma_{zz} = -\frac{\nu\rho g}{E}(z-L)$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{zz} = -\frac{\nu\rho g}{E}(z-L)$$

$$\varepsilon_{zz} = \frac{1}{E}\sigma_{zz} = \frac{\rho g}{E}(z-L)$$

$$\varepsilon_{xy}=\varepsilon_{yz}=\varepsilon_{zx}=0 \qquad \text{(all shear strains components zero)}$$

(So we find all strain components as well. Does the strain satisfy compatibility condition? We will find out by trying to find the displacement solution.)

Step ③ Try to find displacement u_x, u_y, u_z.

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$$

arbitrary function

$$\varepsilon_{zz} = \frac{\partial}{\partial z}u_z = \frac{\rho g}{E}(z-L) \xrightarrow{\text{integrate}} u_z = \frac{\rho g}{E}\left(\frac{z^2}{2}-Lz\right) + f(x,y)$$

$$\varepsilon_{xx} = \frac{\partial}{\partial x}u_x = -\frac{\nu\rho g}{E}(z-L) \longrightarrow u_x = -\frac{\nu\rho g}{E}x(z-L) + g(y,z)$$

$$\varepsilon_{yy} = \frac{\partial}{\partial y}u_y = -\frac{\nu\rho g}{E}(z-L) \longrightarrow u_y = -\frac{\nu\rho g}{E}y(z-L) + h(x,z)$$

Q: How do we determine the unknown functions $f(x,y)$, $g(y,z)$, $h(x,z)$?

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0 \rightarrow \frac{\partial g(y,z)}{\partial y} + \frac{\partial h(x,z)}{\partial x} = 0 \quad \text{---(1)}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0 \rightarrow -\frac{\nu\rho g}{E}y + \frac{\partial h(x,z)}{\partial z} + \frac{\partial f(x,y)}{\partial y} = 0 \quad \text{---(2)}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \rightarrow -\frac{\nu\rho g}{E}x + \frac{\partial g(y,z)}{\partial z} + \frac{\partial f(x,y)}{\partial x} = 0 \quad \text{---(3)}$$

From Eq(2). $$-\frac{\nu\rho g}{E}y + \frac{\partial f(x,y)}{\partial y} = -\frac{\partial h(x,z)}{\partial z} = m(x) \quad \text{---(4)}$$

Left hand side independent of z

Right hand side independent of y

$\therefore$ both sides must be independent of both y and z, i.e. only a function of (x)

$$\frac{\partial f(x,y)}{\partial y} = \frac{\nu\rho g}{E}y + m(x)$$

unknown functions of x but independent of y

$$f(x,y) = \frac{\nu\rho g}{2E}y^2 + m(x)y + p(x) \quad \text{---(5)}$$

Similarly from Eq(3).

$$-\frac{\nu\rho g}{E}x + \frac{\partial f(x,y)}{\partial x} = -\frac{\partial g(y,z)}{\partial z} = n(y) \quad \text{---(6)}$$

independent of z | independent of x | unknown function of y

$$\frac{\partial f(x,y)}{\partial x} = \frac{\nu\rho g}{E}x + n(y)$$

unknown functions of y but independent of x

$$f(x,y) = \frac{\nu\rho g}{2E}x^2 + n(y)x + q(y) \quad \text{---(7)}$$

Combine Eq(5) and (7). $f(x,y)$ must have the following form:

$$\boxed{f(x,y) = \frac{\nu\rho g}{2E}(x^2+y^2) + Cxy + D}$$

i.e. $p(x) = \frac{\nu\rho g}{2E}x^2 + D$

$q(y) = \frac{\nu\rho g}{2E}y^2 + D$

$m(x) = C\cdot x$

$n(y) = C\cdot y$

We still need to determine the unknown constants C and D.

From Eq.(4) $-\frac{\partial h(x,z)}{\partial z} = m(x) = Cx \rightarrow h(x,z) = -Cxz + r(x)$

From Eq.(6) $-\frac{\partial g(y,z)}{\partial z} = n(y) = Cy \rightarrow g(y,z) = -Cyz + s(y)$

Insert these results into Eq.(1): $\frac{\partial g(y,z)}{\partial y} + \frac{\partial h(x,z)}{\partial x} = 0$

$$-Cz + s'(y) - Cz + r'(x) = 0$$

$$s'(y) + r'(x) = 2Cz$$

left hand side independent of z — Right hand side independent of x, y

$\therefore$ $\boxed{C = 0}$

$$s'(y) = -r'(x) = E \leftarrow \therefore \text{ both sides must be a constant } E.$$

independent of x — independent of y

$$s(y) = Ey + G$$
$$r(x) = -Ex + H$$

$\therefore$ $\boxed{\begin{aligned} h(x,z) &= -Ex + H \\ g(y,z) &= Ey + G \end{aligned}}$

$$\begin{cases} u_z = \frac{\rho g}{E}\left(\frac{z^2}{2} - Lz\right) + \frac{\nu \rho g}{2E}(x^2+y^2) + D \\ u_x = -\frac{\nu \rho g}{E} x(z-L) + Ey + G \\ u_y = -\frac{\nu \rho g}{E} y(z-L) - Ex + H \end{cases}$$

We still need to determine the unknown constants

D, E, G, H

Constants D, G, H corresponds to rigid-body translation

e.g. consider $\begin{cases} u_z = D \\ u_x = G \\ u_y = H \end{cases}$

They can be determined by fixing one point in the Rod as a point of reference — otherwise we get infinite number of solutions

For convenience, let's choose the origin as the reference point

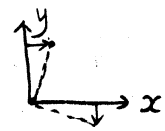
i.e. $u_x(0,0,0) = u_y(0,0,0) = u_z(0,0,0) = 0$.

$\therefore \quad D = G = H = 0.$

also consistent with our B.C.

Constant E corresponds to a rigid-body rotation around z-axis.

e.g. consider $\begin{cases} u_x = Ey \\ u_y = -Ex \end{cases}$



Let's assume the rod does not undergo any rigid-body rotation — otherwise we get infinite number of solutions.

$\therefore \quad E = 0.$

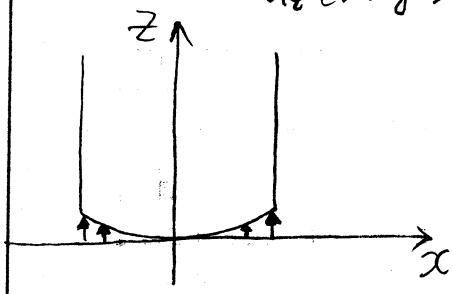
Here is the displacement field of our <u>trial solution</u>:

$$\begin{cases} u_z = \dfrac{\rho g}{E}\left(\dfrac{z^2}{2} - Lz\right) + \dfrac{\nu \rho g}{2E}(x^2+y^2) \\ u_x = -\dfrac{\nu \rho g}{E} x (z-L) \\ u_y = -\dfrac{\nu \rho g}{E} y (z-L) \end{cases}$$

- no more undecided constants
- satisfies both <u>equilibrium</u> and <u>compatibility</u> conditions.
- satisfies <u>all traction</u> boundary conditions.
- does it satisfy the displacement boundary condition $u_z(x,y,0) = 0$?

on the plane $z=0$. violates our B.C.

$$u_z(x,y) = \frac{\nu\rho g}{2E}(x^2+y^2) \neq 0 \ !$$



The bottom of the elastic rod curls up!

How did that happen?

Of course in reality, the bottom of the rod <u>does not</u> curl up. This only shows that the trial solution is not the "true solution". That's too bad ... especially after so much work...

Wait a minute. In elasticity theory, we would say the solution is still useful, in the following sense(s).

First, we can always modify our original problem and let the rod sit on a curved substrate with exactly a shape of

$$u_z(x,y) = \frac{\nu\rho g}{2E}(x^2+y^2) \quad \text{----- i.e. a parabola.}$$

Then our trial solution is the exact solution of this problem.

Second, intuitively we expect our trial solution is pretty good except near the bottom. Because elasticity equations are linear, we can imagine a "correction solution" that can be added to our solution to get the "true solution".

From the Saint Venant's principle, we expect the "correction solution" to be significant only near the rod bottom.

Our trial solution satisfies the displacement B.C. only at a single point $x=0, y=0, z=0$, instead of on the entire plane $z=0$.

Hence we say the displacement B.C. is only satisfied in the <u>weak sense</u>.

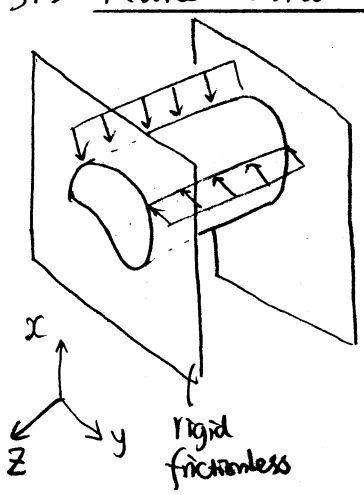
The equations of elasticity can be greatly simplified if we restrict the solution to be 2-Dimensional, i.e. does not depend on one coordinate (e.g. z).

There are several types of 2D elasticity problems:

- plane strain
- plane stress
- anti-plane strain (e.g. *straight screw dislocations)

etc.

§1. Plane Strain



Consider a long bar with ends constrained by rigid-frictionless plates

Let the loads applied on the side of the bar be independent of z.

Then $u_z = 0$ every where

u_x, u_y independent of z.

i.e. the unknowns of this problem is $u_x(x,y)$ and $u_y(x,y)$

We can easily show

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0, \qquad \varepsilon_{yz} = 0, \qquad \varepsilon_{zz} = 0$$

So the only non-zero strain components are

$\varepsilon_{xx}, \quad \varepsilon_{yy}, \quad \varepsilon_{xy}$ — hence the name plane strain.

In isotropic elasticity, $\sigma_{xz} = \sigma_{yz} = 0$.

but in general $\sigma_{zz} \neq 0$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} \rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

equilibrium condition

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \cancel{\frac{\partial \sigma_{zx}}{\partial z}} + F_x = 0 \qquad \left(\frac{\partial}{\partial z} = 0\right)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \cancel{\frac{\partial \sigma_{zy}}{\partial z}} + F_y = 0$$

$$\cancel{\frac{\partial \sigma_{xz}}{\partial x}} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + \cancel{F_z} = 0 \qquad (\sigma_{xz} = \sigma_{yz} = 0,\ F_z = 0)$$

so the only non-trial equations are

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + F_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \end{aligned}}, \qquad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

compatibility condition

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

only one non-trivial equation when $i,j,k,l = x,y$

$$\boxed{\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0}$$

Generalized Hooke's Law

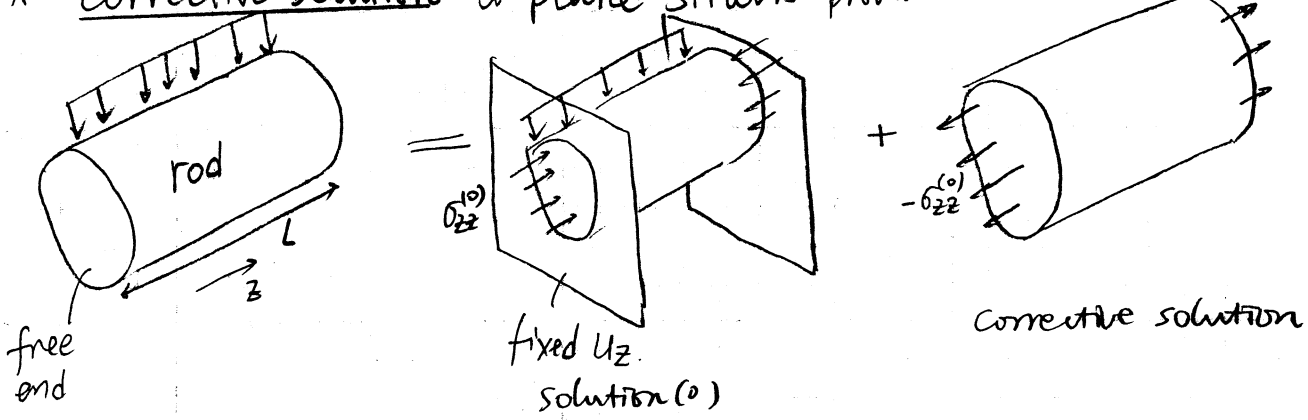
$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} = 0 \rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\therefore \boxed{\begin{aligned} \varepsilon_{xx} &= \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} \\ \varepsilon_{yy} &= -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy} \end{aligned}}$$

* <u>corrective solution</u> to plane strain problem



In reality, we seldom have rods with both ends constrained by rigid, frictionless plates.

However, the plane strain solution is a good approximation to a long rod subjected to lateral loading that is z-independent

To better describe the physical problem (rods with free end), a corrective solution should be added, in which axial forces are applied to the ends.

Because there may be a net force F at the end, the corrective solution is not necessary limited to the neighborhood of the ends.

$$F = -\int_A \sigma_{zz}^{(0)} dA.$$

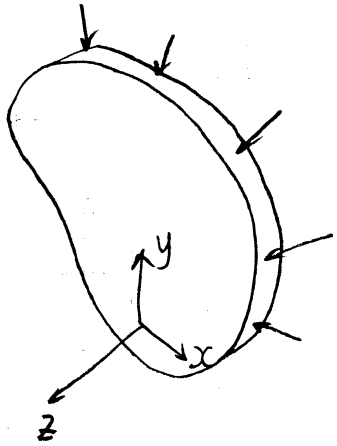
The corrective solution can be approximated by a simple solution

$$\sigma_{zz}^{(1)} = \frac{F}{A} = -\frac{1}{A}\int_A \sigma_{zz}^{(0)} dA.$$

The rod length will change by $\Delta L = \frac{FL}{EA} = -\frac{L}{EA}\int_A \sigma_{zz}^{(0)} dA$

* Of course, the "true" corrective solution is going to be more complicated by this. So we can imagine a correction solution to this correction solution, which should then be localized near the ends.

§2. Plane Stress



The opposite limit of plane strain problem (thick rods) is a thin film — plane stress problem.

Because $\sigma_{xz} = \sigma_{yz} = \sigma_{zz}$ on the surface, if the film is sufficiently thin, we can expect

$$\sigma_{xz} \approx 0, \quad \sigma_{yz} \approx 0, \quad \sigma_{zz} \approx 0$$

everywhere <u>inside</u> the film.

Plane Stress condition:

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \quad , \quad \frac{\partial}{\partial z} = 0.$$

Generalized Hookes Law:

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\ \varepsilon_{yy} &= \frac{1}{E}(-\nu\sigma_{xx} + \sigma_{yy}) \end{aligned}} \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \quad , \quad \varepsilon_{zz} \neq 0.$$

equilibrium condition:

$$\boxed{\begin{aligned} \sigma_{xx,x} + \sigma_{yx,y} + F_x &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + F_y &= 0 \end{aligned}}$$

Same as plane strain

compatibility condition:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

$$\boxed{\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0}$$

$i = j = x, \quad k = l = y$

Same as plane strain

$$* \quad \begin{cases} \varepsilon_{zz,xx} = 0 \\ \varepsilon_{zz,yy} = 0 \\ \varepsilon_{zz,xy} = 0 \end{cases}$$

(cannot be satisfied all at the same time)

$i = j = z, \quad k = l = x$
$i = j = z, \quad k = l = y$
$i = j = z, \quad k = x, \ l = y$

Because $\varepsilon_{zz}(x,y) \neq 0$, we have more compatibility conditions.

§3. Equivalence between Plane Strain and Plane Stress

Both Plane strain and Plane Stress Problems seek solutions for

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ and $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$

as functions of x and y.

They satisfy the same equilibrium and compatibility conditions.

The only difference is in the Generalized Hooke's Law.

Plane Strain	Plane Stress
$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}$	$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}$
$\varepsilon_{yy} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}$	$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy}$
$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$	$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$

introduce Kolosov's constant κ

$$\boxed{\kappa = 3-4\nu} \qquad \boxed{\kappa = \frac{3-\nu}{1+\nu}}$$

Then both Plane Stress and Plane Strain Problems satisfy the same condition

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} \\ \varepsilon_{yy} &= -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{xx} \end{aligned}}$$

Hence we can solve each 2D problem in either plane strain or in plane stress condition and find the other solution by replacing the Kolosov's constant.

§4. Airy Stress function

- Introduce a scalar function $\phi(x,y)$, and write different components of the stress tensor as derivatives of ϕ.
- We need to use 2nd derivatives, so that they transform as rank-2 tensor.
- Assume zero body force: $F_x = F_y = 0$

  then the equilibrium condition becomes

$$\sigma_{xx,x} + \sigma_{xy,y} = 0 \qquad \text{and} \qquad \sigma_{yx,x} + \sigma_{yy,y} = 0$$

- These can be <u>automatically satisfied</u> by defining

$$\boxed{\sigma_{xx} = \phi_{,yy}} \qquad \boxed{\sigma_{yy} = \phi_{,xx}} \qquad \boxed{\sigma_{xy} = -\phi_{,xy}}$$

notice the minus sign!

The equilibrium condition can be easily verified:

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = \phi_{,yyx} - \phi_{xyy} = 0 \\ \sigma_{yx,x} + \sigma_{yy,y} = -\phi_{,xyx} + \phi_{,xxy} = 0 \end{cases}$$

- So the only PDE we need to worry about is the compatibility condition.

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\,\varepsilon_{xy,xy} = 0$$

$$\begin{cases} \varepsilon_{xx} = \frac{\kappa+1}{8\mu}\phi_{,yy} - \frac{3-\kappa}{8\mu}\phi_{,xx} \\ \varepsilon_{yy} = -\frac{3-\kappa}{8\mu}\phi_{,yy} + \frac{\kappa+1}{8\mu}\phi_{,xx} \\ \varepsilon_{xy} = -\frac{1}{2\mu}\phi_{,xy} \end{cases}$$

$$\frac{\kappa+1}{8\mu}(\phi_{,yyyy} + \phi_{,xxxx}) - \frac{3-\kappa}{8\mu}\cdot 2\phi_{,xxyy} + \frac{2}{2\mu}\phi_{,xyxy} = 0$$

$$\frac{\kappa+1}{8\mu}(\phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xxyy}) = 0$$

$$\therefore \quad \boxed{\nabla^4\phi = 0} \qquad \text{or } \nabla^2\nabla^2\phi = 0 \qquad \text{or } \phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xyxy} = 0$$

§5. Airy stress function in the presence of body forces.

When the body force can be written as derivatives of a potential

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y}$$

(e.g. $V = \rho g y$ for gravity)

Then the Airy stress function can be defined as

$$\sigma_{xx} = \phi_{,yy} + V \qquad \sigma_{yy} = \phi_{,xx} + V \qquad \sigma_{xy} = -\phi_{,xy}$$

We can verify that the equilibrium condition

$$\sigma_{xx,x} + \sigma_{xy,y} + F_x = 0$$
$$\sigma_{yx,x} + \sigma_{yy,y} + F_y = 0$$

is automatically satisfied.

The compatibility condition becomes

$$\boxed{\nabla^4 \phi = -2 \cdot \frac{\kappa - 1}{\kappa + 1} \cdot \nabla^2 V}$$

in plane strain	in plane stress
$\nabla^4 \phi = -\left(\frac{1-2\nu}{1-\nu}\right) \nabla^2 V$	?

§6. Example 1

when there is no body force, we have $\nabla^4 \phi = 0$

Let's pick a trial solution

$$\phi(x, y) = \alpha x + \beta y + \gamma$$

It obviously satisfy the condition $\nabla^4 \phi = 0$.

What does this solution mean?

Let's find out the stress field.

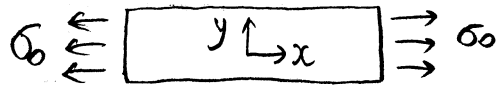
$$\sigma_{xx} = \phi_{,yy} = ?$$
$$\sigma_{yy} = \phi_{,xx} =$$
$$\sigma_{xy} = -\phi_{,xy} =$$

§7. Example 2.

Let's go to higher order polynomials

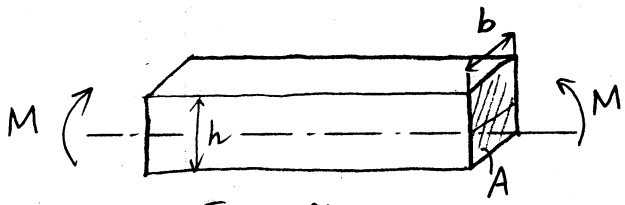
$$\phi(x,y) = \frac{1}{2}Ax^2 + \frac{1}{2}Bxy - Cxy$$

$\sigma_{xx} =$ _____ $\sigma_{yy} =$ _____ $\sigma_{xy} =$ _____



Write down the stress function for a rectangular bar under uniaxial tensile stress σ_0.

$\phi(x,y) =$ _____



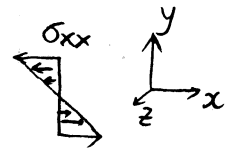
$\sigma_{xx} \propto y$

$$M = -\int_{-h/2}^{h/2} b\,\sigma_{xx}(y)\cdot y\,dy$$

$$\boxed{\sigma_{xx} = -\frac{My}{I_z}}$$

$$I_z = \int_A y^2 dA = \frac{bh^3}{12}$$

↑moment of inertia

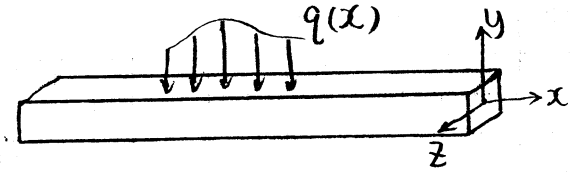


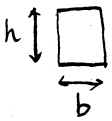
Write down the stress function for a rectangular bar under pure bending M

$\phi(x,y) =$ _____

§1. Review of Euler-Bernoulli Beam Theory

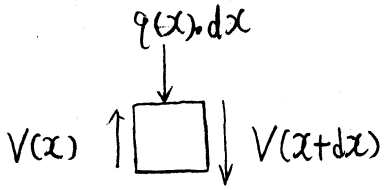
(Mechanics of Materials)





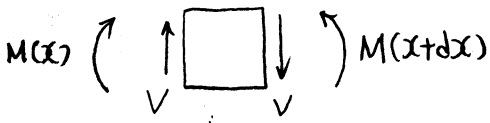
cross section

moment of inertia $I_z = \frac{bh^3}{12}$



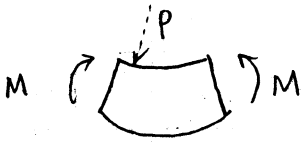
$$\frac{d}{dx}V(x) = -q(x) \quad \text{①}$$

$V(x)$: shear force



$$\frac{d}{dx}M(x) = V(x) \quad \text{②}$$

$M(x)$: bending moment

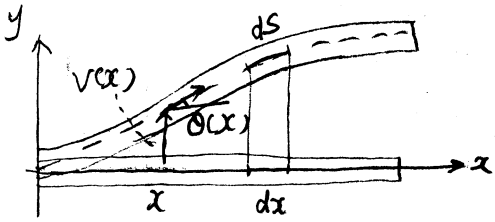


$$K(x) = \frac{M(x)}{EI} \quad \text{③}$$

$K = \frac{1}{\rho}$: curvature

ρ: radius of curvature

E: Young's modulus



from geometry

$$K(x) = \frac{d\theta}{ds}$$

$$\tan\theta(x) = \frac{dv(x)}{dx}$$

$v(x)$: vertical displacement

$\theta(x)$: local beam orientation

in the limit of $\theta(x) \ll 1$

$$K(x) \cong \frac{d^2}{dx^2}v(x) = v''(x) \approx \theta'(x) \quad \text{④}$$

combining ①, ②, ③, ④, we have

4th order ODE, needs 4 B.C.

2 B.C. at each end

$$EI\,v''''(x) = -q(x) \qquad \left(v(x) = u_y(x) \right)$$

our notation

$$EI\,v''(x) = M(x)$$

$$\sigma_{xx}(x,y) = -\frac{M(x)\cdot y}{I_z}$$

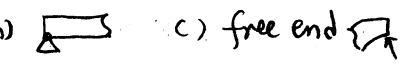
Boundary conditions:

a) cantilever 

$v=0,\ \theta=0$

b) simple (pin) support

$v=0,\ M=0$

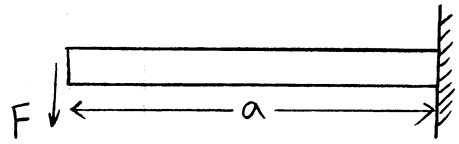
c) free end 

$V=0,\ M=0$

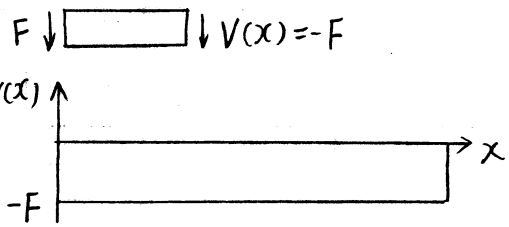
d) 

$\theta=0,\ V=0$

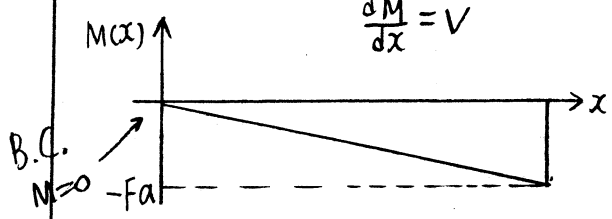
Example 1.



Shear and moment diagram



$$\frac{dM}{dx} = V$$



$$M(x) = -F \cdot x$$

$$\sigma_{xx}(x,y) = -\frac{M(x)\cdot y}{I} = \frac{Fxy}{I}$$

$$\frac{d\theta(x)}{dx} = \frac{M(x)}{EI} = -\frac{F}{EI}(x)$$

B.C. $\theta(x=a)=0$

$$\therefore\ \theta(x) = \frac{F}{2EI}(a^2-x^2)$$

$$\frac{du_y(x)}{dx} = \theta(x) = \frac{F}{2EI}(a^2-x^2)$$

B.C. $u_y(x=a)=0$

$$\therefore\ u_y(x) = \frac{F}{2EI}\left(a^2x - \frac{x^3}{3} - \frac{2a^3}{3}\right)$$

Reading Barber. Chap 5

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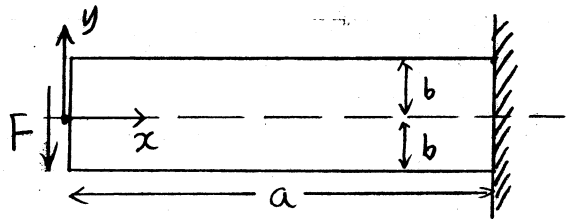
§2. Solve the above problem using <u>Airy Stress function</u>



assume unit thickness in z. $I = \frac{(2b)^3}{12} = \frac{2b^3}{3}$

* can be a thick beam

Mechanics of Materials only applies to thin beam

* we can also obtain all stress components which is difficult in Mech. of Mater.

Boundary conditions,

Strong B.C.
$$\begin{cases} \sigma_{xy}=0 & y=\pm b \\ \sigma_{yy}=0 & y=\pm b \\ \sigma_{xx}=0 & x=0 \end{cases}$$
top, bottom surfaces

weak B.C.
$$\begin{cases} \int_{-b}^{b}\sigma_{xy}\,dy = F & x=0 \\ \int_{-b}^{b}\sigma_{xx}\,y\,dy = 0 & x=0 \\ \int_{-b}^{b}\sigma_{xx}\,dy = 0 & x=0 \end{cases}$$
left end

$$\begin{cases} \int_{-b}^{b}\sigma_{xy}\,dy = F & x=a \\ \int_{-b}^{b}\sigma_{xx}\,y\,dy = Fa & x=a \\ \int_{-b}^{b}\sigma_{xx}\,dy = 0 & x=a \end{cases}$$
right end

From Mechanics of Materials, we expect

$$M = -Fx \quad , \quad \sigma_{xx} = \frac{Fxy}{I}$$

but σ_{xx} alone may not satisfy all the equilibrium and compatibility conditions.

Let stress function be

$$\phi = C_1 x y^3 \qquad \text{(trial solution)}$$

$$\begin{cases} \sigma_{xx} = 6 C_1 x y \\ \sigma_{xy} = -3 C_1 y^2 \\ \sigma_{yy} = 0 \end{cases}$$

← Strong B.C. $\sigma_{xx}=0$ $(x=0)$ is satisfied (left end)

← Strong B.C. $\sigma_{yy}=0$ $(y=\pm b)$ is satisfied (top, bottom)

But B.C. $\sigma_{xy}=0$, $(y=\pm b)$ is violated!

*This is the long side of the beam. We should try to satisfy the strong B.C.

Modify the trial solution

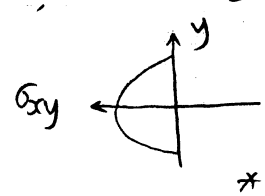
$$\phi = C_1 x y^3 + C_2 x y$$

$$\begin{cases} \sigma_{xx} = 6 C_1 x y \\ \sigma_{xy} = -3 C_1 y^2 - C_2 \\ \sigma_{yy} = 0 \end{cases}$$

← adding a constant to σ_{xy} without changing other stress components

$y=\pm b$. $\quad \sigma_{xy} = -3 C_1 b^2 - C_2 = 0$, $\qquad C_2 = -3 C_1 b^2$

$\therefore \quad \sigma_{xy} = 3 C_1 (b^2 - y^2)$



parabolic distribution of shear stress

* it was difficult to derive in Mech. of Mater

Next, we need to determine the constant C_1.

use weak B.C. at left end $(x=0)$

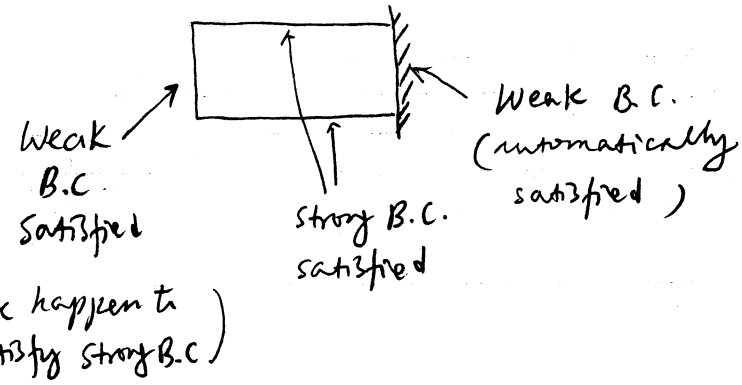
$$F = \int_{-b}^{b} \sigma_{xy}\, dy = \int_{-b}^{b} 3 C_1 (b^2 - y^2)\, dy = 4 C_1 b^3$$

$$C_1 = \frac{F}{4b^3} \qquad \boxed{\phi = \frac{F}{4b^3}\left(xy^3 - 3b^2 xy\right)}$$

The (weak) B.C. at right end is automatically satisfied,

because the stress function approach automatically satisfies equilibrium.

$$\begin{cases} \sigma_{xx} = \dfrac{F}{2b^3} xy \\ \sigma_{xy} = \dfrac{3F}{4b^3}(b^2-y^2) \\ \sigma_{yy} = 0 \end{cases}$$



§3. Displament Field. (Barber. P.111–115 Chap 9)

First obtain the strain field

$$\begin{cases} \varepsilon_{xx} = \dfrac{\sigma_{xx}}{E} - \nu\dfrac{\sigma_{yy}}{E} = \dfrac{3F}{2Eb^3}xy \\ \varepsilon_{yy} = \dfrac{\sigma_{yy}}{E} - \nu\dfrac{\sigma_{xx}}{E} = -\dfrac{3F\nu}{2Eb^3}xy \\ \varepsilon_{xy} = \dfrac{\sigma_{xy}}{E}(1+\nu) = \dfrac{3F(1+\nu)}{4Eb^3}(b^2-y^2) \end{cases}$$

* assume plain stress here, but solution can be easily transformed to plain strain condition

$E = 2\mu(1+\nu)$, $2\mu = \dfrac{E}{1+\nu}$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \rightarrow u_x = \frac{3F}{4Eb^3}x^2y + f(y)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \rightarrow u_y = -\frac{3F\nu}{4Eb^3}xy^2 + g(x)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{3F}{8Eb^3}x^2 + \frac{1}{2}f'(y) - \frac{3F}{8Eb^3}y^2 + \frac{1}{2}g'(x) = \frac{3F(1+\nu)}{4Eb^3}(b^2-y^2)$$

$$\underbrace{\frac{3F}{8Eb^3}x^2 + \frac{1}{2}g'(x)}_{\text{only a function of } x} = \underbrace{\frac{3F\nu}{8Eb^3}y^2 - \frac{1}{2}f'(y)}_{\text{only a function of } y} + \frac{3F(1+\nu)}{4Eb^2}(b^2-y^2) = \underbrace{C}_{\text{so they must be a constant.}}$$

$$\begin{cases} g'(x) = -\dfrac{3Fx^2}{4Eb^3} + C \\ f'(y) = \dfrac{3F\nu}{4Eb^3}y^2 + \dfrac{3F(1+\nu)(b^2-y^2)}{2Eb^3} - C \end{cases}$$

$$\begin{cases} g(x) = -\dfrac{Fx^3}{4Eb^3} + Cx + B \\ f(y) = \dfrac{F\nu y^3}{4Eb^3} + \dfrac{F(1+\nu)(3b^2y - y^3)}{2E} - Cy + A \end{cases}$$

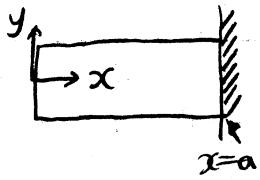
$$\begin{cases} u_x = \dfrac{3Fx^2y}{4Eb^3} + \dfrac{3F(1+\nu)y}{2E} - \dfrac{F(2+\nu)y^3}{4Eb^3} + A - Cy \\ u_y = -\dfrac{3F\nu xy^2}{4Eb^3} - \dfrac{Fx^3}{4Eb^3} + B + Cx \end{cases}$$

constants: A, B — rigid-body translation
C — rigid-body rotation.

* These constants are <u>not</u> necessarily zero.
They should be determined by Boundary Conditions.

---

§4. End condition



Strong boundary condition would be

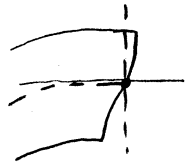
$u_x = 0, \quad u_y = 0 \quad$ on $x = a$.

Since in the above solution both u_x and u_y depend on y, the strong B.C. cannot be satisfied.

We need to modified it to a weak B.C.

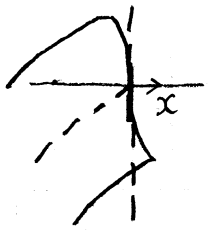
There are several options:

Option a: reduce B.C. to a point



$$\begin{cases} u_x = 0 \\ u_y = 0 \\ \dfrac{\partial u_y}{\partial x} = 0 \end{cases} \quad \text{on } x = a,\ y = 0. \quad \text{(angle of neutral axis} = 0) \quad \rightarrow \quad \begin{cases} A = 0 \\ B = -\dfrac{Fa^3}{2Eb^3} \\ C = \dfrac{3Fa^2}{4Eb^3} \end{cases}$$

Option b: reduce B.C. to a point



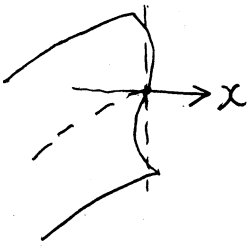
$$\begin{cases} u_x = 0 \\ u_y = 0 \\ \dfrac{\partial u_x}{\partial y} = 0 \end{cases}$$

on $x=a$, $y=0$

(surface normal remain along x-axis)

$$\rightarrow \begin{cases} A = 0 \\ B = -\dfrac{Fa^3}{2Eb^3}\left(1+3(1+\nu)\dfrac{b^2}{a^2}\right) \\ C = \dfrac{3Fa^2}{4Eb^3}\left(1+2(1+\nu)\dfrac{b^2}{a^2}\right) \end{cases}$$

☆ Option c: integral over surface
— probably closest to reality



$$\begin{cases} \int_{-b}^{b} u_x \, dy = 0 \\ \int_{-b}^{b} u_y \, dy = 0 \\ \int_{-b}^{b} u_x \, y \, dy = 0 \end{cases}$$

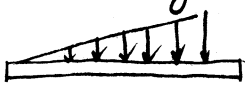
(averaged rotation = 0)

$$\rightarrow \begin{cases} A = 0 \\ B = -\dfrac{Fa^3}{2Eb^3}\left(1+\dfrac{(12+11\nu)}{5}\dfrac{b^2}{a^2}\right) \\ C = \dfrac{3Fa^2}{4Eb^3}\left(1+\dfrac{8+9\nu}{5}\dfrac{b^2}{a^2}\right) \end{cases}$$

§5. General Solution Strategy

— using higher order polynomials

Step 1: determine the maximum order of polynomial using mechanics of materials arguments.



suppose: normal loading $q(x) \sim x^n$

↓

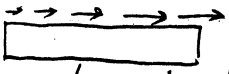
shear force $V(x) \sim x^{n+1}$

bending moment $M(x) \sim x^{n+2}$

$\sigma_{xx} \sim x^{n+2} \cdot y$

$\phi \sim x^{n+2} \cdot y^3$

$\boxed{\text{maximum order} = n+5}$



suppose: shear loading $\sim x^m$

↓

shear force $V(x) \sim x^m$

bending moment $M(x) \sim x^{m+1}$

$\sigma_{xx} \sim x^{m+1} \cdot y$

$\phi \sim x^{m+1} y^3$

$\boxed{\text{maximum order} = m+4}$

Step 2 write down a polynomial function $\phi(x,y)$ that contains all terms up to order

$$\max(n+5,\ m+4)$$

$$\phi(x,y) = C_1 x^2 + C_2 xy + C_3 y^2 + C_4 x^3 + \cdots$$

$$\begin{array}{c} 1 \\ x \quad y \end{array}$$

← no physical meaning

$$\begin{array}{c} x^2 \quad xy \quad y^2 \\ x^3 \quad x^2y \quad xy^2 \quad y^3 \\ x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4 \\ x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5 \\ \cdots \end{array}$$

coefficients:

$$\begin{array}{c} C_1 \quad C_2 \quad C_3 \\ C_4 \quad C_5 \quad C_6 \quad C_7 \\ C_8 \quad C_9 \quad C_{10} \quad C_{11} \quad C_{12} \\ C_{13} \quad C_{14} \quad C_{15} \quad C_{16} \quad C_{17} \quad C_{18} \end{array}$$

Step 3 $\nabla^4 \phi = 0$ (compatibility condition)

This leads to a set of algebraic equation for C_i

* You'd better use a computer program, e.g. Matlab to avoid making mistakes.

Step 4 Boundary conditions (strong & weak)

leads to another set of algebraic equations for C_i

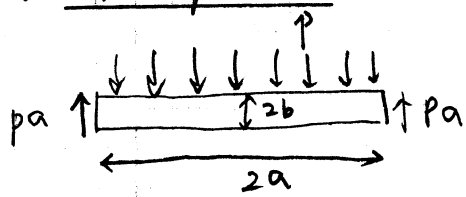
Step 5 solve all equations and determine C_i

* Some of the equations can be redundant, but Matlab can handle that

* If Matlab fails to give a solution, then perhaps a strong B.C. needs to be reduced to a week B.C.

§6. Example 2

(Barber p.57)



$q(x) = p \cdot x^0$

$n = 0.$

maximum order $= 5$

$\phi = C_1 x^2 + C_2 xy + C_3 y^2 + \cdots + C_{18} y^5$

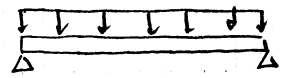
use <u>matlab program</u>

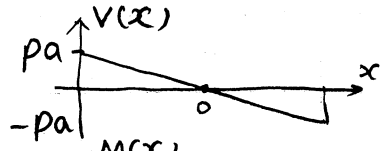
S522a.m

(next page)

* on course work – Homeworks folder.

* Mechanics of Materials Solution





$V(x) = -px$

$M(x) = \frac{1}{2} p(a^2 - x^2)$

$$\sigma_{xx} = -\frac{My}{I} = \frac{3p}{4b^3}(x^2 - a^2) \cdot y$$

define $C_1 \cdots C_{18}$, x, y, a, b, p as symbolic variables.

$$\phi = C_1 x^2 + C_2 xy + C_3 y^2 + \cdots + C_{18} y^5$$

SXX : $\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$

SYY : $\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$ ← they are also polynomial of x, y.

SXY : $\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$

t_1 : $\sigma_{yy}(y=b)$ ← surface tractions, also polynomials.

t_2 : $\sigma_{xy}(y=b)$

t_3 : $\sigma_{yy}(y=-b)$

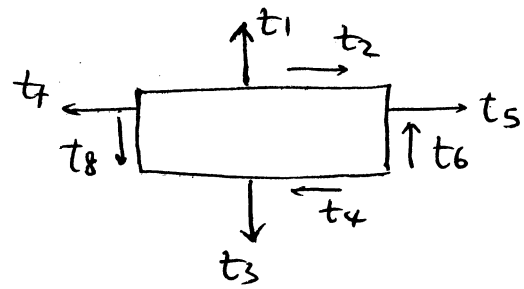
t_4 : $\sigma_{xy}(y=-b)$

t_5 : $\sigma_{xx}(x=a)$

t_6 : $\sigma_{xy}(x=a)$

t_7 : $\sigma_{xx}(x=-a)$

t_8 : $\sigma_{xy}(x=-a)$



introduce polynomial coefficients: $S_1 \cdots S_{16}$, $b_1 \cdots b_3$, ...

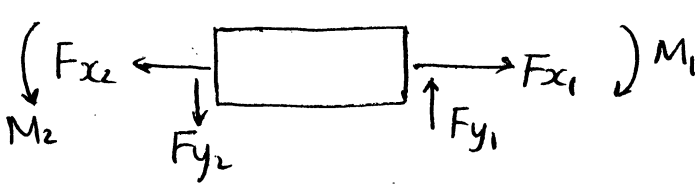
Let
$$t_1 = S_1 x^3 + S_2 x^2 + S_3 x + S_4$$
$$t_2 = S_5 x^3 + S_6 x^2 + S_7 x + S_8$$
$$t_3 = S_9 x^3 + S_{10} x^2 + S_{11} x + S_{12}$$
$$t_4 = S_{13} x^3 + S_{14} x^2 + S_{15} x + S_{16}$$

↑ these coefficients can be found by Matlab.

$$\nabla^4 \phi = b_1 x + b_2 y + b_3$$

total resultant force and moment at ends.

$$F_{x_1} = \int_{-b}^{b} t_5 \, dy$$
$$F_{y_1} = \int_{-b}^{b} t_6 \, dy$$
$$M_1 = \int_{-b}^{b} t_5 \, y \, dy$$



$$F_{x_2} = \int_{-b}^{b} t_7 \, dy$$
$$F_{y_2} = \int_{-b}^{b} t_8 \, dy$$

$S_1 \cdots S_{16}$, b_1, b_2, b_3, $F_{x_1}, F_{y_1}, M_1, F_{x_2}, F_{y_2}, M_2$

are all linear combinations of $C_1 \cdots C_{18}$

Equations to solve:

$S_1=0$, $S_2=0$, $S_3=0$, $S_4+p=0$

$S_5=0$, $S_6=0$, $S_7=0$, $S_8=0$

...

$S_{16}=0$

$b_1=0$, $b_2=0$, $b_3=0$

$F_{x_1}=0$, $M_1=0$, $F_{y_1}-pa=0$

$F_{x_2}=0$, $M_2=0$, $F_{y_2}+pa=0$

more equations than unknowns.

equations contain redundancy

but Matlab can handle it.

Solution:

$$\phi = -\frac{p}{40b^3}\left(10x^2b^3+15x^2yb^2-2y^3b^2+5y^3a^2-5x^2y^3+y^5\right) \quad \text{---} (*)$$

$$\sigma_{xx} = -\frac{py}{20b^3}\left(-6b^2+15a^2-15x^2+10y^2\right)$$

$$\sigma_{yy} = \frac{p}{4b^3}\left(-2b^3-3yb^2+y^3\right)$$

$$\sigma_{xy} = -\frac{3px}{4b^3}\left(-b^2+y^2\right)$$

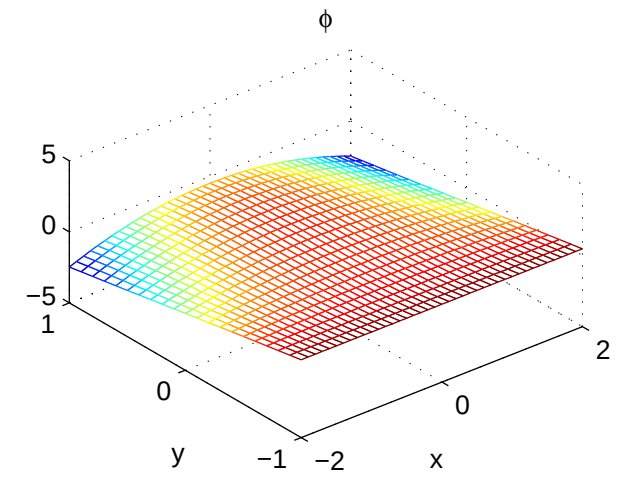
plot S522a-plot.pdf

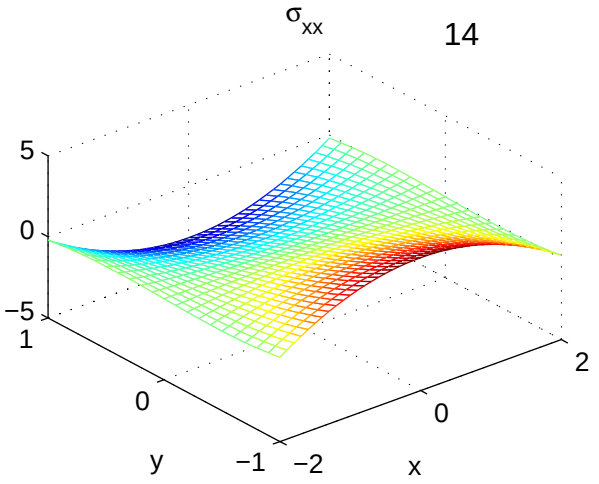
inspect whether B.C. are satisfied.

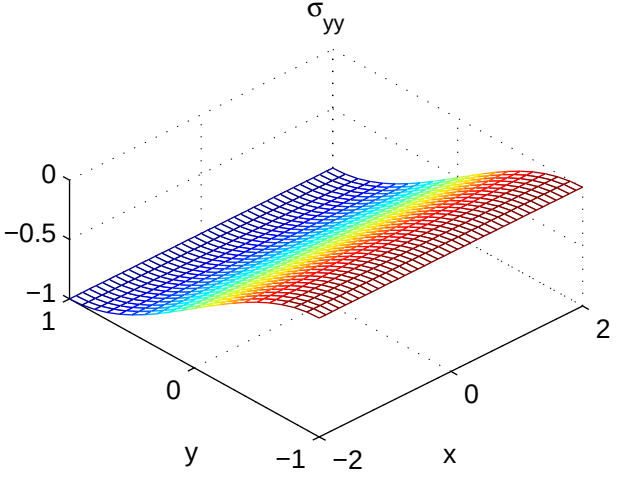
```
%adpated from maple file S522 from J. R. Barber, Elasticity
% http://www-personal.engin.umich.edu/~jbarber/elasticity/maple/S522
%
% adapted by Wei Cai, caiwei@stanford.edu, for ME 340 Elasticity
% Spring 2006, Stanford University
%
%This file gives the solution of the simply-supported beam problem,
%following the strategy of Section 5.2.2.
%

clear all;

syms C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 C11 C12 C13 C14 C15 C16 C17 C18
syms x y a b p

%stress function
phi = C1*x^2+C2*x*y+C3*y^2+C4*x^3+C5*x^2*y+C6*x*y^2+C7*y^3+C8*x^4+ ...
      C9*x^3*y+C10*x^2*y^2+C11*x*y^3+C12*y^4+C13*x^5+C14*x^4*y+ ...
      C15*x^3*y^2+C16*x^2*y^3+C17*x*y^4+C18*y^5;

%stress field
sxx =  diff(diff(phi,y),y);
syy =  diff(diff(phi,x),x);
sxy = -diff(diff(phi,x),y);

%traction force on y=b
t1 = subs(syy,y,b);   %Ty on y=b
t2 = subs(sxy,y,b);   %Tx on y=b

%traction force on y=-b
t3 = subs(syy,y,-b);  %Ty on y=-b
t4 = subs(sxy,y,-b);  %Tx on y=-b

%traction force on x=a
t5 = subs(sxx,x,a);   %Tx on x=a
t6 = subs(sxy,x,a);   %Ty on x=a

%traction force on x=-a
t7 = subs(sxx,x,-a);  %Tx on x=-a
t8 = subs(sxy,x,-a);  %Ty on x=-a

%find coefficients of polynomials t1,t2,t3,t4
s1 = subs(diff(t1,x,3),x,0)/factorial(3);
s2 = subs(diff(t1,x,2),x,0)/factorial(2);
s3 = subs(diff(t1,x,1),x,0)/factorial(1);
s4 = subs(t1,x,0);

s5 = subs(diff(t2,x,3),x,0)/factorial(3);
s6 = subs(diff(t2,x,2),x,0)/factorial(2);
s7 = subs(diff(t2,x,1),x,0)/factorial(1);
s8 = subs(t2,x,0);
```

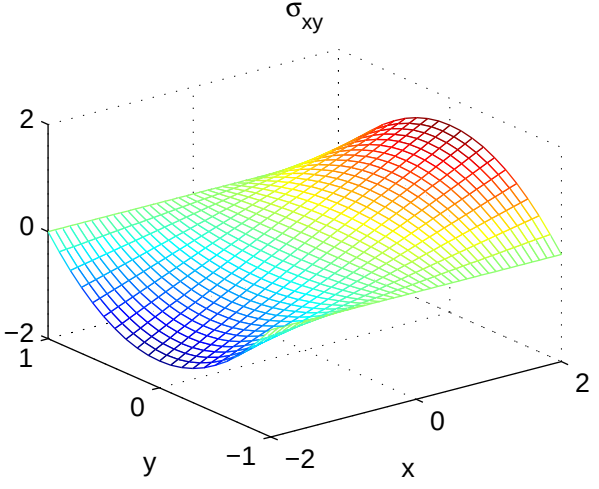
```
s9  = subs(diff(t3,x,3),x,0)/factorial(3);
s10 = subs(diff(t3,x,2),x,0)/factorial(2);
s11 = subs(diff(t3,x,1),x,0)/factorial(1);
s12 = subs(t3,x,0);

s13 = subs(diff(t4,x,3),x,0)/factorial(3);
s14 = subs(diff(t4,x,2),x,0)/factorial(2);
s15 = subs(diff(t4,x,1),x,0)/factorial(1);
s16 = subs(t4,x,0);

%The biharmonic equation is 4th order, so applying it to a 5th order polynomial
%generates a first order polynomial.
%
biharm = diff(phi,x,4)+diff(phi,y,4)+2*diff(diff(phi,x,2),y,2);

%coefficients of biharm
b1 = subs(diff(biharm,x,1),{x,y},{0,0});
b2 = subs(diff(biharm,y,1),{x,y},{0,0});
b3 = subs(biharm,{x,y},{0,0});

%integrated force and torque on x=a
Fx1 = simplify(int(t5, y, -b, b));
Fy1 = simplify(int(t6, y, -b, b));
M1  = simplify(int(t5*y, y, -b, b));
%
%integrated force and torque on x=-a
Fx2 = simplify(int(t7, y, -b, b));
Fy2 = simplify(int(t8, y, -b, b));
M2  = simplify(int(t7*y, y, -b, b));

%Solve all these equations together
s = solve(s1, s2, s3, s4+p, s5, s6, s7, s8,  ...
          s9, s10,s11,s12,  s13,s14,s15,s16, ...
          b1, b2, b3,       Fx1,M1, Fy1-p*a, ...
          'C1','C2','C3','C4','C5','C6','C7','C8','C9',...
          'C10','C11','C12','C13','C14','C15','C16','C17','C18');

%construct cell arrays for future use
coeffs = {C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,C14,C15,C16,C17,C18};
solution = {s.C1,s.C2,s.C3,s.C4,s.C5,s.C6,s.C7,s.C8,s.C9, ...
            s.C10,s.C11,s.C12,s.C13,s.C14,s.C15,s.C16,s.C17,s.C18};

%stress function and stress field solution
phi2 = simplify(subs(phi, coeffs, solution));

sxx2 =  simplify(subs(sxx, coeffs, solution));
syy2 =  simplify(subs(syy, coeffs, solution));
sxy2 =  simplify(subs(sxy, coeffs, solution));

%print out results
disp('phi='); pretty(phi2)
disp('sxx='); pretty(sxx2)
```

```
disp('syy='); pretty(syy2)
disp('sxy='); pretty(sxy2)

%plot stress fields
an = 2.0; bn = 1.0; pn = 1.0;
xn = [-1:0.05:1]*an;
yn = [-1:0.1:1]*bn;
xx = ones(size(yn'))*xn;
yy = yn'*ones(size(xn));
phin = subs(phi2,{a,b,p},{an,bn,pn}); phin = subs(phin,{x,y},{xx,yy});
sxxn = subs(sxx2,{a,b,p},{an,bn,pn}); sxxn = subs(sxxn,{x,y},{xx,yy});
syyn = subs(syy2,{a,b,p},{an,bn,pn}); syyn = subs(syyn,y,yy);
sxyn = subs(sxy2,{a,b,p},{an,bn,pn}); sxyn = subs(sxyn,{x,y},{xx,yy});

figure(1);
subplot(2,2,1); mesh(xn,yn,phin);
title('\phi'); xlabel('x'); ylabel('y');
subplot(2,2,2); mesh(xn,yn,sxxn);
title('\sigma_{xx}'); xlabel('x'); ylabel('y');
subplot(2,2,3); mesh(xn,yn,syyn);
title('\sigma_{yy}'); xlabel('x'); ylabel('y');
subplot(2,2,4); mesh(xn,yn,sxyn);
title('\sigma_{xy}'); xlabel('x'); ylabel('y');
% results
%phi=
%
%                    2  3       2    2      3  2      3  2      2  3    5
%            p (10 x  b  + 15 x  y b  - 2 y  b  + 5 y  a  - 5 x  y  + y )
%      - 1/40 ------------------------------------------------------------
%                                          3
%                                         b
%sxx=
%
%                                 2       2       2       2
%                        p y (-6 b  + 15 a  - 15 x  + 10 y )
%                  - 1/20 -----------------------------------
%                                         3
%                                        b
%syy=
%
%                                    3        2    3
%                            p (-2 b  - 3 y b  + y )
%                        1/4 -----------------------
%                                        3
%                                       b
%sxy=
%
%                                         2    2
%                                 p x (-b  + y )
%                            - 3/4 --------------
%                                         3
%                                        b
```

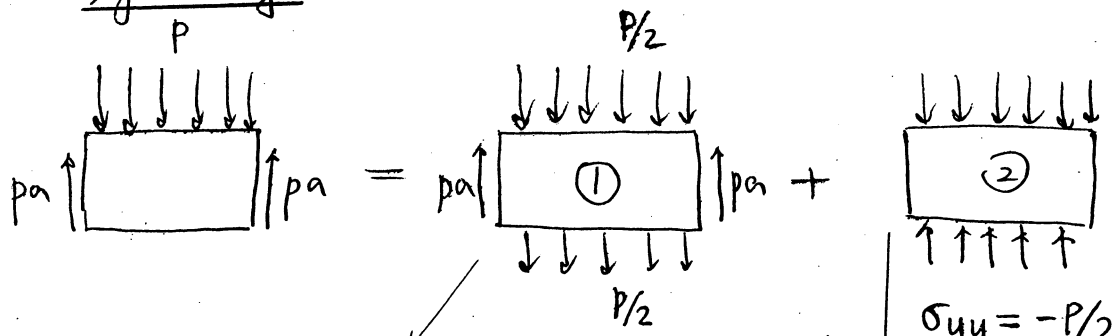

ϕ
5
0
−5
1
0
y
−1
−2
0
2
x


σxx
5
0
−5
1
0
y
−1
−2
0
2
x


σyy
0
−0.5
−1
1
0
y
−1
−2
0
2
x


σxy
2
0
−2
1
0
y
−1
−2
0
2
x

§7. <u>Symmetry</u> considerations.



$\sigma_{yy} = -P/2$ (everywhere)
uniform stress field.

① σ_{yy} — <u>odd</u> function of y : $\frac{\partial^2 \phi}{\partial x^2}$

σ_{xy} — <u>odd</u> function of x : $-\frac{\partial^2 \phi}{\partial x \partial y}$

$\Downarrow$

$\phi(x,y)$ must be an <u>odd</u> function of y (y, y^3, y^5, etc)
and <u>even</u> function of x. (x^0, x^2, x^4, etc)

$$\phi^{①} = C_5 x^2 y + C_7 y^3 + C_{14} x^4 y + C_{16} x^2 y^3 + C_{18} y^5$$

only 5 terms survive!

all other $C_i = 0$.

after solving this problem, add $\begin{cases} \sigma_{yy}^{②} = -\frac{P}{2} \\ \phi^{②} = -\frac{P}{4} x^2 \end{cases}$

to obtain the solution of original problem.

This is the first term in Eq. (*).

- We have seen that a plane strain problem can be represented by a PDE for the stress function: $\nabla^4\phi = 0$ and the corresponding Boundary Conditions.

- We have used polynomials with unknown coefficients as trial solutions to the biharmonic equation
- Another widely used set of solutions is <u>Fourier series</u>,
- Compared with polynomials, solution by Fourier series has the advantage of forward and backward <u>Fourier transform</u>. (convenience) and the easiness to generalize to unbounded domains (e.g. half-space)

§1. Trial Solution

$$\nabla^4\phi = 0$$

suppose $\phi(x,y) = g(x)\,f(y)$ —— separation of variables

furthermore let $g(x) = e^{\alpha x}$, $f(y) = e^{\beta y}$

$$\nabla^4\phi = (\alpha^4 + 2\alpha^2\beta^2 + \beta^4)\, e^{\alpha x} e^{\beta y} = 0$$

$$(\alpha^2+\beta^2)^2 = 0 \rightarrow \alpha^2+\beta^2 = 0 \rightarrow \alpha = \pm i\beta$$

This suggest that

$\boxed{\phi(x,y) = e^{i\lambda x}\, e^{\pm\lambda y}}$ is a solution to $\nabla^4\phi = 0$

—— this can be easily verified.

* Notice that for each λ, this only corresponds to 2-independent solutions. $e^{i\lambda x}e^{\lambda y}$, $e^{i\lambda x}e^{-\lambda y}$

Since $\nabla^4\phi = 0$ is 4th order, we expect two more solutions of the similar form.

It can be easily verified that $\boxed{\phi(x,y) = e^{i\lambda x}\cdot y\cdot e^{\pm\lambda y}}$ is also a solution to $\nabla^4\phi = 0$

A linear combination of these 4 solutions is still a solution to the biharmonic equation

$$\phi(x,y) = e^{i\lambda x}\left[(C_1 + C_2 y)e^{\lambda y} + (C_3 + C_4 y)e^{-\lambda y}\right]$$

* The stress function corresponding to a physical solution must be real. Hence only the <u>real part</u> of the above solution will be considered.

* Notice that $e^{i\lambda x} = \cos\lambda x + i\sin\lambda x$ and C_1, C_2, C_3, C_4 are (in general) complex numbers.

* Therefore, the above general solution can also be rewritten (in real numbers) as

$$\phi(x,y) = \cos\lambda x \cdot f(y) \quad \longleftarrow \text{ an even function of } x$$

or

$$\phi(x,y) = \sin\lambda x \cdot f(y) \quad \longleftarrow \text{ an odd function of } x$$

where $f(y) = (A + By)e^{\lambda y} + (C + Dy)e^{-\lambda y}$

* Using the definitions $\cosh\lambda y \equiv \dfrac{e^{\lambda y} + e^{-\lambda y}}{2}$ ← even func. of y

$\sinh\lambda y \equiv \dfrac{e^{\lambda y} - e^{-\lambda y}}{2}$ ← odd func. of y

$f(y)$ can also be written into even and odd functions of y

$$f(y) = (A' + B'y)\cosh\lambda y + (C' + D'y)\sinh\lambda y$$

* $A'\cosh\lambda y + D'y\sinh\lambda y \longleftarrow$ an even function of y

$B'y\cosh\lambda y + C'\sinh\lambda y \longleftarrow$ an odd function of y

Summary: using sin, cos, sinh, cosh, we have obtained a series of general solutions to the biharmonic equation that can be either even (symmetric) or odd (anti symmetric) in x and/or y. For example,

$$\phi(x,y) = \cos\lambda x\,(A\cosh\lambda y + D\,y\sinh\lambda y)$$

satisfies $\nabla^4\phi = 0$ and is an even function for both x and y.

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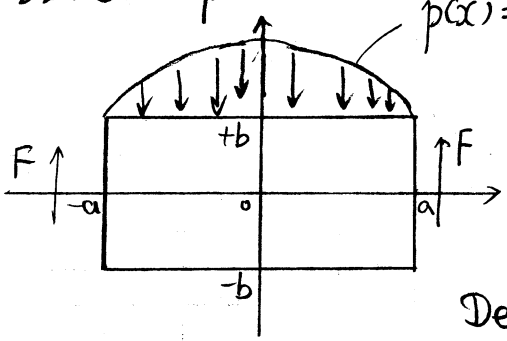
§2. Example 1

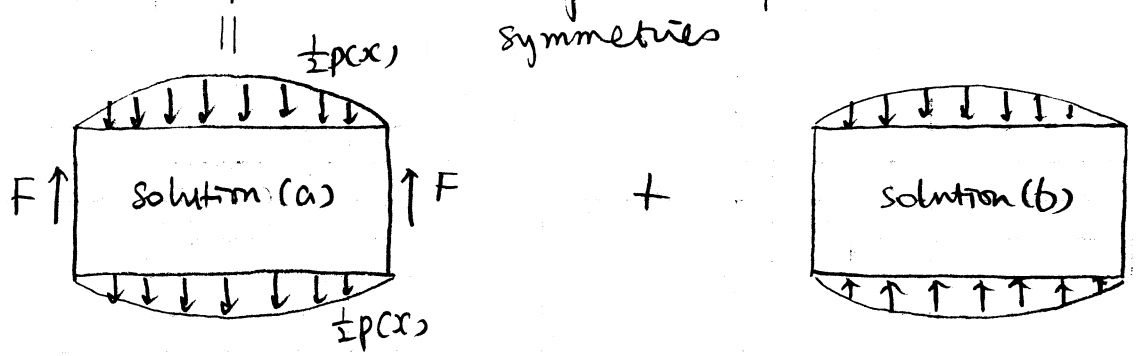


Let $\lambda = \frac{\pi}{2a}$.

$\therefore\ p(x) = p_0\cos\lambda x = \sigma_{xx}(x, y=b)$

* B.C. motivates a Fourier trial solution

Decompose the problem into different symmetries



Symmetries:

σ_{yy} — odd in y, even in x	σ_{yy} — even in y, even in x
σ_{xy} — odd in x	σ_{xy} —
σ_{xx} — _____	σ_{xx} —
$\phi(x,y)$ — _____	$\phi(x,y)$ — _____

trial solution:

$\phi = \cos\lambda x\,(By\cosh\lambda y + C\sinh\lambda y)$

$\phi = \cos\lambda x\,(A\cosh\lambda y + Dy\sinh\lambda y)$

Let's first find solution (a).

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = -\lambda^2 \cos\lambda x \,(By\cosh\lambda y + C\sinh\lambda y)$$

B.C. $\sigma_{yy} = -\frac{1}{2}p(x) = -\frac{1}{2}p_0 \cos\lambda x \qquad y = b$

$\sigma_{yy} = \frac{1}{2}p(x) = \frac{1}{2}p_0 \cos\lambda x \qquad y = -b.$

$$\boxed{-\lambda^2(Bb\cosh\lambda b + C\sinh\lambda b) = -\frac{1}{2}p_0}$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \cos\lambda x\,[B\lambda^2 y\cosh\lambda y + (2B\lambda + C\lambda^2)\sinh\lambda y]$$

B.C. $\sigma_{xx} = 0 \qquad x = \pm a.$ (strong B.C.)

automatically satisfied by the trial solution!

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \lambda\sin\lambda x\,[(B + C\lambda)\cosh\lambda y + B\lambda y\sinh\lambda y]$$

B.C. $\sigma_{xy} = 0 \qquad y = b$ (another strong B.C.)

$$\boxed{(B + C\lambda)\cosh\lambda b + B\lambda b\sinh\lambda b = 0}$$

The two equations in the box can be solved together to find coefficients B and C.

Introduce short hand notation: $\cosh\lambda b \equiv c, \quad \sinh\lambda b \equiv s.$

$$\begin{cases} Bb\cdot c + C\cdot s = \frac{p_0}{2\lambda^2} \\ (B + C\lambda)\cdot c + B\lambda b\cdot s = 0 \end{cases}$$

also notice $\cosh^2\lambda b - \sinh^2\lambda b = 1$

solve by Matlab $\longrightarrow$

$$\begin{cases} B = \dfrac{\frac{p_0}{2\lambda}\cosh\lambda b}{\lambda b - \cosh\lambda b\sinh\lambda b} \\ C = -\dfrac{p_0}{\quad}\,\dfrac{\lambda b\sinh\lambda b + \cosh\lambda b}{\quad} \end{cases}$$

Solution (b) can be found similarly.

B.C.

$y = b$

$y = -b$

$x = \pm a$

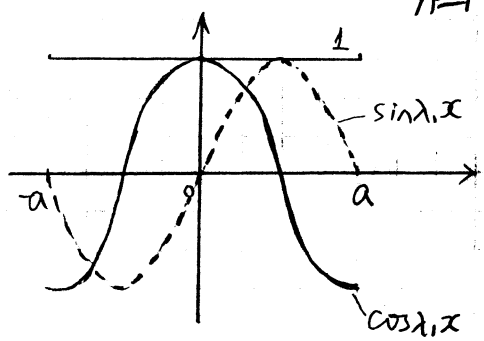
$x = \pm a$.

The solution to the original problem is obtained by superimposing solution (a) and solution (b).

§3. Fourier Series for $f(x)$ on $-a \le x \le a$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\lambda_n x + b_n \sin\lambda_n x$$

$$\lambda_n = n\frac{\pi}{a}.$$



$$a_0 = \frac{1}{a}\int_{-a}^{a} f(x)\,dx$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos(\lambda_n x)\,dx$$

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin(\lambda_n x)\,dx$$

Here we represent an arbitrary function $f(x)$ over the domain $x \in [-a, a]$ in terms of basis functions,

$1, \cos\lambda_n x, \sin\lambda_n x.$

The constant term is needed here because all other basis functions have zero average.

$$\int_{-a}^{a} \cos\lambda_n x\,dx = \frac{1}{\lambda_n}\sin\lambda_n x\Big|_{-a}^{a} = 0 \qquad \sin\lambda_n a = 0$$

$$\int_{-a}^{a} \sin\lambda_n x\,dx = -\frac{1}{\lambda_n}\cos\lambda_n x\Big|_{-a}^{a} = 0 \qquad \cos(-\lambda_n a) = \cos\lambda_n a$$

The properties of the Fourier Series stem from the orthogonal conditions among basis functions e.g.

$$\frac{1}{a}\int_{-a}^{a} \cos\lambda_n x \cos\lambda_m x\,dx = \frac{1}{2a}\int_{-a}^{a} \cos(\lambda_n+\lambda_m)x + \cos(\lambda_n-\lambda_m)x\,dx$$

$$= \delta_{nm}.$$

We say that $\{1, \cos\lambda_n x, \sin\lambda_n x, n=1,2,\cdots\}$ form a complete basis on the domain $x \in [-a, a]$, i.e. it can represent any function on this domain.

However, the Fourier Series above is not the only way to represent an arbitrary function in the domain of $[-a, a]$

The multiplicity of representation is related to the finiteness of the domain, over which $f(x)$ is defined.

In the above Fourier series, the represented function extends beyond the original domain $[-a, a]$ in a periodic manner.

If we use a different representation, the function may look different beyond the domain $[-a, a]$, but that is not important.

In general, we are seeking a set of basis functions $\varphi_i(x)$ $i = 1, 2, \cdots$ such that

$$\boxed{\frac{1}{a}\int_{-a}^{a} \varphi_i(x)\,\varphi_j(x)\,dx = \delta_{ij}}$$

and that $\{\varphi_i(x)\}$ form a <u>complete</u> (which is more difficult to prove) basis over the domain $[-a, a]$

Then $$\boxed{f(x) = \sum_{i=1}^{\infty} c_i\,\varphi_i(x)}$$

$$\frac{1}{a}\int_{-a}^{a} f(x)\,\varphi_j(x)\,dx = \sum_{i=1}^{\infty} c_i \frac{1}{a}\int_{-a}^{a} \varphi_i(x)\,\varphi_j(x)\,dx$$

$$= \sum_{i=1}^{\infty} c_i\,\delta_{ij} = c_j$$

$$c_j = \frac{1}{a}\int_{-a}^{a} f(x)\,\varphi_j(x)\,dx$$

or equivalently

$$\boxed{c_i = \frac{1}{a}\int_{-a}^{a} f(x)\,\varphi_i(x)\,dx}$$

In the above, we have seen that $\{\varphi_i(x)\} = \{1, \cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, \cdots\}$ qualifies as such a basis set

$\lambda_n = n\frac{\pi}{a}$ $n = 1, 2, \cdots$

But there are other basis sets as well

§4. Another Basis Set

- Notice that in §2 Example 1, the traction $p(x) = p_0 \cos \frac{\pi x}{2a}$ does not belong to the basis set described in §3.

  This means that if we were to represent $p(x)$ in this basis set, it will be a linear superposition of more than one (in fact, infinite) basis functions. — that would be inconvenient

- If we use the basis set in §3, the boundary condition $\sigma_{xx}=0$, $x=\pm a$ wouldn't be automatically satisfied

$$\cos \lambda_n a = \cos n\pi = (-1)^n$$

- This motivates the search for a different basis set.

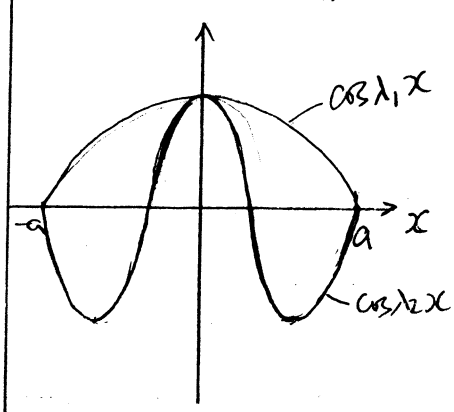
$$\varphi_n(x) = \cos \lambda_n x \qquad n=1,2,3,\cdots$$

$$f(x) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x$$

(neglet sine terms if we constrain $f(x)$ to be an even function of x)

$$\lambda_n = \frac{(2n-1)\pi}{2a}$$

$$\lambda_1 = \frac{\pi}{2a}, \quad \lambda_2 = \frac{3\pi}{2a}, \quad \lambda_3 = \frac{5\pi}{2a}, \cdots$$



notice the constant term $(\frac{a_0}{2})$ is gone.

Again, one can show that

$$\frac{1}{a}\int_{-a}^{a} \cos\lambda_n x \cos\lambda_m x \, dx = \delta_{mn}$$

(orthogonal cond.)

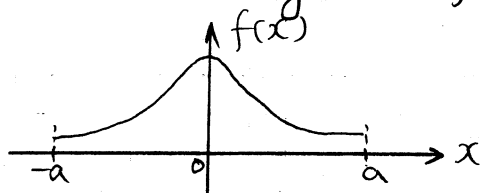
and that $\{\cos\lambda_n x\}$ — notice the different definition of λ_n

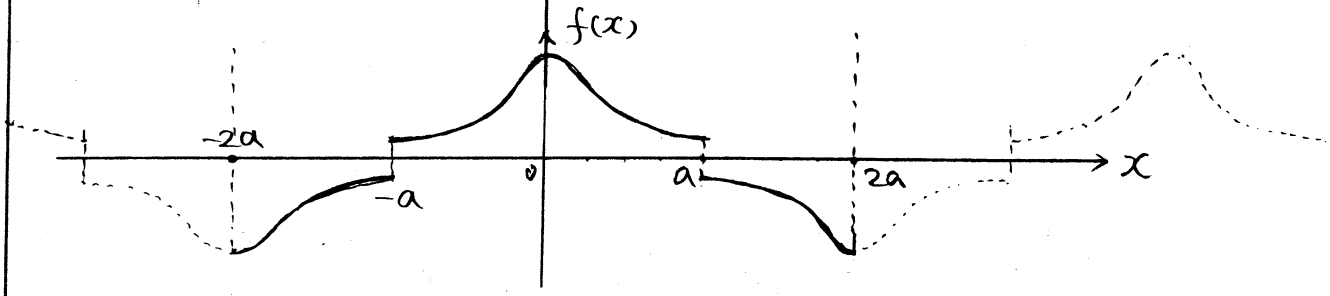
forms a <u>complete</u> basis for even functions on domain $[-a, a]$ (more difficult to prove)

To show that $\{\varphi_n(x) = \cos \lambda_n x\}$ $\lambda_n = \frac{(2n-1)}{2a}\pi$, $n = 1, 2, 3, \cdots$

form a complete basis set for even functions in $[-a, a]$

Let's consider an arbitrary even function $f(x)$.





We now extend the domain of $f(x)$ to $-2a \le x \le 2a$. as illustrated above.

(thanks to the suggestion by Chris Weinberger)

$$f(x+2a) = -f(x)$$

$$f(x+4a) = -f(x+2a) = f(x)$$

We can apply the Fourier series expansion over the extended domain $x \in [-2a, 2a]$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2a}$$

$$a_n = \frac{1}{2a}\int_{-2a}^{2a} f(x)\cos\frac{n\pi x}{2a}\,dx.$$

Because of the anti-symmetry of $f(x)$ on domain $[-2a, 2a]$,

$a_n = 0$ if n is an even number.

$$a_0 = \frac{1}{2a}\int_{-2a}^{2a} f(x)\,dx = 0.$$

$$\therefore f(x) = \sum_{n=1,3,5,\cdots} a_n \cos\frac{n\pi x}{2a} = \sum_{k=1}^{\infty} a_k \cos\frac{(2k-1)\pi x}{2a}$$

where

$$a_k = \frac{1}{2a}\int_{-2a}^{2a} f(x)\cos\frac{(2k-1)\pi x}{2a}\,dx$$

$$= \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{(2k-1)\pi x}{2a}\,dx \qquad \text{(also by symmetry)}$$

Therefore, any even function $f(x)$ in $-a \le x \le a$ can be represented by

$$f(x) = \sum_{k=1}^{\infty} a_k \cos\frac{(2k-1)\pi x}{2a}$$

Hence $\left\{\varphi_k(x) = \cos\frac{(2k-1)\pi x}{2a}\right\}$ is a complete basis set.

In summary, an even function $f(x)$ in $-a \le x \le a$ can be represented as a Fourier series either as

① $$\boxed{\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \\ a_n &= \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx \\ a_0 &= \frac{1}{a}\int_{-a}^{a} f(x)\,dx \end{aligned}}$$

, i.e. basis function $\left\{\psi_n(x) = \cos\frac{n\pi x}{a}\right\}$ $n = 0, 1, 2, \cdots$

or as

② $$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2a} \\ a_n &= \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{(2n-1)\pi x}{2a} \end{aligned}}$$

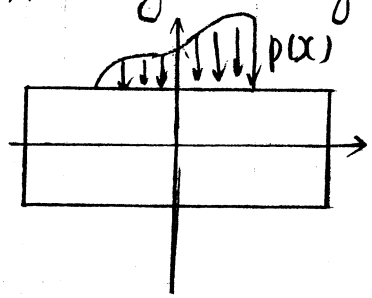
, i.e. basis function $\left\{\varphi_n(x) = \cos\frac{(2n-1)\pi x}{2a}\right\}$ $n = 1, 2, \cdots$

The two approach work equally well in their ability to represent an arbitrary even function.

But the ②nd representation is prefered if we want to solve a beam under arbitrary loading using the Fourier method

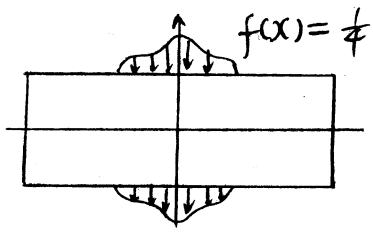
* because one of the strong B.C. ($\sigma_{xx}=0$, $x=\pm a$) can be satisfied automatically.

§5. Arbitrary Loading



An arbitrary loading on a symmetric structure (e.g. a rectangle) can always be decomposed into superpositions of loadings that are either even or odd in x and/or y.

For example, one of the four terms may look like this:



Symmetries:

σ_{yy} : even in x, odd in y

ϕ : even in x, odd in y.

In the following, we will solve the problem with this symmetry. Problems with a different symmetry (e.g. odd in x, even in y) can be solved in a similar way.

Decompose loading into Fourier modes (2nd approach)

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\lambda_n x \qquad \lambda_n = \frac{(2n-1)\pi}{2a}$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\lambda_n x \, dx \qquad \left(\text{there is no } n=0 \text{ term}\right)$$

Trial Solution

odd in y — even in x

$$\phi(x,y) = \sum_{n=1}^{\infty} \left(B_n y \cosh\lambda_n y + C_n \sinh\lambda_n y\right) \underline{\cos\lambda_n x}$$

$$\sigma_{yy} = \phi_{,xx} = \sum_{n=1}^{\infty} \left(B_n y \cosh\lambda_n y + C_n \sinh\lambda_n y\right)(-\lambda_n^2)\cos\lambda_n x$$

$$\sigma_{xx} = \phi_{,yy} = \sum_{n=1}^{\infty} \left(B_n\lambda_n^2 y \cosh\lambda_n y + (2B_n\lambda_n + C_n\lambda_n^2)\sinh\lambda_n y\right)\cos\lambda_n x$$

$$\sigma_{xy} = -\phi_{,xy} = \sum_{n=1}^{\infty} \left((B_n + C_n\lambda_n)\cosh\lambda_n y + B_n\lambda_n y \sinh\lambda_n y\right)\lambda_n \sin\lambda_n x$$

Boundary conditions:

$\sigma_{xx} = 0, \quad x = \pm a \leftarrow$ automatically satisfied because $\cos\lambda_n a = 0$.

$$\begin{cases} \sigma_{yy}(x, y=b) = f(x) = \sum_{n=1}^{\infty} a_n \cos\lambda_n x \\ \sigma_{xy}(x, y=b) = 0 \end{cases}$$

$$\rightarrow \begin{cases} B_n b \cosh\lambda_n b + C_n \sinh\lambda_n b = -\frac{a_n}{\lambda_n^2} \\ (B_n + C_n\lambda_n)\cosh\lambda_n b + B_n\lambda_n b \sinh\lambda_n b = 0 \end{cases} \quad \text{for } n = 1, 2, 3, \cdots$$

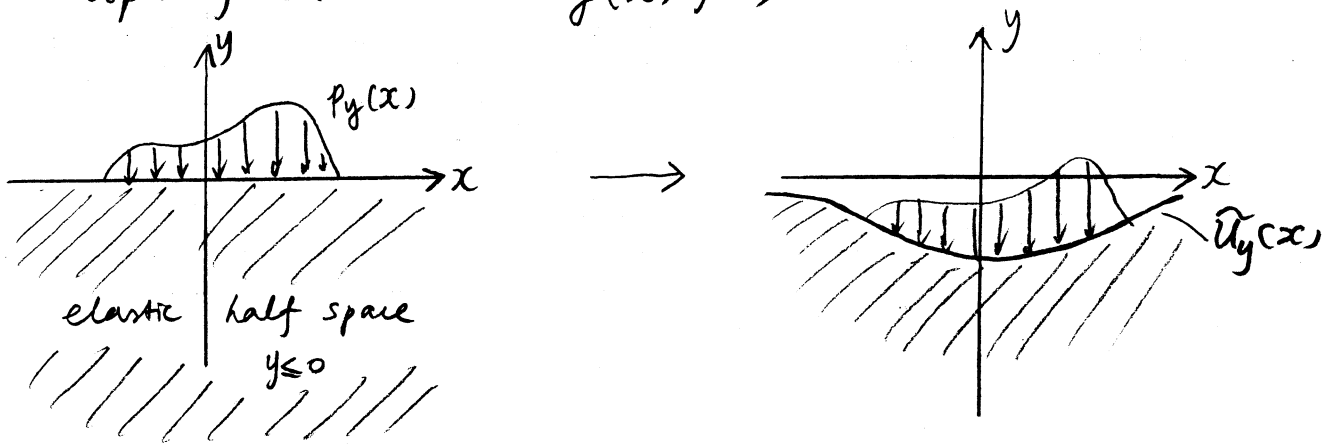
$$\xrightarrow{\text{solve}} \begin{cases} B_n = \cdots \\ C_n = \cdots \end{cases} \quad \text{for every mode } n = 1, 2, 3, \cdots$$

* Notice that this approach would not work if $f(x)$ is expanded in the "usual" Fourier series ①

Because $\cos n\pi = \pm 1 \neq 0$, each Fourier mode does not satisfy the strong B.C. $\sigma_{xx} = 0$, $x = \pm a$ individually.

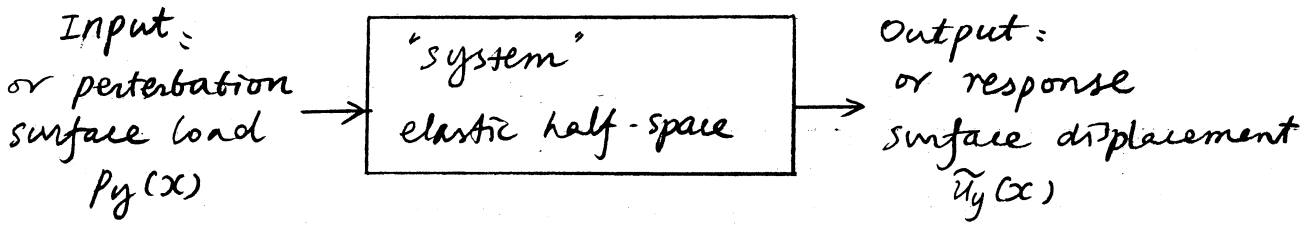
§1. Problem Statement

Suppose the top surface ($y=0$) of an elastic half-space is subjected to an arbitrary load distribution $p_y(x)$, what will be the displacement field (shape change) of the top surface? i.e. $u_y(x, y=0)$?



introduce new notation $\tilde{u}_y(x) \equiv u_y(x, y=0)$ for surface displacement.

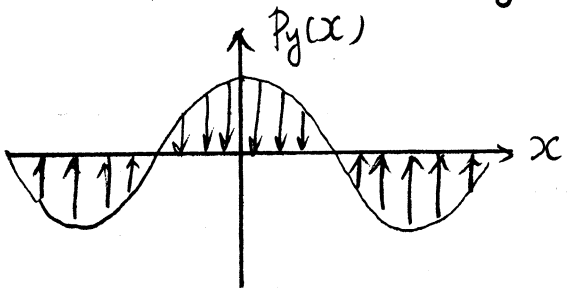
We can consider the half-space as a linear system.



* Barber solved this problem following the solution he obtained in polar coordinates

Here we will solve it using the Fourier method we developed before.

§2 Sinusoidal Loading



consider the special case of

$$P_y(x) = P_0 \cos\lambda x$$

for some arbitrary λ.

(alternating compressive and tensile loading)

B.C. $\sigma_{yy}(x, y=0) = -P_y(x) = -P_0 \cos\lambda x$

$\sigma_{xy}(x, y=0) = 0$ (frictionless, no tangential force)

Trial Solution:

$$\boxed{\phi(x,y) = \cos\lambda x \cdot (A + By) \cdot e^{\lambda y}} \qquad (\lambda > 0)$$

* We do not include the $e^{-\lambda y}$ term because we want the solution to remain finite when $y \to -\infty$

$$\sigma_{xx} = \phi_{,yy} = \cos\lambda x \left[(A\lambda^2 + 2B\lambda) + B\lambda^2 y \right] e^{\lambda y}$$

$$\sigma_{yy} = \phi_{,xx} = -\cos\lambda x \cdot \lambda^2 \cdot (A + By)\, e^{\lambda y}$$

$$\sigma_{xy} = -\phi_{,xy} = \sin\lambda x \cdot \lambda \cdot \left[(A\lambda + B) + B\lambda y \right] e^{\lambda y}$$

at $y = 0$, $\sigma_{xy} = (A\lambda + B) \cdot \lambda \cdot \sin\lambda x = 0$

$\therefore A\lambda + B = 0, \quad B = -A\lambda$

$$\sigma_{yy} = -\cos\lambda x \cdot \lambda^2 \cdot A = -P_0 \cos\lambda x$$

$$\therefore A = \frac{P_0}{\lambda^2}$$

$$\boxed{\begin{aligned} \sigma_{xx} &= -P_0 \cos\lambda x\, (1 + \lambda y)\, e^{\lambda y} \\ \sigma_{yy} &= -P_0 \cos\lambda x\, (1 - \lambda y)\, e^{\lambda y} \\ \sigma_{xy} &= -P_0 \lambda \sin\lambda x \cdot y \cdot e^{\lambda y} \end{aligned}}$$

$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$

(plane strain)

The next step is to find the displacement field.

Start with strain field (Generalized Hooke's Law in Plane strain)

$$\begin{cases} \varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} \\ \varepsilon_{yy} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy} \\ \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy} \end{cases}$$

$$\boxed{\begin{aligned} \varepsilon_{xx} &= -\frac{p_0}{E}\cos\lambda x\left[(1-\nu-2\nu^2)+(1+\nu)\lambda y\right]e^{\lambda y} \\ \varepsilon_{yy} &= -\frac{p_0}{E}\cos\lambda x\left[(1-\nu-2\nu^2)-(1+\nu)\lambda y\right]e^{\lambda y} \\ \varepsilon_{xy} &= -\frac{p_0\lambda}{E}\sin\lambda x\cdot(1+\nu)\cdot y\cdot e^{\lambda y} \end{aligned}}$$

Next, integrate to find displacement field

$$\boxed{\begin{aligned} u_x &= -\frac{p_0}{\lambda E}\sin\lambda x\left[(1-\nu-2\nu^2)+(1+\nu)\lambda y\right]e^{\lambda y} + C \\ u_y &= -\frac{p_0}{\lambda E}\cos\lambda x\left[(2-2\nu^2)-(1+\nu)\lambda y\right]e^{\lambda y} + D \end{aligned}}$$

the displacement field on the top surface ($y=0$) is:

$$\boxed{\begin{aligned} \tilde{u}_x(x) &= -\frac{p_0}{\lambda E}\sin\lambda x\,(1-\nu-2\nu^2) + C \\ \tilde{u}_y(x) &= -\frac{p_0}{\lambda E}\cos\lambda x\,(2-2\nu^2) + D \end{aligned}}$$

rigid-body translation, set to zero

Recall that the input (perturbation) is $p_y(x) = p_0\cos\lambda x$,

the output (response) is

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{\lambda E}\cdot p_y(x)$$

* Notice that the response is inversely proportional to the wave vector λ of the perturbation

§3. Fourier Transform

Just as <u>Fourier Series</u> can be applied to a function defined on a finite domain, the <u>Fourier Transform</u> can be applied to a function $f(x)$ defined on the infinite domain $-\infty < x < \infty$.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k)\, e^{ikx}\, dk \qquad \leftarrow \text{inverse F.T.}$$

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx \qquad \leftarrow \text{Fourier transform}$$

common Fourier Transform Pairs

$f(x)$	$\hat{f}(k)$
$\delta(x)$	1
1	$2\pi\,\delta(k)$
$\frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{x^2}{2\sigma^2}}$	$e^{-\frac{\sigma^2 k^2}{2}}$
$e^{-\lambda\lvert x\rvert}$	$\frac{2\lambda}{k^2+\lambda^2}$
$\frac{1}{x}$	$-\pi i\, \mathrm{sgn}(k)$
$\log x$	$-\frac{\pi}{\lvert k\rvert}$
$f'(x)$	$ik\cdot\hat{f}(k)$

* $\delta(x)$ is Dirac-delta function defined as

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{+\infty} \delta(x)\, f(x)\, dx = f(0)$$

* $\mathrm{sgn}(k) = \begin{cases} +1 & k > 0 \\ 0 & k = 0 \\ -1 & k < 0 \end{cases}$

$$\hat{f}(k) = \mathcal{F}[f(x)]$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}(k)]$$

Recall that when

$$p_y(x) = p_0 \cos \lambda x$$

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{\lambda} \cdot p_0 \cos \lambda x \qquad (\text{for } \lambda > 0)$$

Obviously, if λ can be either positive or negative. then

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|\lambda|} \cdot p_0 \cos \lambda x .$$

when $p_y(x) = p_0 \sin \lambda x$, we can easily show that

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|\lambda|} \cdot p_0 \sin \lambda x .$$

These two expression can be written together as

$$\boxed{\begin{aligned} &\text{when } p_y(x) = p_0 e^{ik_0 x} \\ &\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|k_0|} \cdot (p_0 e^{ik_0 x}) \end{aligned}}$$

Introduce the Fourier transform of $p_y(x)$ and $\tilde{u}_y(x)$

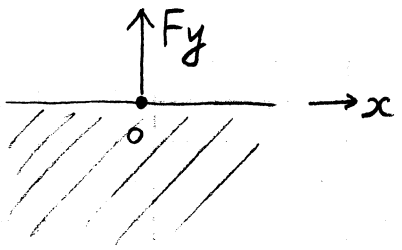
$$\hat{p}_y(k) = \int_{-\infty}^{+\infty} p_y(x) e^{-ikx} dx, \quad p_y(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{p}_y(k) e^{ikx} dk$$

$$\hat{u}_y(k) = \int_{-\infty}^{+\infty} \tilde{u}_y(x) e^{-ikx} dx, \quad \tilde{u}_y(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}_y(k) e^{ikx} dk$$

The above relation in the box can be rewritten in the Fourier space as

$$\boxed{\hat{u}_y(k) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|k|} \cdot \hat{p}_y(k)}$$

§4. Point Force Loading



$$P_y(x) = -F_y\,\delta(x) \quad \left(\begin{array}{l}\text{concentrated}\\ \text{load at } x=0\end{array}\right)$$

In Fourier space:

$$\hat{P}_y(k) = -F_y$$

$$\hat{u}_y(k) = \frac{2(1-\nu^2)}{E}\cdot\frac{F_y}{|k|}$$

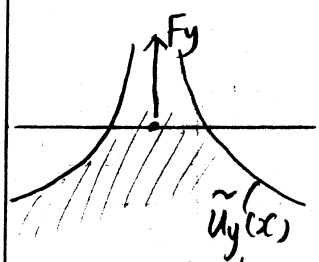
Back to real space (inverse F.T.)

$$\tilde{u}_y(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}_y(k)\, e^{ikx}\, dk$$

$$= \frac{2(1-\nu^2)}{E}\cdot F_y \cdot \mathrm{Re}\left[\mathcal{F}^{-1}\left[\frac{1}{|k|}\right]\right] \quad \left(\begin{array}{l}\text{only take the}\\ \text{real part}\end{array}\right)$$

$$= \frac{2(1-\nu^2)}{E}\cdot F_y \cdot \mathrm{Re}\left[-\frac{1}{\pi}\log x\right]$$

$$= -\frac{2(1-\nu^2)}{\pi E}\cdot F_y \cdot \log|x| \quad \left(\begin{array}{l}* \text{ when } x<0\\ \log x = \log|x| + \pi i\end{array}\right)$$



surface displacement

$$\therefore\ \tilde{u}_y(x) = -\frac{2(1-\nu^2)}{\pi E}\cdot F_y \cdot \log|x| + C$$

arbitrary constant

notice that $E = 2\mu(1+\nu)$

also in plane strain $\kappa = 3-4\nu$, $\kappa+1 = 4(1-\nu)$

$$\frac{2(1-\nu^2)}{E} = \frac{2(1-\nu)(1+\nu)}{2\mu(1+\nu)} = \frac{1-\nu}{\mu} = \frac{\kappa+1}{4\mu}$$

κ: Kolosov constant

$$\boxed{\tilde{u}_y(x) = -\frac{\kappa+1}{4\pi\mu}\cdot F_y \cdot \log|x|}$$

What is the surface displacement in the x-direction?

Recall when $p_y(x) = p_0 \cos\lambda x$

$$\tilde{u}_x(x) = -\frac{1-\nu-2\nu^2}{E}\cdot\frac{1}{\lambda}\cdot p_0 \sin\lambda x$$

We can show that when $p_y(x) = p_0 e^{ik_0 x}$

$$\tilde{u}_x(x) = -\frac{1-\nu-2\nu^2}{E}\cdot\frac{-i}{|k_0|}\cdot(p_0 e^{ik_0 x})$$

Hence

$$\boxed{\hat{u}_x(k) = -\frac{1-\nu-2\nu^2}{E}\cdot\frac{-i}{|k_0|}\cdot\hat{p}_y(k)}$$

In response to a point load, $p_y(x) = -F_y\,\delta(x)$,

$$\hat{u}_x(k) = \frac{1-\nu-2\nu^2}{E}\cdot\frac{-i}{|k_0|}\cdot F_y$$

$$\tilde{u}_x(x) = \frac{1-\nu-2\nu^2}{E}\cdot F_y\cdot \mathrm{Re}\left[\mathcal{F}^{-1}\left[\frac{-i}{|k|}\right]\right]$$

$$= \frac{1-\nu-2\nu^2}{E}\cdot F_y\cdot \mathrm{Im}\left[\frac{1}{\pi}\log x\right]$$

$$= -\frac{1-\nu-2\nu^2}{E}\cdot F_y\cdot H(-x)$$

$\left(* \ \mathrm{Im}[\log(x)] = \begin{cases} 0 & x>0 \\ \pi & x<0 \end{cases}\right.$

$H(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$ step function$\Big)$

Remember that we can add an arbitrary constant to $\tilde{u}_y(x)$

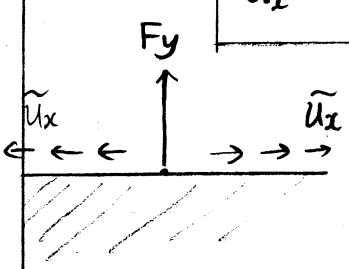
$$\therefore\ \tilde{u}_x(x) = \frac{1-\nu-2\nu^2}{E}\cdot F_y\cdot\frac{\pi}{2}\cdot \mathrm{sgn}(x) + C'$$

arbitrary constant

notice that $\frac{1-\nu-2\nu^2}{E} = \frac{(1+\nu)(1-2\nu)}{2\mu(1+\nu)} = \frac{1-2\nu}{2\mu}$

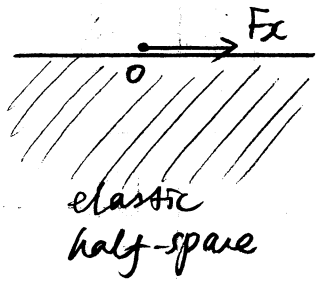
also in plane strain $\kappa = 3-4\nu$, $1-2\nu = \frac{\kappa-1}{2}$, $\frac{1-2\nu}{2\mu} = \frac{\kappa-1}{4\mu}$

$$\boxed{\tilde{u}_x(x) = \frac{\kappa-1}{8\mu}\cdot F_y\cdot \mathrm{sgn}(x)}$$



$(\nu < 0.5)$

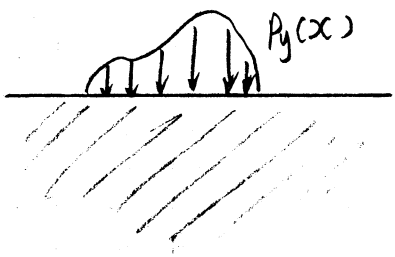
In a similar approach, we can obtain the surface displacement in response to a concentrated tangential force F_x



$$\tilde{u}_x(x) = -\frac{\kappa+1}{4\pi\mu}\cdot F_x \cdot \log|x|$$

$$\tilde{u}_y(x) = -\frac{\kappa-1}{8\mu}\cdot F_y \cdot \mathrm{sgn}(x)$$

§5. Arbitrary Load Distribution $p_y(x)$



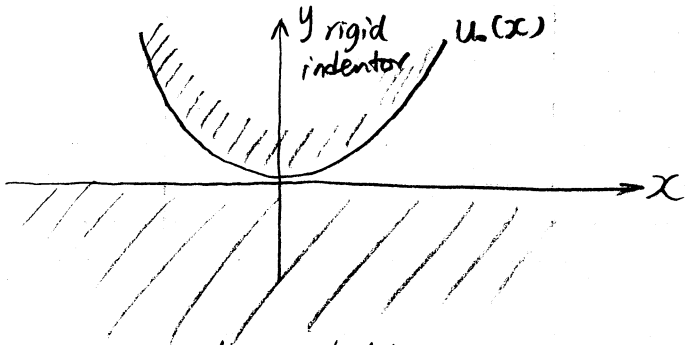
$$\tilde{u}_y(x) = \frac{\kappa+1}{4\pi\mu}\int_{-\infty}^{+\infty} p_y(x')\cdot \log|x-x'|\cdot dx'$$

$$\tilde{u}_x(x) = -\frac{\kappa-1}{8\mu}\int_{-\infty}^{+\infty} p_y(x')\cdot \mathrm{sgn}(x-x')\cdot dx$$

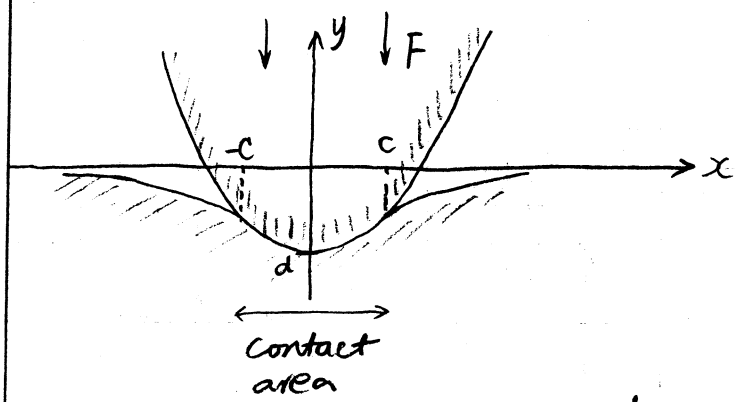
(linear superposition in real space)

* A similar expression exists for distributed tangential loading.

For simplicity, let us consider the problem of a rigid indentor in contact with an elastic half-space.



$u_0(x)$: shape (profile) of the indentor



intuitively, the contact area 2c would increase with increasing indenting force F.

§1. Boundary condition of contact Problem

"contact area"
$-c \le x \le c$, $y=0$

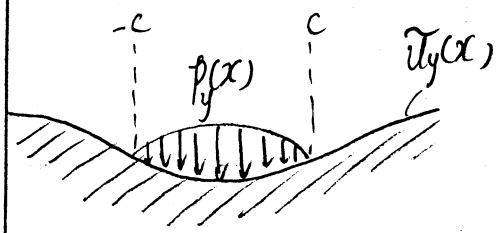
$$\begin{cases} u_y = u_0(x) + d & \text{— the shape of elastic medium conforms to that of indentor,} \\ \sigma_{yy} \le 0 & \text{* d is still unknown} \\ \sigma_{xy} = 0 & \text{← frictionless compression} \end{cases}$$

"gap" area
$|x| > c$, $y=0$

$$\begin{cases} u_y < u_0(x) + d & \text{— existence of a gap} \\ \sigma_{yy} = 0 & \text{— free surface, zero tractions} \\ \sigma_{xy} = 0 & \end{cases}$$

- The contact area 2c and indentation depth d are <u>not</u> specified a priori
- The goal is to solve for c and d given the indentor shape $u_0(x)$ and indenting force F.

* usually very difficult.



Let $p_y(x)$ be the pressure exerted by the indentor onto the half-space.

$$\begin{cases} p_y(x) \geq 0 & -c \leq x \leq c \quad \text{in contact area} \\ p_y(x) = 0 & |x| > c \quad \text{outside} \end{cases}$$

Let $\tilde{u}_y(x)$ be the surface displacement in y direction

$$\begin{cases} \tilde{u}_y(x) = u_0(x) + d & -c \leq x \leq c \\ \tilde{u}_y(x) < u_0(x) + d & |x| > c. \end{cases}$$

From Lecture Notes on "Half-space", we have

$$\tilde{u}_y(x) = \frac{\kappa+1}{4\pi\mu} \int_{-\infty}^{+\infty} p_y(x') \cdot \log|x-x'| \cdot dx'$$

$$\tilde{u}_y(x) = \frac{\kappa+1}{4\pi\mu} \int_{-c}^{c} p_y(x') \cdot \log|x-x'| \cdot dx'$$

* so we constrain $|x'| \leq c$

If we also constrain $|x| \leq c$, then

$$u_0(x) + d = \frac{\kappa+1}{4\pi\mu} \int_{-c}^{c} p_y(x') \cdot \log|x-x'| \cdot dx'$$

To get rid of the unknown constant d, $\frac{d}{dx}$ on both sides.

$$\frac{du_0(x)}{dx} = \frac{\kappa+1}{4\pi\mu} \int_{-c}^{c} \frac{p_y(x')}{x-x'} dx' \quad (\text{for } |x| \leq c)$$

$$* \ \frac{d}{dx} \log|x-x'| = \frac{1}{x-x'}$$

Our task is to invert this equation to find $p_y(x')$, after that we will find $\tilde{u}_y(x)$ everywhere (outside the contact area)

§2.

To invert an integral equation is difficult, and we will need some special mathematical tools.

Now that both x and x' are limited to the domain $[c, c]$, It is natural to introduce angular variables ϕ, θ such that $x = c\cos\phi$ and $x' = c\cos\theta$

and $\phi, \theta \in [0, \pi]$

$$dx = -c\sin\phi\, d\phi \quad , \quad dx' = -c\sin\theta\, d\theta$$

$u_0(x)$ can be rewritten as a function of ϕ, $u_0(\phi)$

$p_y(x')$ can be rewritten as a function of θ, $p_y(\theta)$

$$-\frac{1}{c\sin\phi}\cdot\frac{du_0(\phi)}{d\phi} = \frac{\kappa+1}{4\pi\mu}\int_{\pi}^{0}\frac{p_y(\theta)}{c(\cos\phi-\cos\theta)}(-c\sin\theta)\cdot d\theta$$

$$-\frac{1}{c\sin\phi}\frac{du_0(\phi)}{d\phi} = \frac{\kappa+1}{4\pi\mu}\int_{0}^{\pi}\frac{p_y(\theta)\sin\theta\, d\theta}{\cos\phi-\cos\theta}$$

Now we need a special mathematical formula.

$$\boxed{-\frac{\sin n\phi}{\sin\phi} = \frac{1}{\pi}\int_0^{\pi}\frac{\cos n\theta\, d\theta}{\cos\phi-\cos\theta}} \qquad n = 0, 1, 2, \cdots$$

* This formula will be proved at the end of this lecture note.

In order to take advantage of this formula, we need to expand

$$\frac{du_0(\phi)}{d\phi} = \sum_{n=1}^{\infty} w_n \sin n\phi$$

(the $n=0$ term makes zero contribution)

$$p_y(\theta)\sin\theta = \sum_{n=0}^{\infty} p_n \cos n\theta$$

i.e. $$p_y(\theta) = \sum_{n=0}^{\infty}\frac{p_n\cos n\theta}{\sin\theta}$$

$$-\frac{1}{c}\sum_{n=1}^{\infty}\frac{w_n \sin n\phi}{\sin\phi} = \frac{\kappa+1}{4\mu}\cdot\frac{1}{\pi}\sum_{n=0}^{\infty}\int_0^{\pi}\frac{p_n\cos(n\theta)\,d\theta}{\cos\phi-\cos\theta}$$

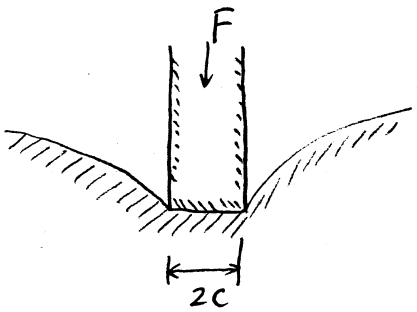
$$= -\frac{\kappa+1}{4\mu}\sum_{n=1}^{\infty}\frac{p_n \sin n\phi}{\sin\phi}$$

$$\therefore \quad w_n = \frac{\kappa+1}{4\mu}\cdot p_n\cdot c \qquad \text{for } n=1,2,3,\cdots$$

$$\boxed{p_n = \frac{4\mu}{(\kappa+1)\cdot c}\cdot w_n} \qquad n=1,2,3,\cdots$$

p_0 is arbitrary

§3. Flat Punch



The flat punch problem is easier than a general contact problem because $2c$ is already known.

Flat Punch $\Leftrightarrow$ $u_0 = \text{const} \quad -c\le x\le c$.

$$\frac{du_0(\phi)}{d\phi}=0$$

i.e. when we expand $\dfrac{du_0(\phi)}{d\phi} = \sum_{n=1}^{\infty} w_n \sin n\phi$

then $w_n = 0$ for all $n=1,2,3,\cdots$

Therefore $p_n = 0$ for all $n=1,2,3,\cdots$

The only non-zero component is p_0

$$p_y(\theta) = \sum_{n=0}^{\infty}\frac{p_n\cos n\theta}{\sin\theta} = \frac{p_0}{\sin\theta}$$

Recall that $x' = c\cos\theta$, hence $\sin\theta = \sqrt{1-\left(\frac{x'}{c}\right)^2}$

$$p_y(x') = \frac{p_0}{\sqrt{1-\left(\frac{x'}{c}\right)^2}}$$

How to determine the unknown coefficient p_0 and $2c$?

Intuitively, we expect the contact pressure $p_y(x)$ to be non-singular everywhere (unlike the flat punch).

This is because a singular solution has very high energy. The system should automatically remove the singularity by adjusting the contact area $2c$.

Notice that $\frac{1}{\sin\theta} = \frac{1}{\sqrt{1-(\frac{x'}{c})^2}}$ is singular at $x' = \pm c$

Hence the numerator $p_0 + p_2\cos 2\theta$ must contain a multiplication factor $\sin\theta$ to cancel the denominator.

This is the case when $p_0 = -p_2$

$$p_0 + p_2\cos 2\theta = p_0(1-\cos 2\theta) = 2p_0\sin^2\theta$$

$$p(\theta) = \frac{p_0 + p_2\cos 2\theta}{\sin\theta} = 2p_0\sin\theta$$

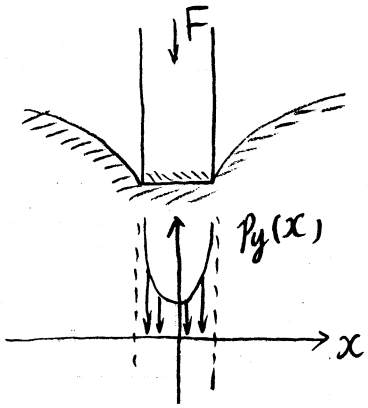
$$p_y(x') = 2p_0\sqrt{1-(\tfrac{x'}{c})^2}$$ — the inverse of flat punch problem

The total indenting force

$$F = \int_{-c}^{c} p_y(x')\,dx' = p_0\pi c$$

$$p_0 = \frac{F}{\pi c}$$

$$p_y(x') = \frac{2F}{\pi c}\sqrt{1-(\tfrac{x'}{c})^2} = \frac{2F}{\pi c^2}\sqrt{c^2-x'^2}$$



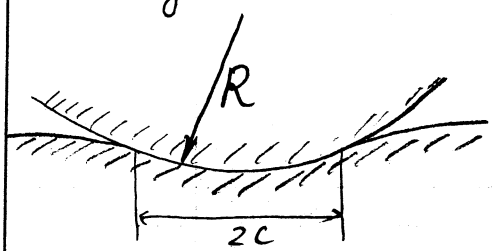
Total indenting force

$$F = \int_{-c}^{c} p_y(x')\,dx' = p_0 \pi c$$

$$\therefore \quad p_y(x') = -\frac{F}{\pi c\sqrt{1-\left(\frac{x'}{c}\right)^2}}$$

$$\boxed{p_y(x') = -\frac{F}{\pi\sqrt{c^2-x'^2}}}$$

§4. Cylindrical Punch (Hertz contact problem)



Let R be the radius of curvature of the rigid indentor.

We can approximately write

$$u_0(x) = \frac{x^2}{2R}$$

Since $x = c\cos\phi$, $\quad u_0(\phi) = \frac{c^2}{2R}\cos^2\phi$

We shall rewrite $\frac{du_0(\phi)}{d\phi}$ into the form of $\sum_{n=1}^{\infty} w_n \sin n\phi$.

$$u_0(\phi) = \frac{c^2}{4R}(\cos 2\phi - 1)$$

$$\frac{du_0(\phi)}{d\phi} = -\frac{c^2}{2R}\sin 2\phi = \sum_{n=1}^{\infty} w_n \sin n\phi$$

$$\therefore \quad w_2 = -\frac{c^2}{2R} \qquad w_1 = w_3 = w_4 = \cdots = 0$$

$$p_2 = -\frac{4\mu}{(K+1)\cdot c}\cdot\frac{c^2}{2R} = -\frac{2\mu c}{(K+1)\cdot R} \qquad p_1 = p_3 = p_4 = \cdots = 0$$

p_0 is still unknown

$$p(\theta) = \frac{p_0}{\sin\theta} + \frac{p_2\cos 2\theta}{\sin\theta} = \frac{p_0 + p_2\cos 2\theta}{\sin\theta}$$

Recall $p_2 = -\frac{2\mu c}{(\kappa+1)\cdot R}$, $p_0 = -p_2$, $p_0 = \frac{F}{\pi c}$

$$\therefore \quad -\frac{2\mu c}{(\kappa+1)R} = -\frac{F}{\pi c}$$

$$F = \frac{2\pi\mu}{\kappa+1}\cdot\frac{c^2}{R} \quad , \quad \boxed{c = \sqrt{\frac{F(\kappa+1)R}{2\pi\mu}}}$$

* contact area $\sim \sqrt{F}$

Q: What is the indentation depth d ?

Q: What if the indentor profile is $u_0(x) = x^4$?

*§5. Hilbert transform. (David Hilbert 1862-1943 German mathematician)

The Hilbert transform of a function $s(t)$ is defined as

$$r(t) = \mathcal{H}\{s(t)\} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{s(t')}{t-t'}\,dt'$$

Hilbert transform is its own inverse (with a minus sign)

$$s(t) = -\mathcal{H}\{r(t)\} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{r(t')}{t-t'}\,dt'$$

Some Hilbert transform pairs

Signal $s(t)$	Hilbert transform $\mathcal{H}\{s(t)\}$
$\delta(t)$	$\frac{1}{\pi t}$
$\sin t$	$-\cos t$
$\cos t$	$\sin t$
$e^{i\omega t}$	$-i\,\mathrm{sgn}(\omega)\,e^{i\omega t}$

Recall that $\frac{d\tilde{u}_y(x)}{dx} = \frac{\kappa+1}{4\pi\mu}\int_{-\infty}^{+\infty} \frac{p_y(x')}{x-x'}\,dx'$

Hence $\frac{d\tilde{u}_y(x)}{dx}$ is the Hilbert transform of $\frac{\kappa+1}{4\mu}p_y(x)$

Applying the reverse Hilbert transform

$$\frac{\kappa+1}{4\mu}p_y(x) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\frac{d\tilde{u}_y(x')}{dx'}}{x-x'}\,dx'$$

Unfortunately, this does not help us finding $p_y(x)$ — which is zero if $|x|>c$,

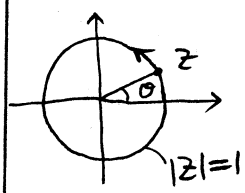
$\frac{d\tilde{u}_y(x)}{dx}$ is non-zero and unknown if $|x|>c$.

* §6. Proof of the formula

$$-\frac{\sin n\phi}{\sin\phi} = \frac{1}{\pi}\int_0^{\pi}\frac{\cos n\theta}{\cos\phi-\cos\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos n\theta}{\cos\phi-\cos\theta}\,d\theta$$

by complex analysis.

Define complex variable $z \equiv e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$



$$z + \frac{1}{z} = 2\cos\theta, \qquad dz = e^{i\theta}\cdot i\cdot d\theta = iz\,d\theta$$

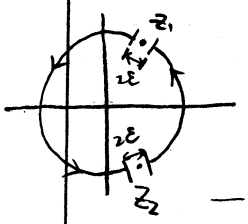
$$z^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos n\theta + i\sin n\theta}{\cos\phi-\cos\theta}\,d\theta = \frac{1}{2\pi}\oint_{|z|=1}\frac{z^n\cdot\frac{dz}{iz}}{\cos\phi-\frac{1}{2}(z+\frac{1}{z})}$$

Define complex variable

$$z_1 = \cos\phi + i\sin\phi = e^{i\phi}$$
$$z_2 = z_1^* = \cos\phi - i\sin\phi = e^{-i\phi}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos n\theta + i\sin n\theta}{\cos\phi-\cos\theta}\,d\theta = \frac{1}{\pi}\oint_{|z|=1}\frac{iz^n\,dz}{z^2-2\cos\phi\cdot z+1}$$



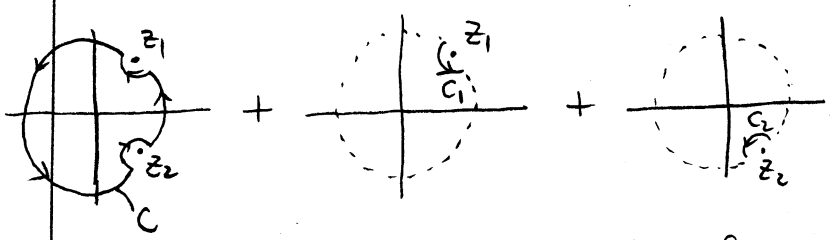
original integration contour

— Cauchy Principal value

$$= \frac{1}{\pi}\oint_{|z|=1}\frac{iz^n\,dz}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{\pi}\left\{\oint_C + \int_{C_1} + \int_{C_2}\right\}\frac{iz^n\,dz}{(z-z_1)(z-z_2)}$$

||



The contour integral $\oint_C = 0$ because the integrand does not contain any singularity inside C.

On the other hand, for the integral $\int_{C_1}$, we can introduce a new angular variable α, such that

$$z = z_1 + \varepsilon e^{i\alpha} \quad , \quad \varepsilon \to 0.$$

$$\int_{C_1} \frac{i z^n}{(z-z_1)(z-z_2)} dz = \int_{C_1} \frac{i z_1^n \varepsilon e^{i\alpha} i d\alpha}{\varepsilon e^{i\alpha}(z_1 - z_2)}$$

$$= -\frac{z_1^n}{z_1 - z_2} \int_{C_1} d\alpha = -\frac{\pi z_1^n}{2i\sin\phi}$$

Similarly,

$$\int_{C_2} \frac{i z^n}{(z-z_1)(z-z_2)} dz = \frac{\pi z_2^n}{2i\sin\phi}$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos n\theta + i\sin n\theta}{\cos\phi - \cos\theta} d\theta = \frac{1}{\pi}\left\{-\frac{\pi z_1^n}{2i\sin\phi} + \frac{\pi z_2^n}{2i\sin\phi}\right\}$$

$$= -\frac{\sin n\phi}{\sin\phi} \qquad \left(z_1^n - z_2^n = 2i\sin n\phi\right)$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos n\theta}{\cos\phi - \cos\theta} d\theta = -\frac{\sin n\phi}{\sin\phi} \qquad n = 0, 1, 2, \cdots$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\sin n\theta}{\cos\phi - \cos\theta} d\theta = 0$$

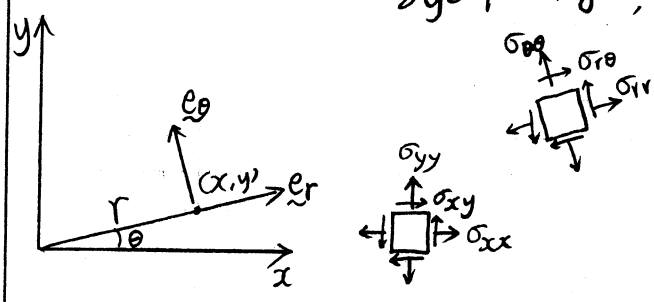
There are many 2D elasticity problems which are easier to discuss using polar coordinates than cartesian coordinates.

Here we discuss the stress function approach in polar coordinates. The physics behind this approach is identical to what we discussed before (i.e. equilibrium is automatically satisfied, $\nabla^4\phi = 0$ corresponds to compatibility condition, etc), only the mathematical expressions are difficult.

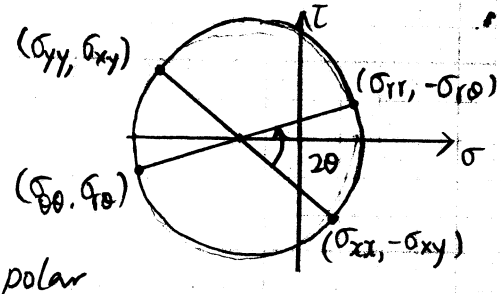
§1. The relationship between stress and stress function

Recall in cartesian coordinate, given a stress function $\phi(x,y)$ the stress field is

$$\sigma_{xx} = \frac{\partial^2}{\partial y^2}\phi(x,y), \quad \sigma_{yy} = \frac{\partial^2}{\partial x^2}\phi(x,y), \quad \sigma_{xy} = -\frac{\partial^2}{\partial x \partial y}\phi(x,y)$$



Mohr's circle



Notice that the "stress cube" for the polar coordinate system is rotated by angle θ from that for the cartesian coordinate system.

Hence

$$\begin{cases} \sigma_{rr} = \dfrac{\sigma_{xx}+\sigma_{yy}}{2} + \dfrac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \\ \sigma_{\theta\theta} = \dfrac{\sigma_{xx}+\sigma_{yy}}{2} - \dfrac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta \\ \sigma_{r\theta} = -\dfrac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta \end{cases}$$

(See lecture note "Tensors", p. 8)

Combine the equations above, we can express $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$ in terms of $\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x \partial y}$.

However, this is not enough. To solve 2D elasticity problems in polar coordinates, we need to express $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$ in terms of $\frac{\partial^2}{\partial r^2}, \frac{\partial^2}{\partial \theta^2}, \frac{\partial^2}{\partial r \partial \theta}$.

This can be done by establishing the relationship between $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$.

Notice that $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$

$$\underset{\sim}{e}_r \cdot \underset{\sim}{e}_x = \cos\theta \qquad \underset{\sim}{e}_r \cdot \underset{\sim}{e}_y = \sin\theta$$

$$\underset{\sim}{e}_\theta \cdot \underset{\sim}{e}_x = -\sin\theta \qquad \underset{\sim}{e}_\theta \cdot \underset{\sim}{e}_y = \cos\theta$$

The same stress function $\phi(x,y)$ can also be written as a function of r and θ, i.e. $\phi(r,\theta)$

$$\begin{cases} \frac{\partial}{\partial x}\phi(r,\theta) = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta \frac{\partial \phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \phi}{\partial \theta} \\ \frac{\partial}{\partial y}\phi(r,\theta) = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \frac{\partial \phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta} \end{cases}$$

* This means that the gradient operator $\underset{\sim}{\nabla}$ can be written as

$$\underset{\sim}{\nabla} = \underset{\sim}{e}_x \cdot \frac{\partial}{\partial x} + \underset{\sim}{e}_y \cdot \frac{\partial}{\partial y} = \underset{\sim}{e}_r \frac{\partial}{\partial r} + \underset{\sim}{e}_\theta \frac{1}{r} \cdot \frac{\partial}{\partial \theta}$$

$$\underset{\sim}{e}_r = \cos\theta\, \underset{\sim}{e}_x + \sin\theta\, \underset{\sim}{e}_y$$

$$\underset{\sim}{e}_\theta = -\sin\theta\, \underset{\sim}{e}_x + \cos\theta\, \underset{\sim}{e}_y$$

$$\frac{\partial^2\phi}{\partial x^2} = \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial x}\right)$$

$$= \cos\theta \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial\phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$$\frac{\partial^2\phi}{\partial x^2} = \cos^2\theta\frac{\partial^2\phi}{\partial r^2} + \sin^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi + 2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)\phi$$

similarly

$$\frac{\partial^2\phi}{\partial y^2} = \sin^2\theta\frac{\partial^2\phi}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi - 2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)\phi$$

$$\frac{\partial^2\phi}{\partial x\partial y} = \sin\theta\cos\theta\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi - (\cos^2\theta - \sin^2\theta)\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)\phi$$

now recall the relationship between $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$ and $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$

we have

$$\boxed{\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\\
\sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2}\\
\sigma_{r\theta} &= \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)
\end{aligned}}$$

* This means that the tensorial operator $\underset{\sim}{\nabla}\underset{\sim}{\nabla}$ can be written as

$$\underset{\sim}{\nabla}\underset{\sim}{\nabla} = \left(\underset{\sim}{e}_x\frac{\partial}{\partial x} + \underset{\sim}{e}_y\frac{\partial}{\partial y}\right)\left(\underset{\sim}{e}_x\frac{\partial}{\partial x} + \underset{\sim}{e}_y\frac{\partial}{\partial y}\right) = \underset{\sim}{e}_x\underset{\sim}{e}_x\frac{\partial^2}{\partial x^2} + \underset{\sim}{e}_y\underset{\sim}{e}_y\frac{\partial^2}{\partial y^2} + (\underset{\sim}{e}_x\underset{\sim}{e}_y + \underset{\sim}{e}_y\underset{\sim}{e}_x)\frac{\partial^2}{\partial x\partial y}$$

$$= \left(\underset{\sim}{e}_r\frac{\partial}{\partial r} + \underset{\sim}{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\underset{\sim}{e}_r\frac{\partial}{\partial r} + \underset{\sim}{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \underset{\sim}{e}_r\underset{\sim}{e}_r\frac{\partial^2}{\partial r^2} + \underset{\sim}{e}_\theta\underset{\sim}{e}_\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) + (\underset{\sim}{e}_\theta\underset{\sim}{e}_r + \underset{\sim}{e}_r\underset{\sim}{e}_\theta)\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

$$\frac{\partial\underset{\sim}{e}_r}{\partial\theta} = e_\theta, \qquad \frac{\partial\underset{\sim}{e}_\theta}{\partial\theta} = -\underset{\sim}{e}_r, \qquad \frac{\partial\underset{\sim}{e}_r}{\partial r} = \frac{\partial\underset{\sim}{e}_\theta}{\partial r} = 0$$

* The Laplace operator $\nabla^2 = \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}$ can be written as

$$\nabla^2 = \left(\underset{\sim}{e}_x\frac{\partial}{\partial x} + \underset{\sim}{e}_y\frac{\partial}{\partial y}\right)\cdot\left(\underset{\sim}{e}_x\frac{\partial}{\partial x} + \underset{\sim}{e}_y\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$= \left(\underset{\sim}{e}_r\frac{\partial}{\partial r} + \underset{\sim}{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\underset{\sim}{e}_r\frac{\partial}{\partial r} + \underset{\sim}{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

§2. Biharmonic equation.

Recall that in the stress function approach, equilibrium condition is automatically satisfied.

The compatibility + Generalized Hooke's Law lead to the biharmonic equation $\nabla^4\phi \equiv \nabla^2(\nabla^2\phi) = 0$

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0$$

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0$$

Next, we discuss the relationship between fundamental quantities in elasticity in polar coordinates

displacement field	$\leftrightarrow$ strain field	$\leftrightarrow$ stress field	$\leftrightarrow$ Traction force
$u_r(r,\theta)$	$\varepsilon_{rr}(r,\theta)$	$\sigma_{rr}(r,\theta)$	$T_r(r,\theta)$
$u_\theta(r,\theta)$	$\varepsilon_{r\theta}(r,\theta)$	$\sigma_{r\theta}(r,\theta)$	$T_\theta(r,\theta)$
	$\varepsilon_{\theta\theta}(r,\theta)$	$\sigma_{\theta\theta}(r,\theta)$	

§3. Displacement field and Strain Tensor

In Cartesian coordinate $\quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

In tensor notation $\quad \underset{\approx}{\varepsilon} = \frac{1}{2}\left[\underset{\sim}{\nabla}\,\underset{\sim}{u} + (\underset{\sim}{\nabla}\,\underset{\sim}{u})^T\right]$

$$\underset{\sim}{u} = u_x\,\underset{\sim}{e_x} + u_y\,\underset{\sim}{e_y}, \quad \underset{\sim}{\nabla} = \underset{\sim}{e_x}\frac{\partial}{\partial x} + \underset{\sim}{e_y}\frac{\partial}{\partial y}$$

$$\underset{\sim}{\nabla}\,\underset{\sim}{u} = \left(\underset{\sim}{e_x}\frac{\partial}{\partial x} + \underset{\sim}{e_y}\frac{\partial}{\partial y}\right)(u_x\,\underset{\sim}{e_x} + u_y\,\underset{\sim}{e_y})$$

$$= \underset{\sim}{e_x}\underset{\sim}{e_x}\frac{\partial u_x}{\partial x} + \underset{\sim}{e_y}\underset{\sim}{e_y}\frac{\partial u_y}{\partial y} + \underset{\sim}{e_x}\underset{\sim}{e_y}\frac{\partial u_y}{\partial x} + \underset{\sim}{e_y}\underset{\sim}{e_x}\frac{\partial u_x}{\partial y}$$

$$\underset{\approx}{\varepsilon} = \frac{1}{2}\left[\underset{\sim}{\nabla}\,\underset{\sim}{u} + (\underset{\sim}{\nabla}\,\underset{\sim}{u})^T\right] = \underset{\sim}{e_x}\underset{\sim}{e_x}\frac{\partial u_x}{\partial x} + \underset{\sim}{e_y}\underset{\sim}{e_y}\frac{\partial u_y}{\partial y} + (\underset{\sim}{e_x}\underset{\sim}{e_y} + \underset{\sim}{e_y}\underset{\sim}{e_x})\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

In polar coordinates,

$$\underset{\sim}{u} = u_r \underset{\sim}{e_r} + u_\theta \underset{\sim}{e_\theta}, \quad \underset{\sim}{\nabla} = \underset{\sim}{e_r}\frac{\partial}{\partial r} + \underset{\sim}{e_\theta}\frac{1}{r}\frac{\partial}{\partial \theta}$$

$$\underset{\sim}{\nabla}\underset{\sim}{u} = \left(\underset{\sim}{e_r}\frac{\partial}{\partial r} + \underset{\sim}{e_\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right)(u_r \underset{\sim}{e_r} + u_\theta \underset{\sim}{e_\theta})$$

$$\underset{\sim}{\nabla}\underset{\sim}{u} = \underset{\sim}{e_r}\underset{\sim}{e_r}\frac{\partial u_r}{\partial r} + \underset{\sim}{e_\theta}\underset{\sim}{e_\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

typo in Sadd, p.24 Eq (1.9.21)

$$+ \underset{\sim}{e_\theta}\underset{\sim}{e_r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)$$

$$+ \underset{\sim}{e_r}\underset{\sim}{e_\theta}\frac{\partial u_\theta}{\partial r}$$

$$\underset{\approx}{\varepsilon} = \frac{1}{2}\left[(\underset{\sim}{\nabla}\underset{\sim}{u}) + (\underset{\sim}{\nabla}\underset{\sim}{u})^T\right]$$

$$= \underset{\sim}{e_r}\underset{\sim}{e_r}\frac{\partial u_r}{\partial r} + \underset{\sim}{e_\theta}\underset{\sim}{e_\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + (\underset{\sim}{e_\theta}\underset{\sim}{e_r} + \underset{\sim}{e_r}\underset{\sim}{e_\theta})\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)$$

$$\boxed{\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\ \varepsilon_{\theta\theta} &= \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) \\ \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \end{aligned}}$$

§4. Generalized Hooke's Law

In Cartesian Coordinate (isotropic medium)

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

* in component form

$$\begin{aligned} \sigma_{xx} &= (\lambda+2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz} \\ \sigma_{yy} &= \lambda\varepsilon_{xx} + (\lambda+2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz} \\ \sigma_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda+2\mu)\varepsilon_{zz} \\ \sigma_{xy} &= 2\mu\varepsilon_{xy} \\ \sigma_{yz} &= 2\mu\varepsilon_{yz} \\ \sigma_{xz} &= 2\mu\varepsilon_{xz} \end{aligned}$$

In tensor notation

$$\underset{\approx}{\sigma} = \lambda \underset{\text{trace}}{\mathrm{Tr}}[\underset{\approx}{\varepsilon}] \cdot \underset{\text{identity tensor}}{\underset{\approx}{I}} + 2\mu\underset{\approx}{\varepsilon}$$

$$\mathrm{Tr}[\underset{\approx}{\varepsilon}] = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$= \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$$

In polar coordinates

$$\boxed{\begin{aligned}
\sigma_{rr} &= (\lambda+2\mu)\,\varepsilon_{rr} + \lambda\,\varepsilon_{\theta\theta} + \lambda\,\varepsilon_{zz} \\
\sigma_{\theta\theta} &= \lambda\,\varepsilon_{rr} + (\lambda+2\mu)\,\varepsilon_{\theta\theta} + \lambda\,\varepsilon_{zz} \\
\sigma_{zz} &= \lambda\,\varepsilon_{rr} + \lambda\,\varepsilon_{\theta\theta} + (\lambda+2\mu)\,\varepsilon_{zz} \\
\sigma_{r\theta} &= 2\mu\,\varepsilon_{r\theta} \\
\sigma_{\theta z} &= 2\mu\,\varepsilon_{\theta z} \\
\sigma_{rz} &= 2\mu\,\varepsilon_{rz}
\end{aligned}}$$

§4. Stress field and <u>traction force</u>

In Cartesian coordinate $\quad T_j = \sigma_{ij}\, n_i$

In tensor notation $\quad \underset{\sim}{T} = \underset{\approx}{\sigma} \cdot \underset{\sim}{n}$

In polar coordinates

$$T_r = \sigma_{rr}\, n_r + \sigma_{r\theta}\, n_\theta + \sigma_{rz}\, n_z$$

$$T_\theta = \sigma_{r\theta}\, n_r + \sigma_{\theta\theta}\, n_\theta + \sigma_{\theta z}\, n_z$$

$$T_z = \sigma_{rz}\, n_r + \sigma_{\theta z}\, n_\theta + \sigma_{zz}\, n_z$$

stress equilibrium

$\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \underset{\sim}{F} = 0$ (tensor notation)

$$\left(\underset{\sim}{e}_r \frac{\partial}{\partial r} + \underset{\sim}{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\sigma_{rr}\, \underset{\sim}{e}_r \underset{\sim}{e}_r + \sigma_{\theta\theta}\, \underset{\sim}{e}_\theta \underset{\sim}{e}_\theta + \sigma_{r\theta}\, \underset{\sim}{e}_r \underset{\sim}{e}_\theta + \sigma_{r\theta}\, \underset{\sim}{e}_\theta \underset{\sim}{e}_r\right) + F_r\, \underset{\sim}{e}_r + F_\theta\, \underset{\sim}{e}_\theta = 0$$

(component form)

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}\sigma_{rr} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} - \frac{\sigma_{\theta\theta}}{r} + F_r = 0$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r}\sigma_{r\theta} + F_\theta = 0$$

(in 2D)

§5. General Solution to the biharmonic equation

$$\nabla^4 \phi(r,\theta) = 0$$

trial solution: $\quad \phi(r,\theta) = f(r)\, e^{in\theta} \qquad n = 0, 1, 2, 3, \cdots$

obviously, $\phi(r,\theta)$ must be periodic in θ

$$\phi(r,\theta) = \phi(r, \theta + 2\pi)$$

$$\frac{\partial}{\partial \theta}\phi = in\phi, \qquad \frac{\partial^2}{\partial \theta^2}\phi = -n^2\phi$$

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\phi = 0$$

$$\therefore \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) f(r) = 0$$

trial solution $f(r) = r^m$ — polynomial

$$\frac{\partial}{\partial r} f(r) = m r^{m-1}, \quad \frac{\partial^2}{\partial r^2} f(r) = m(m-1) r^{m-2}$$

$$\frac{1}{r}\frac{\partial}{\partial r} f(r) = m r^{m-2}, \quad \frac{n^2}{r^2} f(r) = n^2 r^{m-2}$$

$$\therefore \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) r^m = \left(m(m-1) + m - n^2\right) r^{m-2}$$

$$= (m^2 - n^2) r^{m-2}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) r^{m-2} = \left[(m-2)^2 - n^2\right] r^{m-4}$$

$$\therefore \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) r^m = \left[(m-2)^2 - n^2\right](m^2 - n^2) r^{m-4}$$

In general, there are 4 independent solutions:

$m = n, \ -n, \ 2+n, \ 2-n.$

This leads to a general solution to $\nabla^4 \phi(r,\theta) = 0$

$\phi(r,\theta) = f_n(r) e^{in\theta}$, where $f_n(r) = A_{n1} r^{n+2} + A_{n2} r^{-n+2} + A_{n3} r^n + A_{n4} r^{-n}$

This is called <u>Michell solution</u>, where $A_{n1}, A_{n2}, A_{n3}, A_{n4}$ are free coefficients to be determined by the <u>boundary conditions</u>

* When $n=0$ or 1, some of the solutions degenerate

$n=0$	$m=$	n,	$-n$,	$2+n$,	$2-n$	
		0,	0,	2,	2	degenerate
$n=1$,	$m=$	1,	-1,	3,	1	degenerate

In this case, we need to find additional solutions to $\nabla^4\phi = 0$ to form a trial solution that is flexible enough to satisfy all boundary conditions. The Michel solution for $n=0, 1$ is

$$f_0(r) = A_{01} r^2 + \underline{A_{02} r^2 \ln r} + \underline{A_{03} \ln r} + A_{04}$$

new solutions

$$f_1(r) = A_{11} r^3 + \underline{A_{12} r \ln r} + A_{13} r + A_{14} r^{-1}$$

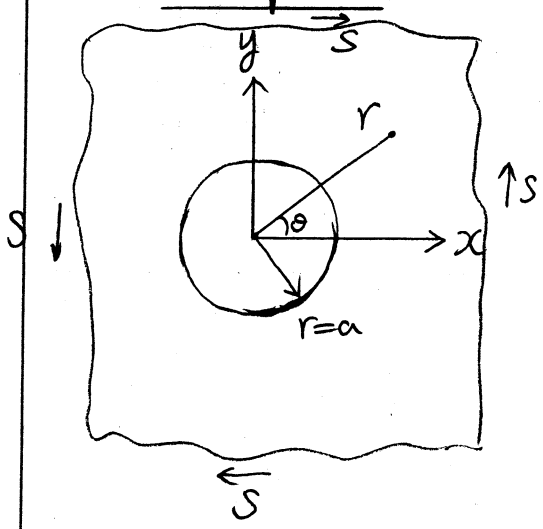
We will discuss the procedure to generate additional solutions when two solutions degenerate in the near future.

As a summary, we may compare a general solution of the biharmonic equation in polar and cartesian coordinates.

$$\nabla^4 \phi = 0$$

$\phi(r,\theta) = r^m e^{in\theta}$	$\phi(x,y) = e^{ikx}(A+By)e^{\pm ky}$
$m = n, -n, 2+n, 2-n$	
oscillation in θ determines the order of polynomial in r	oscillation in x determines the speed of exponential decay in y.

§6. Example 1.



A circular hole in a shear field

Boundary conditions:

$$\begin{cases} \sigma_{xx} = \sigma_{yy} \to 0, & r \to \infty \\ \sigma_{xy} \to S, & r \to \infty \end{cases}$$

$$\begin{cases} \sigma_{r\theta} = 0, & r = a \\ \sigma_{rr} = 0, & r = a \end{cases}$$

Let's first imagine the case where the hole does not exist. Obviously, in this case, we should have uniform stress field $\sigma_{xy} = S$ everywhere as the solution.

Let's express $\sigma_{xy} = S$ in polar coordinates

$$\sigma_{xy} = S \Leftrightarrow \phi(x,y) = -Sxy \qquad \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$$

$$\Leftrightarrow \phi(r,\theta) = -Sr^2 \sin\theta\cos\theta$$

$$= -\frac{1}{2} S r^2 \sin 2\theta$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = S\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -S\sin 2\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = S\cos 2\theta$$

* We would get the same answer from the Mohr's circle.

Of course, this solution would not satisfy the B.C. at $r=a$ when the hole is present.

In this case, we will let the final solution be

$$\phi(r,\theta) = \phi^{(0)}(r,\theta) + \phi^{(1)}(r,\theta)$$

where $\phi^{(0)}(r,\theta) = -\frac{1}{2} S r^2 \sin 2\theta$ — the solution when hole is not present

and $\phi^{(1)}(r,\theta)$ is the <u>change</u> of stress function when the hole is created.

Since $\phi^{(0)}$ satisfies the Boundary Condition at $r \to \infty$, we require the stress field of $\phi^{(1)}$ to go to zero at $r \to \infty$.

Trial solution:

Since $\phi^{(0)}$ contains $\sin 2\theta$, which is part of $e^{in\theta}$, $n=2$,

Let's try $\phi^{(1)} = f_2(r)\sin 2\theta$ where

$$f_2(r) = A_{21} r^4 + A_{22} + A_{23} r^2 + A_{24} r^{-2}$$

— Michel solution for $n=2$.

The stress field for $\phi^{(1)}(r,\theta)$ is

$$\sigma_{\theta\theta}^{(1)} = \frac{\partial^2 \phi^{(1)}}{\partial r^2} = \left(12 A_{21} r^2 + 2 A_{23} + 6 A_{24} r^{-4}\right) \sin 2\theta$$

Since we expect $\sigma_{\theta\theta}^{(1)} \to 0$ as $r \to \infty$,

$$A_{21} = A_{23} = 0$$

Hence we can rewrite $f_2(r)$ as

$$f_2(r) = A + B r^{-2}$$

$$\phi(r,\theta) = \phi^{(0)} + \phi^{(1)} = -\frac{1}{2} S r^2 \sin 2\theta + A \sin 2\theta + \frac{B}{r^2} \sin 2\theta$$

$$\begin{cases} \sigma_{rr} = \left(S - \frac{4A}{r^2} - \frac{6B}{r^4}\right) \sin 2\theta \\ \sigma_{r\theta} = \left(S + \frac{2A}{r^2} + \frac{6B}{r^4}\right) \cos 2\theta \\ \sigma_{\theta\theta} = \left(-S + \frac{6B}{r^4}\right) \sin 2\theta \end{cases}$$

B.C. $\sigma_{rr} = \sigma_{r\theta} = 0$ at $r = a$.

$$\begin{cases} S - \frac{4A}{a^2} - \frac{6B}{a^4} = 0 \\ S + \frac{2A}{a^2} + \frac{6B}{a^4} = 0 \end{cases} \Rightarrow \begin{cases} A = S a^2 \\ B = -\frac{1}{2} S a^4 \end{cases}$$

$\therefore$ Final Solution:

$$\phi(r,\theta) = \left(-\frac{1}{2} S r^2 + S a^2 - \frac{1}{2} S \frac{a^4}{r^2}\right) \sin 2\theta$$

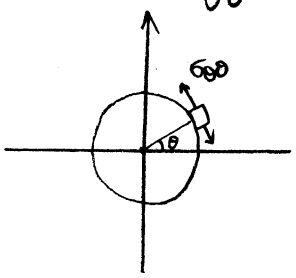
$$\begin{cases} \sigma_{rr} = S\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right) \sin 2\theta \\ \sigma_{r\theta} = S\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right) \cos 2\theta \\ \sigma_{\theta\theta} = S\left(-1 - \frac{3a^4}{r^4}\right) \sin 2\theta \end{cases}$$

On the surface of the hole ($r=a$)

$\sigma_{rr}=0, \quad \sigma_{r\theta}=0$

The only non-zero stress component is

$\sigma_{\theta\theta} = -4S\sin 2\theta \qquad (r=a)$



$$\max_{0\le\theta\le 2\pi} |\sigma_{\theta\theta}| = 4S$$

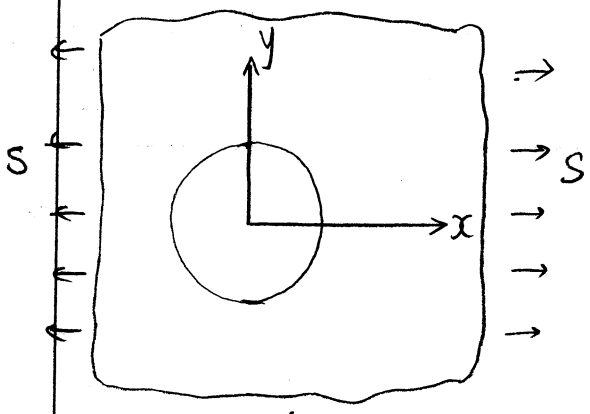
maximum shear stress $= 2S$

Recall the applied shear stress (at $r\to\infty$) is S.

Hence the $\boxed{\text{stress concentration factor} = 2.}$

§7. Example 2.

A circular hole in tensile field (Barber. P108)



B.C.

$$\begin{cases} \sigma_{xx}=S, \quad \sigma_{xy}=\sigma_{yy}=0 & r\to\infty \\ \sigma_{r\theta}=\sigma_{rr}=0 & r=a \end{cases}$$

without hole $\phi^{(0)} = \frac{1}{2}Sy^2 = \frac{1}{2}Sr^2\sin^2\theta$

$= \frac{1}{4}Sr^2(1-\cos 2\theta)$

with hole: $\phi = \phi^{(0)} + \phi^{(1)}$

we shall select terms containing both $n=0$ and $n=2$ for $\phi^{(1)}$.
at the same time, their stress field should $\to 0$ when $r\to\infty$

$$\phi^{(1)} = A\ln r + B\theta + C\cos 2\theta + \frac{D}{r^2}\cos 2\theta$$

B.C. $\Rightarrow$ $A = -\frac{Sa^2}{2}, \quad B=0, \quad C=\frac{Sa^2}{2}, \quad D=-\frac{Sa^4}{4}$

$$\begin{cases} \sigma_{rr} = \frac{S}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{S\cos 2\theta}{2}\left(\frac{3a^4}{r^4} - \frac{4a^2}{r^2} + 1\right) \\ \sigma_{r\theta} = \frac{S\sin 2\theta}{2}\left(\frac{3a^4}{r^4} - \frac{2a^2}{r^2} - 1\right) \\ \sigma_{\theta\theta} = \frac{S}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{S\cos 2\theta}{2}\left(\frac{3a^4}{r^4}+1\right) \end{cases}$$

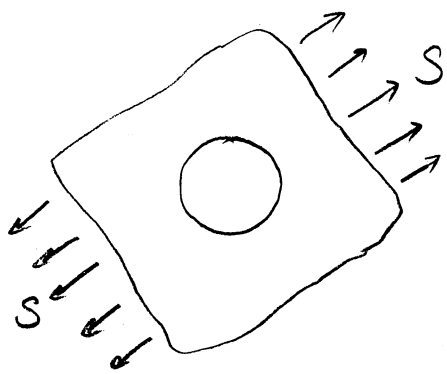
$$\max_{0\le\theta\le 2\pi} \sigma_{\theta\theta} = 3S \qquad (r=a)$$

at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore$ Stress concentration factor $= 3$

§8. Example 3

rotate the solution in Example 2 by $\frac{\pi}{4}$, we get



Solution (a)

$\theta \to \theta - \frac{\pi}{4}$

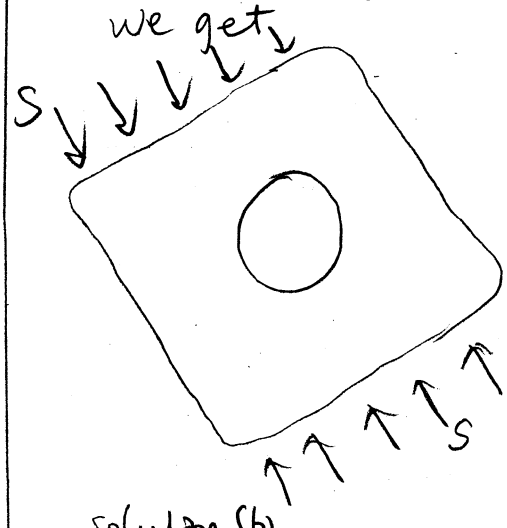
$2\theta \to \theta - \frac{\pi}{2}$

$\cos 2\theta \to \sin 2\theta$

$\sin 2\theta \to -\cos 2\theta$

$$\begin{cases} \sigma_{rr} = \frac{S}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{S\sin 2\theta}{2}\left(\frac{3a^4}{r^4} - \frac{4a^2}{r^2} + 1\right) \\ \sigma_{r\theta} = -\frac{S\cos 2\theta}{2}\left(\frac{3a^4}{r^4} - \frac{2a^2}{r^2} - 1\right) \\ \sigma_{\theta\theta} = \frac{S}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{S\sin 2\theta}{2}\left(\frac{3a^4}{r^4}+1\right) \end{cases}$$

rotate the solution in Example 2 by $-\pi/4$, and reverse sign, we get



Solution (b)

$S \to -S$

$\theta \to \theta + \frac{\pi}{4}$

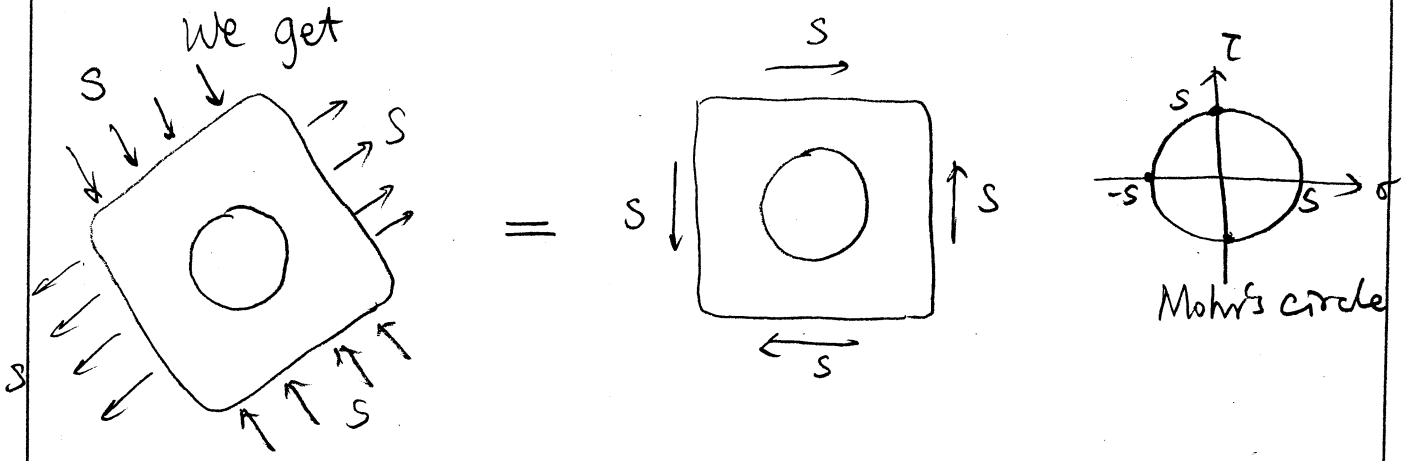
$2\theta \to 2\theta + \frac{\pi}{2}$

$\cos 2\theta \to -\sin 2\theta$

$\sin 2\theta \to \cos 2\theta$

$$\begin{cases} \sigma_{rr} = -\frac{S}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{S\sin 2\theta}{2}\left(\frac{3a^4}{r^4} - \frac{4a^2}{r^2} + 1\right) \\ \sigma_{r\theta} = -\frac{S}{2}\cos 2\theta\left(\frac{3a^4}{r^4} - \frac{2a^2}{r^2} - 1\right) \\ \sigma_{\theta\theta} = -\frac{S}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{S\sin 2\theta}{2}\left(\frac{3a^4}{r^4}+1\right) \end{cases}$$

Add solution (a) and solution (b) together, we get



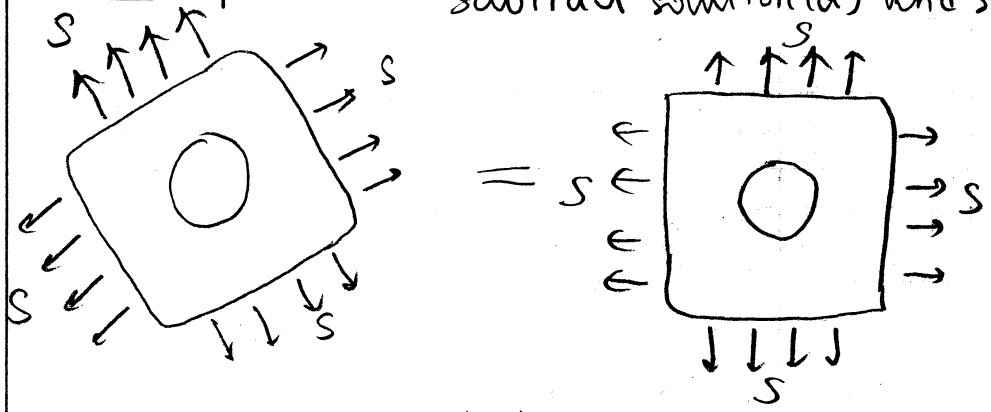
Mohr's circle

$$\begin{cases} \sigma_{rr} = S\sin 2\theta\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right) \\ \sigma_{r\theta} = S\cos 2\theta\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right) \\ \sigma_{\theta\theta} = -S\sin 2\theta\left(1 + \frac{3a^4}{r^4}\right) \end{cases}$$

consistent with example 1, P. 10.

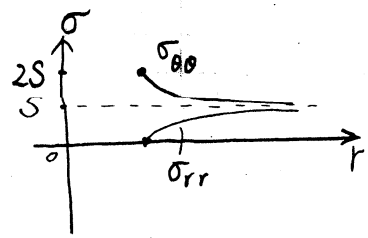
§9. Example 4

Subtract solution (a) and solution (b), we get



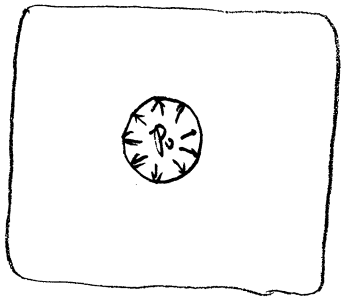
only the θ-independent terms remain

$$\begin{cases} \sigma_{rr} = S\left(1 - \frac{a^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = S\left(1 + \frac{a^2}{r^2}\right) \end{cases}$$



— circular hole in bi-axial tension

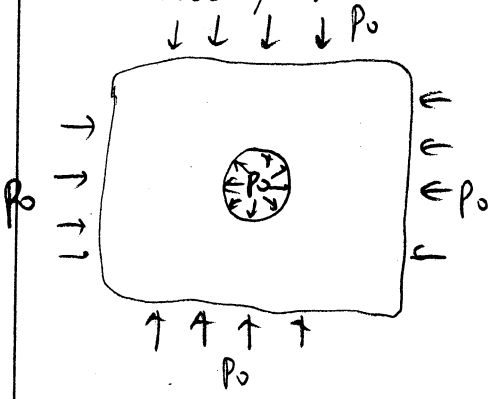
§10. Example 5



pressurized hole in an otherwise infinite medium

B.C. $\begin{cases} \sigma_{rr} = p_0, \quad \sigma_{r\theta} = 0, & r = a \\ \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0, & r \to \infty \end{cases}$

This problem can be thought of as a superposition of two problems



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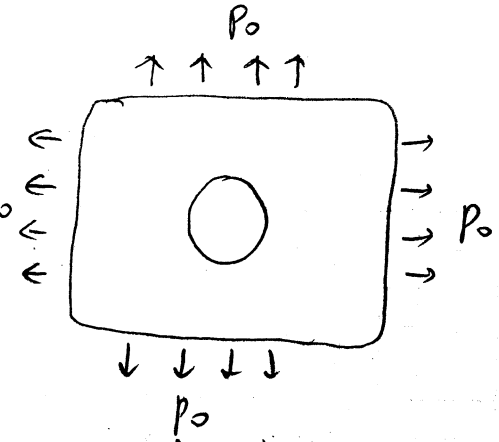


uniform stress field

$\sigma_{xx} = \sigma_{yy} = -p_0, \quad \sigma_{xy} = 0$

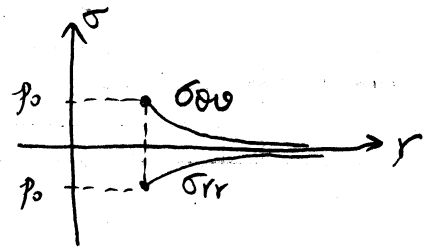
$\sigma_{rr} = \sigma_{\theta\theta} = -p_0, \quad \sigma_{r\theta} = 0$

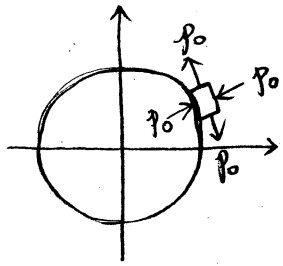
circular hole in biaxial loading

$\begin{cases} \sigma_{rr} = p_0 \left(1 - \frac{a^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = p_0 \left(1 + \frac{a^2}{r^2}\right) \end{cases}$

Final solution:-

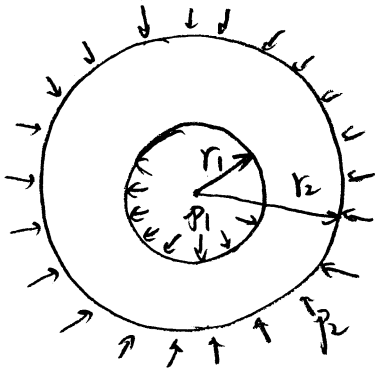
$\begin{cases} \sigma_{rr} = -p_0 \frac{a^2}{r^2} \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = p_0 \frac{a^2}{r^2} \end{cases}$





Q: what is the magnitude and orientation of the maximum shear stress?

§11. Example 6. Thick-walled pressure vessel

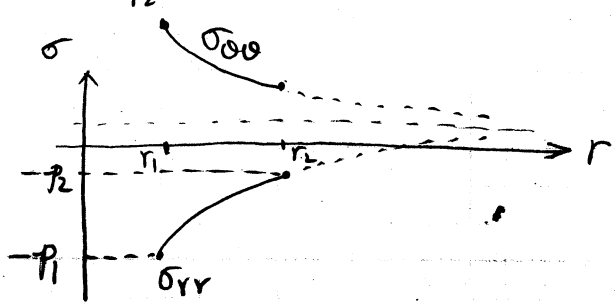


B.C. $\begin{cases} \sigma_{rr} = -p_1 & \sigma_{r\theta}=0, \quad r=r_1 \\ \sigma_{rr} = -p_2 & \sigma_{r\theta}=0, \quad r=r_2 \end{cases}$

This problem can be easily solved by superimposing the two solutions in Example 5 after multiplying them with different constants.

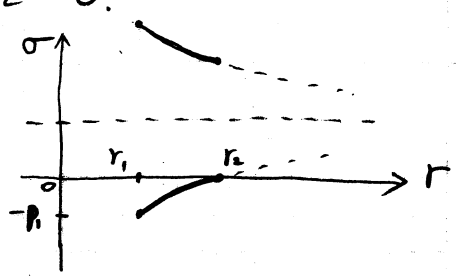
$$\begin{cases} \sigma_{rr} = A - \dfrac{B}{r^2} \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = A + \dfrac{B}{r^2} \end{cases} \longrightarrow \text{B.C.} \begin{cases} A - \dfrac{B}{r_1^2} = -p_1 \\ A - \dfrac{B}{r_2^2} = -p_2 \end{cases}$$

$$\begin{cases} A = \dfrac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} \\ B = \dfrac{p_2 - p_1}{\frac{1}{r_1^2} - \frac{1}{r_2^2}} \end{cases}$$



consider the special case of $p_2 = 0$.

$$\begin{cases} \sigma_{rr} = \dfrac{p_1 r_1^2}{r_2^2 - r_1^2}\left(1 - \dfrac{r_2^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \\ \sigma_{\theta\theta} = \dfrac{p_1 r_1^2}{r_2^2 - r_1^2}\left(1 + \dfrac{r_2^2}{r^2}\right) \end{cases}$$



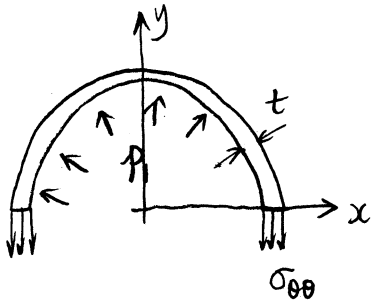
max($\sigma_{\theta\theta}$) occur at $r = r_1$ (inner wall)

$$= p_1 \frac{r_1^2 + r_2^2}{r_2^2 - r_1^2}$$

In the thin wall limit $r_2 - r_1 = t \ll r_1$, $r_2^2 \approx r_1^2 + 2r_1 t$

$$\max(\sigma_{\theta\theta}) \approx p_1 \cdot \frac{r_1}{t}$$

compare this solution with thin-wall approximation in mechanics of materials.



Force balance in y-direction

$$\sigma_{\theta\theta} \cdot 2t = p_1 \cdot 2r_1$$

$$\sigma_{\theta\theta} = p_1 \cdot \frac{r_1}{t}$$

agree with the exact solution in the limit of $t \ll r_1$.

Now that we have the exact solution from Elasticity theory, we can see by how much the mechanics of material theory underestimate the maximum stress for thick-wall pressure vessels.

How to generate new solutions to the biharmonic equation $\nabla^4\phi=0$, when two solutions become <u>degenerate</u>?

In polar coordinates $\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi$

a general solution is $\phi(r,\theta)=r^m e^{in\theta}$ Table 8.1 (P107) 9.1 (P118)

where $m=n,\ -n,\ 2+n,\ 2-n.$

barbertable.m on Courseworx

*typo in Table 8.1, P107, Barber.

In general, there are 4 independent solutions for each n.

But when $n=0$ or $n=2$, there are less than 4 independent solutions of the form $r^m e^{in\theta}$.

In this case, we need a procedure to generate more solutions to the equation $\nabla^4\phi=0$, to make sure that our trial solution is flexible enough to satisfy all boundary conditions.

In the following, let us first discuss what people do in similar situations when solving simpler equations.

§1. Homogeneous Linear Equations with Constant Coefficients.

— from "Fundamentals of Differential Equations and Boundary Value Problems", Nagle & Saff. 2nd ed. (1996). P. 329.

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0$$

where $y^{(n)}(x) \equiv \frac{d^n}{dx^n} y(x)$

trial solution $y(x) = e^{\lambda x} \rightarrow y^{(n)}(x) = \lambda^n e^{\lambda x}$

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$ — $\boxed{\text{auxiliary equation}}$

Suppose this equation has n different roots.

$\lambda_1, \lambda_2, \cdots, \lambda_n$.

then a general solution to the original equation is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}$$

where $\{C_1, C_2, \cdots C_n\}$ will be determined by boundary conditions.

<u>Example 1</u> $\quad y''' - 2y'' - 5y' + 6y = 0$

$$\downarrow$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-1)(\lambda+2)(\lambda-3) = 0$$

general solution: $y(x) = C_1 e^{x} + C_2 e^{-2x} + C_3 e^{3x}$

Sometimes the auxiliary equation can have repeated roots.
i.e. we will have less than n distinct values for λ_i.

Suppose λ_1 is a root of the auxiliary equation of multiplicity m. then

$$y(x) = e^{\lambda_1 x},\ x e^{\lambda_1 x},\ x^2 e^{\lambda_1 x},\ \cdots\ x^{m-1} e^{\lambda_1 x}$$

are m independent solution to the original differential equation

Notice that these solutions can be written as

$$y(x) = e^{\lambda_1 x},\ \frac{d}{d\lambda_1} e^{\lambda_1 x},\ \frac{d^2}{d\lambda_1^2}(e^{\lambda_1 x}),\ \cdots\ \frac{d^{m-1}}{d\lambda_1^{m-1}}(e^{\lambda_1 x})$$

i.e. the new solutions are obtained by taking derivatives with respect to λ_1.

<u>Example 2.</u> $y'''' - y''' - 3y'' - 5y' - 2y = 0$

$$\downarrow$$

$$\lambda^4 - \lambda^3 - 3\lambda^2 - 5\lambda - 2 = 0$$

$$(\lambda - 1)^3 (\lambda + 2) = 0$$

root $\lambda = 1$ has multiplicity 3

General solution: $y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 e^{-2x}$

§2. Cauchy-Euler Equations

(or equidimensional equations)

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_0 y(x) = 0$$

(* more similar our biharmonic equation in polar coordinates. $\phi = f(r) e^{in\theta}$.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) f(r) = 0 \quad . \ )$$

variable transformation: $x = e^t$ $\quad (t = \ln x)$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = x\frac{dy}{dx} \quad \rightarrow \quad x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dy}{dt} + x^2\frac{d^2y}{dx^2} \rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{dy}{dt} + x^2\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dt^2} + 2x\frac{dx}{dt}\frac{d^2y}{dx^2} + x^2\frac{d^3y}{dx^3}\frac{dx}{dt}$$

$$= \frac{d^2y}{dt^2} + 2x^2\frac{d^2y}{dx^2} + x^3\frac{d^3y}{dx^3}$$

$$= \frac{d^2y}{dt^2} - 2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + x^3\frac{d^3y}{dx^3}$$

$$= -\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + x^3\frac{d^3y}{dx^3}$$

$$\rightarrow x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} - 2\frac{dy}{dt}$$

Hence the Cauchy-Euler equation can always be transformed into the following form.

$$b_n y^{(n)}(t) + b_{n-1} y^{(n-1)}(t) + \cdots + b_1 y'(t) + b_0 y(t) = 0$$

for function $y(t)$. $\quad y^{(n)}(t) = \frac{d^n}{dt^n} y(t)$.

In general, the trial solution is $y(t) = e^{\lambda t} \; (= x^{\lambda})$

This leads to the auxiliary solution for λ:

$$b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots b_1 \lambda + b_0 = 0$$

Suppose there are n distinct roots: $\lambda_1 \cdots \lambda_n$.

then the general solution is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}$$

i.e. $y(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2} + \cdots + c_n x^{\lambda_n}$

Suppose λ_1 is the root of the auxiliary equation with multiplicity m. then

$$y(t) = e^{\lambda_1 t}, \quad t e^{\lambda_1 t}, \quad t^2 e^{\lambda_1 t}, \quad \cdots, t^{m-1} e^{\lambda_1 t}$$

are independent solutions to the differential equation.

i.e. $y(x) = x^{\lambda_1}, \quad \boxed{x^{\lambda_1} \ln x}, \quad x^{\lambda_1} (\ln x)^2, \cdots x^{\lambda_1} (\ln x)^{m-1}$

are independent solutions to the Cauchy-Euler equation.

<u>Example 3</u>

$$3x^2 \frac{d^2y}{dx^2} + 11x \frac{dy}{dx} - 3y = 0$$

$x = e^t$, $\quad \rightarrow \quad 3 \frac{d^2y}{dt^2} + 8 \frac{dy}{dt} - 3y = 0$

$$3\lambda^2 + 8\lambda - 3 = 0, \quad (3\lambda - 1)(\lambda + 3) = 0$$

$\lambda_1 = \frac{1}{3}, \quad \lambda_2 = -3, \quad y(t) = c_1 e^{\frac{1}{3}t} + c_2 e^{-3t}$

$y(x) = c_1 x^{1/3} + c_2 x^{-3}$

Again, the new solutions can be written as derivatives with respect to λ_i; e.g

$$\frac{d}{d\lambda_1}(x^{\lambda_1}) = \frac{d}{d\lambda_1}\left[e^{\lambda_1 \cdot \ln x}\right] = \ln x \cdot e^{\lambda_1 \ln x} = \ln x \cdot x^{\lambda_1}$$

$$\frac{d^2}{d\lambda_1^2}(x^{\lambda_1}) = (\ln x)^2 \cdot x^{\lambda_1}$$

$$\cdots$$

§3. Radial Component of the Biharmonic Equation

$\nabla^4 \phi = 0$. trial solution $\phi(r,\theta) = f(r)\, e^{in\theta}$

$f(r)$ satisfies the following equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) f(r) = 0$$

(* This is equivalent to the Cauchy-Euler equation if we multiply r^4 on both sides)

trial solution: $f(r) = r^\lambda$

auxiliary equation: $(\lambda^2 - n^2)((\lambda-2)^2 - n^2) = 0$

$$\lambda = n,\ -n,\ 2+n,\ 2-n.$$

$n=0$	$\lambda = 0,\ 0,\ 2,\ 2$
$n=1$	$\lambda = \underline{\underline{1}},\ -1,\ 3,\ \underline{\underline{1}}$
$n=2$	$\lambda = 2,\ -2,\ 4,\ 0$

Notice degenerate roots when $n=0$ and $n=1$.

<u>Example 4</u> Consider the case of $n=1$.

$\lambda_1 = 1$ is a root with multiplicity 2.

two independent solutions are

$$r^{\lambda_1}, \quad \frac{d}{d\lambda_1} r^{\lambda_1} = \ln r \cdot r^{\lambda_1}$$

i.e. r and $\underline{\underline{r \ln r}}$

So the general solution for $f(r)$ when $n=1$ is

$$f(r) = c_1 r + \underbrace{\underline{\underline{c_2 r \ln r}}}_{\text{new solution.}} + c_3 r^{-1} + c_4 r^3$$

<u>Example 5</u> Consider the case of $n=0$.

$\lambda_1 = 0$ is a root with multiplicity 2.

two independent solutions are

$$r^{\lambda_1} = 1, \quad \frac{d}{d\lambda_1} r^{\lambda_1} = \ln r$$

$\lambda_2 = 2$ is a root with multiplicity 2

two independent solutions are

$$r^{\lambda_2} = r^2, \quad \frac{d}{d\lambda_2} r^{\lambda_2} = r^2 \ln r$$

So the general solution for $f(r)$ when $n=0$ is

$$f(r) = c_1 + \underline{\underline{c_2 \ln r}} + c_3 r^2 + c_4 r^2 \ln r$$

<u>Summary</u>: When two solutions for $f(r)$ degenerate, we can obtain a new solution by multiplying $\ln r$.

§4. Geometric Interpretation of $\frac{d}{d\lambda_1}(\cdots)$ procedure to generate new solutions.

Consider equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}\right)f(r)=0$$

and imagine $n=1+\varepsilon \qquad (0<\varepsilon\ll 1)$

(* for now let us imagine that n does not have to be an integer)

as long as $\varepsilon>0$, the two solutions

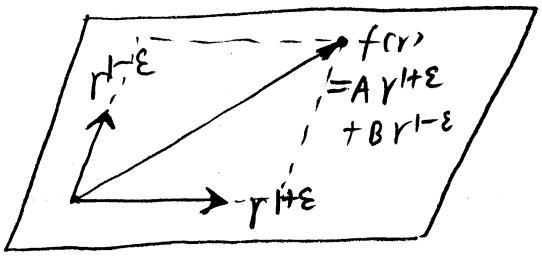
$$r^{2-n}=r^{1-\varepsilon}$$
$$r^{n}=r^{1+\varepsilon}$$

do not degenerate. But these two solutions degenerate when $\varepsilon=0$.

Let $f(r)=A\,r^{1+\varepsilon}+B\,r^{1-\varepsilon}$

be the family of solutions formed by superposition of $r^{1+\varepsilon}$ and $r^{1-\varepsilon}$. with all possible values of A & B.

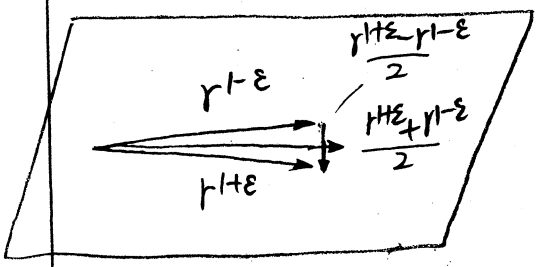
Geometrically, the family of solutions $f(r)$ is equivalent to a plane. $r^{1-\varepsilon}$ and $r^{1+\varepsilon}$ are the two "basis vectors" of the plane.



Equivalently,

$$\frac{r^{1+\varepsilon}+r^{1-\varepsilon}}{2} \quad \text{and} \quad \frac{r^{1+\varepsilon}-r^{1-\varepsilon}}{2}$$

can also be used as "basis vectors".



i.e. any function $f(r)$ in this plane can be represented as

$$f(r)=C_1\,\frac{r^{1+\varepsilon}+r^{1-\varepsilon}}{2}+C_2\,\frac{r^{1+\varepsilon}-r^{1-\varepsilon}}{2}$$

In the limit of $\varepsilon \to 0$ (but $\varepsilon \neq 0$).

$$\frac{r^{1+\varepsilon} + r^{1-\varepsilon}}{2} = r + O(\varepsilon^2)$$

$$\frac{r^{1+\varepsilon} - r^{1-\varepsilon}}{2} = \varepsilon \cdot \frac{d}{d\varepsilon}(r^{1+\varepsilon}) = \varepsilon \cdot r \ln r + O(\varepsilon^3)$$

Therefore, any function $f(r)$ in this plane can be represented as

$$f(r) = C_1 r + C_2 \varepsilon \cdot r \ln r + O(\varepsilon^2)$$

Let $D = C_2 \varepsilon$

Then any function $f(r)$ in this plane can be represented as

$$f(r) \approx C r + D r \ln r$$

When $\varepsilon = 0$,

$f(r) = C \cdot r + D r \ln r$ is the solution of the original partial differential equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) f(r) = 0,$$

<u>Summary</u>: In the limit of $\varepsilon \to 0$. (but $\varepsilon \neq 0$)

the plane spanned by all functions $f(r) = A r^{1+\varepsilon} + B r^{1-\varepsilon}$ can be well approximated by the plane spanned by all functions $f(r) = C r + D r \ln r$

This approximation becomes better as ε becomes smaller.

Hence, it is reasonable to expect that $f(r) = C r + D r \ln r$ is a solution of the original differential equation when $n = 1$ ($\varepsilon = 0$).

§5. Generalization to $\phi(r,\theta)$

We have seen that for $n=0$, a general solution for $f(r)$ is

$$f(r) = C_1 + C_2 \ln r + C_3 r^2 + C_4 r^2 \ln r \qquad (p.6)$$

This corresponds to a general solution to $\nabla^4\phi = 0$:

$$\phi(r,\theta) = f(r)\, e^{in\theta} = f(r) \qquad \text{(i.e. independent of } \theta \text{ because } n=0\text{)}$$

Notice the term C_1 is a constant, which will not lead to any stress field — it is a trivial solution.

This indicates that we need to find a new solution to replace the constant term C_1.

Notice that $e^{in\theta} = \cos n\theta + i \sin n\theta$

this means that both

$$\phi(r,\theta) = f(r) \cos n\theta$$

and

$$\phi(r,\theta) = f(r) \sin n\theta$$

are solutions to the biharmonic equation $\nabla^4\phi = 0$.

But when $n=0$, $f(r)\sin n\theta = 0$, i.e. these solutions vanishes!

When this happens, we need to find new solutions to the equation $\nabla^4\phi = 0$.

To extract the missing solution corresponding to

$e^{i\lambda\theta}$, as $\lambda \to 0$.

We take its derivative with respect to λ.

$$\frac{d}{d\lambda} e^{i\lambda\theta} = i\theta \qquad \left(\text{or } \frac{d}{d\lambda}\sin\lambda\theta\Big|_{\lambda=0} = \theta\right)$$

Removing the constant i, the new solution is θ.

It can be easily verified that

$\phi(r,\theta) = f(r)\cdot\theta$ can be used as a trial solution

to $\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0.$

because $\frac{\partial^2}{\partial\theta^2}\phi(r,\theta) = 0.$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) f(r) = 0.$$

$$f(r) = C_1 + C_2 \ln r + C_3 r^2 + C_4 r^2 \ln r.$$

Hence, both

$$\phi(r,\theta) = C_1 + C_2 \ln r + C_3 r^2 + C_4 r^2 \ln r$$

and

$$\phi(r,\theta) = (C_1 + C_2 \ln r + C_3 r^2 + C_4 r^2 \ln r)\cdot\theta$$

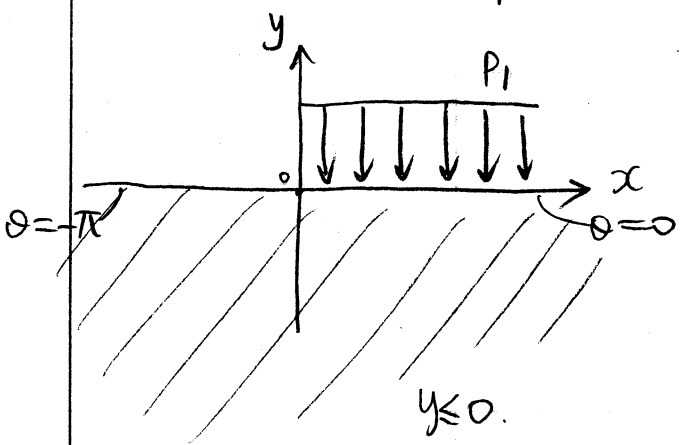
are solutions to the biharmonic equation $\nabla^4\phi = 0$.

The mitchel solution is

$$\phi(r,\theta) = C_1\theta + C_2 \ln r + C_3 r^2 + C_4 r^2 \ln r.$$

which is a "mixture" of the above two solutions.

§6. Half-space under uniform loading in semi-infinite domain

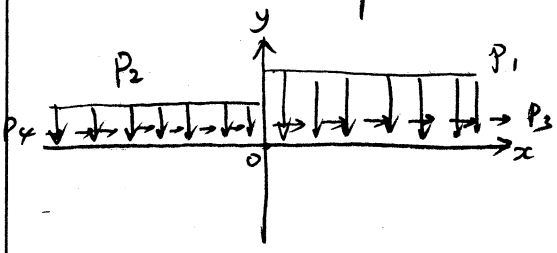


$$p_y(x) = \begin{cases} P_1 & x>0 \\ 0 & x<0. \end{cases}$$

B.C.
$$\begin{cases} \sigma_{\theta\theta} = -P_1 & \theta=0 \\ \sigma_{r\theta} = 0 & \theta=0 \end{cases}$$

$$\begin{cases} \sigma_{\theta\theta} = 0 & \theta=-\pi \\ \sigma_{r\theta} = 0 & \theta=-\pi \end{cases}$$

This is a special case of a more general problem



B.C.
$$\begin{cases} \sigma_{\theta\theta} = -P_1 & \theta=0 \\ \sigma_{r\theta} = P_3 & \theta=0 \end{cases}$$

$$\begin{cases} \sigma_{\theta\theta} = -P_2 & \theta=-\pi \\ \sigma_{r\theta} = P_4 & \theta=-\pi. \end{cases}$$

Since P_1, P_2, P_3, P_4 are arbitrary constants we need to find a general solution $\phi(r,\theta)$ with 4 independent coefficients.

Look for all solutions to $\nabla^4\phi=0$ whose stress field is independent of r (see B.C. has no r-dependence)

Mitchel solution:

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2\cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^2\sin 2\theta$	$-2\sin 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$

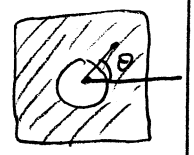
in Barber's table 8.1 P.107

missing one solution?

Solution of the type $\phi(r,\theta) = f(r)\cdot\theta$

	ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
*	$r^2\theta$	2θ	-1	2θ
	$r^2\ln r\cdot\theta$	$(2\ln r+1)\theta$	$-\ln r-1$	$(2\ln r+3)\theta$
	$\ln r\ \theta$	θ/r^2	$(\ln r-1)/r^2$	$-\theta/r^2$

← stress independent of r



$\phi(r,\theta) = r^2\theta$ is the desired solution.
it is not allowed if θ can go from 0 to 2π continuously, in the elastic medium.
But here it is allowed because for the elastic half space θ is limited to $[-\pi, 0]$.

General Solution:

$$\phi(r,\theta) = r^2(A_1\cos 2\theta + A_2 + A_3\sin 2\theta + A_4\theta)$$

$$\sigma_{rr} = (-2\cos 2\theta)A_1 + 2A_2 + (-2\sin 2\theta)A_3 + (2\theta)A_4$$

B.C. $\begin{cases} \sigma_{r\theta} = (2\sin 2\theta)A_1 + 0\cdot A_2 + (-2\cos 2\theta)A_3 - A_4 \\ \sigma_{\theta\theta} = (2\cos 2\theta)A_1 + 2A_2 + (2\sin 2\theta)A_3 + (2\theta)A_4 \end{cases}$

$\theta = 0$. $\begin{cases} \sigma_{r\theta} = -2A_3 - A_4 = P_3 \\ \sigma_{\theta\theta} = 2A_1 + 2A_2 = -P_1 \end{cases}$ $\qquad$ $\theta = -\pi$ $\begin{cases} \sigma_{r\theta} = -2A_3 - A_4 = P_4 \\ \sigma_{\theta\theta} = 2A_1 + 2A_2 - 2\pi A_4 = -P_2 \end{cases}$

<u>Example 6.</u> $\quad P_2 = P_3 = P_4 = 0.$

$$A_1 = -\frac{P_1}{4}, \quad A_2 = -\frac{P_1}{4}, \quad A_3 = \frac{P_1}{4\pi}, \quad A_4 = -\frac{P_1}{2\pi}$$

Q: Notice that the above solution cannot satisfy the B.C. when $P_3 \neq P_4$. What shall we do?

To generate missing solution, try taking $\frac{d}{d\lambda}$ from an existing solution.

Consider solution

$$\phi(r,\theta) = r^{\lambda} e^{i\lambda\theta} = e^{\lambda(\ln r + i\theta)}$$

$$\frac{d}{d\lambda}\phi(r,\theta) = (\ln r + i\theta)\, e^{\lambda(\ln r + i\theta)}$$

$$= (\ln r + i\theta)(r^{\lambda} e^{i\lambda\theta})$$

$$= (\ln r + i\theta)(r^{\lambda}\cos\lambda\theta + i\, r^{\lambda}\sin\lambda\theta)$$

when $\lambda = 2$,

$$\phi(r,\theta) = r^2(\cos 2\theta + i\sin 2\theta)$$

$$\frac{d}{d\lambda}\phi(r,\theta) = (\ln r + i\theta)\cdot r^2\cdot(\cos 2\theta + i\sin 2\theta)$$

$$= r^2\left[(\cos 2\theta \ln r - \theta\sin 2\theta) + i(\sin 2\theta \ln r + \theta\cos 2\theta)\right]$$

It can be verified that

$$\phi = r^2(\cos 2\theta \ln r - \theta\sin 2\theta)$$

and

$$\phi = r^2(\sin 2\theta \ln r + \theta\cos 2\theta)$$

are solutions of $\nabla^4\phi = 0$.

barbertable_ext.m on courseworK

Homework: use these solutions to solve the case of

$P_1 = P_2 = P_4 = 0, \quad P_3 \neq 0,$ (P. 11)

ss components do not

up the deficit due to the
ew stress functions

(8.57)

in θ

(8.58)

solution of the elasticity
ıts form a Fourier series

$_3 r\theta \sin\theta$
$_3 r\theta \cos\theta$

$r^{-n}) \cos(n\theta)$

$\sin(n\theta)$ (8.59)

components are easily
enience, we give them
ı few components from
terms in the Michell
'Michell', from which

ch term in the Fourier
olid, there is no inner
ıat give stresses which

degeneracy discussed
ar, some of the stress
ent fields and cannot
n requirements place

with application to the

Table 8.1: The Michell solution — stress components

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2\ln(r)$	$2\ln(r)+1$	0	$2\ln(r)+3$
$\ln(r)$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^3\cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln(r)\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln(r)\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$	$(n+1)(n+2)r^n\cos n\theta$
$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n}\cos n\theta$	$-n(n-1)r^{-n}\sin n\theta$	$(n-1)(n-2)r^{-n}\cos n\theta$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2}\cos n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$	$(n+1)(n+2)r^n\sin n\theta$
$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n}\sin n\theta$	$n(n-1)r^{-n}\cos n\theta$	$(n-1)(n-2)r^{-n}\sin n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$	$n(n+1)r^{-n-2}\sin n\theta$

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Table 9.1: The Michell solution — displacement components

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$r^2\ln(r)$	$(\kappa-1)r\ln(r)-r$	$(\kappa+1)r\theta$
$\ln(r)$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$\frac{1}{2}\{(\kappa-1)\theta\sin\theta-\cos\theta$ $+(\kappa+1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta-\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$
$r\ln(r)\cos\theta$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta-\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\cos\theta-\sin\theta$ $-(\kappa-1)\ln(r)\sin\theta\}$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$r\theta\cos\theta$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta+\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{-(\kappa-1)\theta\sin\theta-\cos\theta$ $-(\kappa+1)\ln(r)\cos\theta\}$
$r\ln(r)\sin\theta$	$\frac{1}{2}\{-(\kappa+1)\theta\cos\theta-\sin\theta$ $+(\kappa-1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta+\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^{-n+2}\cos n\theta$	$(\kappa+n-1)r^{-n+1}\cos n\theta$	$-(\kappa-n+1)r^{-n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{-n}\cos n\theta$	$nr^{-n-1}\cos n\theta$	$nr^{-n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^{-n+2}\sin n\theta$	$(\kappa+n-1)r^{-n+1}\sin n\theta$	$(\kappa-n+1)r^{-n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{-n}\sin n\theta$	$nr^{-n-1}\sin n\theta$	$-nr^{-n-1}\cos n\theta$

For plane strain

$$\kappa = 3 - 4\nu$$

whilst for plane stress

$$\kappa = \left(\frac{3-\nu}{1+\nu}\right)$$

Polar Coordinates II

```
% double check Table 8.1 (p.107) and Table 9.1 (p.118) in Barber
% t stands for theta, k stands for kappa (Kolosov's constant)
% Note: original Table 8.1 line 10 has a typo, should be -2*sin(t)/r
%
% ME340 Elasticity, Stanford University
% Wei Cai, caiwei@stanford.edu


syms r t n k mu


%Table 8.1
tab81 = [
% phi          srr          srt          stt
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 r^2           2            0            2
 r^2*log(r)    2*log(r)+1   0            2*log(r)+3
 log(r)        1/r^2        0            -1/r^2
 t             0            1/r^2        0
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 r^3*cos(t)        2*r*cos(t)      2*r*sin(t)      6*r*cos(t)
 r*t*sin(t)        2*cos(t)/r      0               0
 r*log(r)*cos(t)   cos(t)/r        sin(t)/r        cos(t)/r
 cos(t)/r          -2*cos(t)/r^3   -2*sin(t)/r^3   2*cos(t)/r^3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 r^3*sin(t)        2*r*sin(t)      -2*r*cos(t)     6*r*sin(t)
 r*t*cos(t)        -2*sin(t)/r      0               0
 r*log(r)*sin(t)   sin(t)/r        -cos(t)/r       sin(t)/r
 sin(t)/r          -2*sin(t)/r^3   2*cos(t)/r^3    2*sin(t)/r^3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 r^(n+2)*cos(n*t)  -(n+1)*(n-2)*r^n*cos(n*t)  n*(n+1)*r^n*sin(n*t)     (n+1)*(n+2)*r^n*cos ↙
(n*t)
 r^(2-n)*cos(n*t)  -(n+2)*(n-1)/r^n*cos(n*t) -n*(n-1)/r^n*sin(n*t)     (n-1)*(n-2)/r^n*cos ↙
(n*t)
 r^n*cos(n*t)      -n*(n-1)*r^(n-2)*cos(n*t)  n*(n-1)*r^(n-2)*sin(n*t) n*(n-1)*r^(n-2)*cos ↙
(n*t)
 cos(n*t)/r^n      -n*(n+1)/r^(n+2)*cos(n*t) -n*(n+1)/r^(n+2)*sin(n*t) n*(n+1)/r^(n+2)*cos ↙
(n*t)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 r^(n+2)*sin(n*t)  -(n+1)*(n-2)*r^n*sin(n*t) -n*(n+1)*r^n*cos(n*t)     (n+1)*(n+2)*r^n*sin ↙
(n*t)
 r^(2-n)*sin(n*t)  -(n+2)*(n-1)/r^n*sin(n*t)  n*(n-1)/r^n*cos(n*t)     (n-1)*(n-2)/r^n*sin ↙
(n*t)
 r^n*sin(n*t)      -n*(n-1)*r^(n-2)*sin(n*t) -n*(n-1)*r^(n-2)*cos(n*t) n*(n-1)*r^(n-2)*sin ↙
(n*t)
 sin(n*t)/r^n      -n*(n+1)/r^(n+2)*sin(n*t)  n*(n+1)/r^(n+2)*cos(n*t) n*(n+1)/r^(n+2)*sin ↙
(n*t)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
];

diff81 = tab81;
```

Polar Coordinates II

```
 for i=1:length(tab81(:,1)),
     phi = tab81(i,1);
     srr = simplify( diff(phi,r)/r + diff(phi,t,2)/r^2 );
     srt = simplify(-diff(diff(phi,t)/r,r));
     stt = simplify(diff(phi,r,2));
     diff81(i,2:4)=[simplify(srr-tab81(i,2)),  ...
                    simplify(srt-tab81(i,3)),  ...
                    simplify(stt-tab81(i,4)) ];
 end


 %Table 9.1
 tab91 = [
 % phi          2*mu*ur           2*mu*ut
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  r^2           (k-1)*r           0
  r^2*log(r)    (k-1)*r*log(r)-r  (k+1)*r*t
  log(r)        -1/r              0
  t             0                 -1/r
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  r^3*cos(t)        (k-2)*r^2*cos(t)    (k+2)*r^2*sin(t)
  r*t*sin(t)        1/2*((k-1)*t*sin(t)-cos(t)+(k+1)*log(r)*cos(t))  ...
                    1/2*((k-1)*t*cos(t)-sin(t)-(k+1)*log(r)*sin(t))
  r*log(r)*cos(t)   1/2*((k+1)*t*sin(t)-cos(t)+(k-1)*log(r)*cos(t))  ...
                    1/2*((k+1)*t*cos(t)-sin(t)-(k-1)*log(r)*sin(t))
  cos(t)/r          cos(t)/r^2          sin(t)/r^2
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  r^3*sin(t)        (k-2)*r^2*sin(t)    -(k+2)*r^2*cos(t)
  r*t*cos(t)        1/2*((k-1)*t*cos(t)+sin(t)-(k+1)*log(r)*sin(t))  ...
                    1/2*((1-k)*t*sin(t)-cos(t)-(k+1)*log(r)*cos(t))
  r*log(r)*sin(t)   1/2*(-(1+k)*t*cos(t)-sin(t)+(k-1)*log(r)*sin(t))  ...
                    1/2*((k+1)*t*sin(t)+cos(t)+(k-1)*log(r)*cos(t))
  sin(t)/r          sin(t)/r^2          -cos(t)/r^2
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  r^(n+2)*cos(n*t)  (k-n-1)*r^(n+1)*cos(n*t)   (k+n+1)*r^(n+1)*sin(n*t)
  r^(2-n)*cos(n*t)  (k+n-1)*r^(1-n)*cos(n*t)   (n-k-1)*r^(1-n)*sin(n*t)
  r^n*cos(n*t)      -n*r^(n-1)*cos(n*t)        n*r^(n-1)*sin(n*t)
  cos(n*t)/r^n      n/r^(n+1)*cos(n*t)         n/r^(n+1)*sin(n*t)
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  r^(n+2)*sin(n*t)  (k-n-1)*r^(n+1)*sin(n*t)   -(k+n+1)*r^(n+1)*cos(n*t)
  r^(2-n)*sin(n*t)  (k+n-1)*r^(1-n)*sin(n*t)    (k-n+1)*r^(1-n)*cos(n*t)
  r^n*sin(n*t)      -n*r^(n-1)*sin(n*t)        -n*r^(n-1)*cos(n*t)
  sin(n*t)/r^n       n/r^(n+1)*sin(n*t)        -n/r^(n+1)*cos(n*t)
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 ];

 diff91 = tab81;

 for i=1:length(tab91(:,1)),
     phi = tab91(i,1);
     ur  = tab91(i,2)/(2*mu);
```

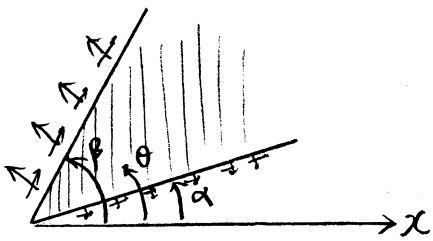
```
    ut  = tab91(i,3)/(2*mu);
    err = diff(ur,r);
    ert = (diff(ur,t)/r + diff(ut,r) - ut/r)/2;
    ett = diff(ut,t)/r + ur/r;
    srr = mu/(k-1)*((k+1)*err + (3-k)*ett);   %invert Eq.(3.19), p.38
    stt = mu/(k-1)*((3-k)*err + (k+1)*ett);
    srt = 2*mu*ert;
%    srr = simplify( diff(phi,r)/r + diff(phi,t,2)/r^2 );
%    srt = simplify(-diff(diff(phi,t)/r,r));
%    stt = simplify(diff(phi,r,2));
    diff91(i,2:4)=[simplify(srr-tab81(i,2)),  ...
                   simplify(srt-tab81(i,3)),  ...
                   simplify(stt-tab81(i,4)) ];
end

disp('Table 8.1 (p.107) of Barber' );
tab81
disp('difference in stress' );
diff81


disp('Table 9.1 (p.118) of Barber' );
tab91
disp('difference in stress' );
diff91
```

We can extend our study of the half-space problem by polar coordinates to wedges of arbitrary angle

§1. Example 1



wedge under uniform traction

σ_{rr}, $\sigma_{r\theta}$ independent of r
(at least on the boundary)

trial solution

$$\phi = r^2 (A_1 \cos 2\theta + A_2 + A_3 \sin 2\theta + A_4 \theta)$$

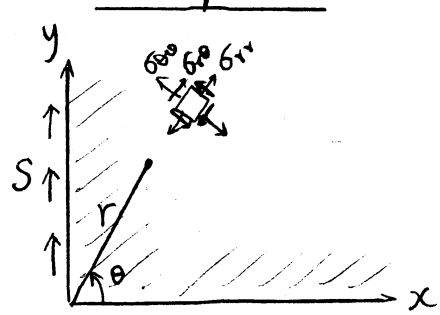
$$\sigma_{rr} = -2A_1 \cos 2\theta + 2A_2 - 2A_3 \sin 2\theta + 2A_4 \theta$$

$$\sigma_{r\theta} = 2A_1 \sin 2\theta + 0 - 2A_3 \cos 2\theta - A_4$$

$$\sigma_{\theta\theta} = 2A_1 \cos 2\theta + 2A_2 + 2A_3 \sin 2\theta + 2A_4 \theta$$

* We know at some point, the solutions
$r^2(\cos 2\theta \ln r - \theta \sin 2\theta)$, $r^2(\sin 2\theta \ln r + \theta \cos 2\theta)$
are needed. But they don't appear here. Why?
When are these solutions needed?
(See Lecture notes "Polar Coordinates II", P12.13.)

§2. Example 2



Uniform shear on a right-angle wedge

$\alpha = 0$, $\beta = \pi/2$

B.C. $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ $\theta = 0$

$\sigma_{r\theta} = S$, $\sigma_{\theta\theta} = 0$, $\theta = \pi/2$

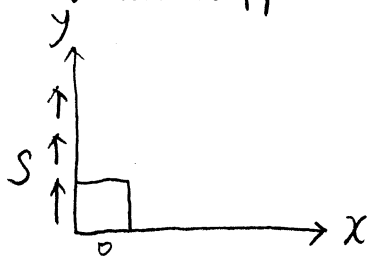
$\theta = 0$: $\sigma_{r\theta} = -2A_3 - A_4 = 0$
$\sigma_{\theta\theta} = 2A_1 + 2A_2 = 0$

$\theta = \frac{\pi}{2}$: $\sigma_{r\theta} = 2A_3 - A_4 = S$
$\sigma_{\theta\theta} = -2A_1 + 2A_2 + \pi A_4 = 0$

$\Rightarrow$ $A_1 = -\frac{\pi S}{8}$, $A_2 = \frac{\pi S}{8}$
$A_3 = \frac{S}{4}$, $A_4 = -\frac{S}{2}$

$$\phi = S\left(-\frac{\pi r^2 \cos 2\theta}{8} + \frac{\pi r^2}{8} + \frac{r^2 \sin 2\theta}{4} - \frac{r^2 \theta}{2}\right)$$

What happens at corner?

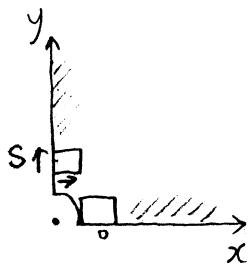


$\sigma_{xy} \neq \sigma_{yx}$?

$$\phi = S\left(-\frac{\pi}{8}(x^2-y^2) + \frac{\pi}{8}(x^2+y^2) + \frac{xy}{2} + \frac{x^2+y^2}{2}\arctan\frac{y}{x}\right)$$

answer:

$$\sigma_{xy} = \sigma_{yx} = -\frac{\partial^2\phi}{\partial x \partial y} = -\frac{Sy^2}{x^2+y^2}$$

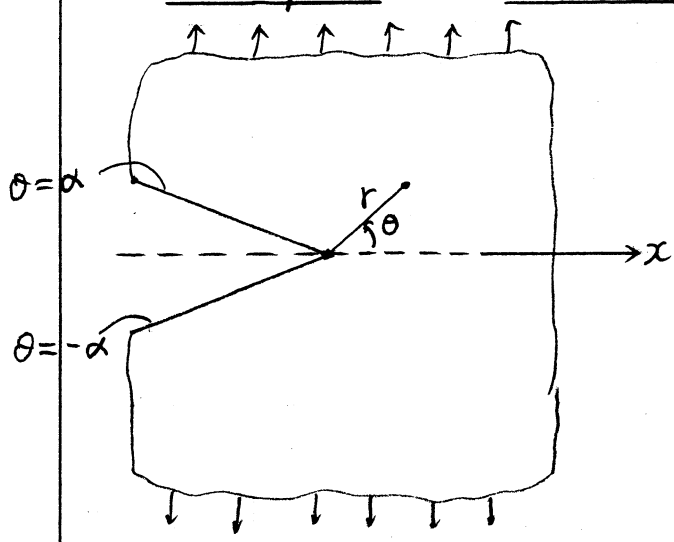


if $y=0,\ x>0,\quad \sigma_{xy}=0$

if $x=0,\ y>0,\quad \sigma_{xy}=-S$

if $x=0,\ y=0,\quad \sigma_{xy}$ undefined.

§3. Example 3 — Notch & Crack : re-entrant corner



intuitively we expect stress field to be singular at the corner. ($r \to 0$)

At the same time, the notch surface ($\theta = \pm\alpha$) is traction-free.

i.e. $\sigma_{r\theta} = \sigma_{\theta\theta} = 0, \quad \theta = \pm\alpha$

We would like to know "how singular" is the stress field at the corner.

→ Look for stress functions that produce singular stress fields.
William's solutions.

$$\phi = r^{n+2}\{A_1\cos(n+2)\theta + A_2\cos n\theta + A_3\sin(n+2)\theta + A_4\sin n\theta\}$$

Let $n = \lambda - 1$. (* λ does NOT have to be an integer!)

$$\phi = r^{\lambda+1}\{A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta\}$$

B.C. $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$, $\theta = \pm\alpha$.

$$\underbrace{\begin{bmatrix} (\lambda+1)\sin(\lambda+1)\alpha & (\lambda-1)\sin(\lambda-1)\alpha \\ (\lambda+1)\cos(\lambda+1)\alpha & (\lambda+1)\cos(\lambda-1)\alpha \end{bmatrix}}_{M_1} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

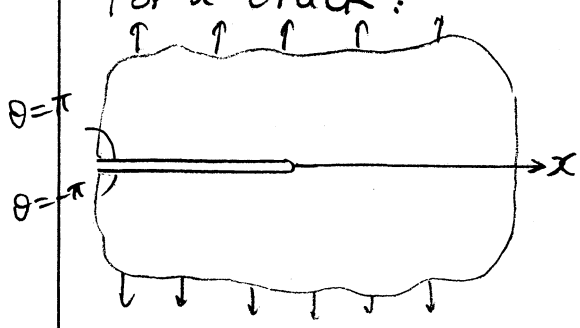
$$\underbrace{\begin{bmatrix} (\lambda+1)\cos(\lambda+1)\alpha & (\lambda-1)\cos(\lambda-1)\alpha \\ (\lambda+1)\sin(\lambda+1)\alpha & (\lambda+1)\sin(\lambda-1)\alpha \end{bmatrix}}_{M_2} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = 0$$

to have a non-trivial solution. (trivial solution is: $A_1 = A_2 = A_3 = A_4 = 0$)

we need:

$$\left.\begin{aligned} \det(M_1) = 0 &\longrightarrow \lambda \sin 2\alpha + \sin 2\lambda\alpha = 0 \\ \text{or} \\ \det(M_2) = 0 &\longrightarrow \lambda \sin 2\alpha - \sin 2\lambda\alpha = 0 \end{aligned}\right\} \lambda \sin 2\alpha \pm \sin 2\lambda\alpha = 0$$

For a crack:



$\alpha = \pi. \rightarrow \sin 2\alpha = 0.$

$\sin 2\pi\lambda = 0$

$\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

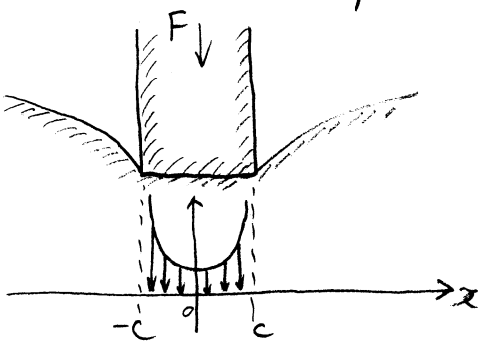
* Notice that $\lambda = 0$ is always a solution to $\lambda \sin 2\alpha \pm \sin 2\lambda\alpha = 0$. It corresponds to $\sigma \sim \frac{1}{r}$ and we will show later on that this singularity is too strong — unphysical.

* $\lambda = 1, \frac{3}{2}, \cdots$ corresponds to non-singular stress fields.

Hence the leading <u>singularity at the crack tip</u> is

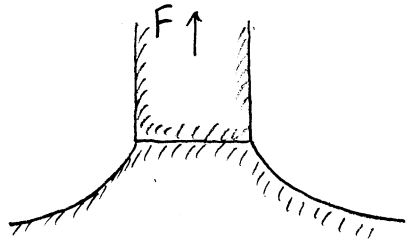
$\phi \sim \sqrt{r}$, $\begin{matrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{matrix} \sim \frac{1}{\sqrt{r}}$ square root singularity.

* §4. Crack tip singularity from flat punch solution



① From lecture note "Contact" P.5 the pressure arising from the flat punch:

$$p_y(x) = \frac{F}{\pi\sqrt{c^2-x^2}} = -\sigma_{yy}(x, y=0)$$

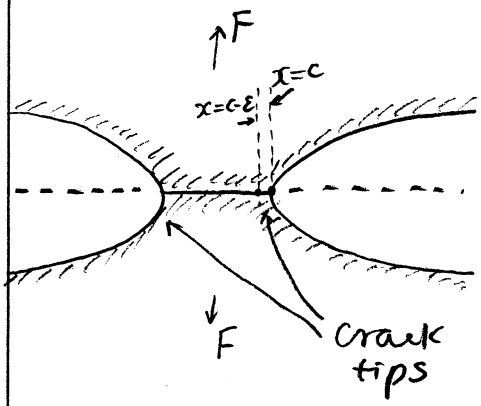


② How is this solution related to a crack?

Imagine we "glue" the flat punch to the half space and pull it.

The stress field will simply change its sign. i.e.

$$\sigma_{yy}(x, y=0) = \frac{F}{\pi\sqrt{c^2-x^2}}$$



③ Next imagine we glue together two half-spaces over the region $-c \le x \le c$, and then pull them apart with force F.

By symmetry, the y-displacement in the region $-c \le x \le c$ must be 0.

So the solution is identical to that in step ②.

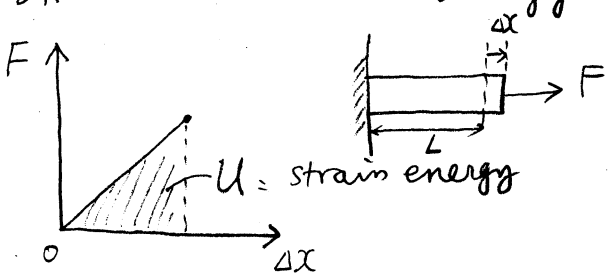
Consider the point $x = c-\varepsilon$, $y=0$. $\varepsilon \ll c$.

$$\sigma_{yy}(x=c-\varepsilon, y=0) = \frac{F}{\pi\sqrt{c^2-(c^2-2c\varepsilon+\varepsilon^2)}} = \frac{F}{\pi\sqrt{2c\varepsilon-\varepsilon^2}}$$

$$\approx \frac{F}{\pi\sqrt{2c\varepsilon}} \propto \frac{1}{\sqrt{\varepsilon}}$$ square root singularity.

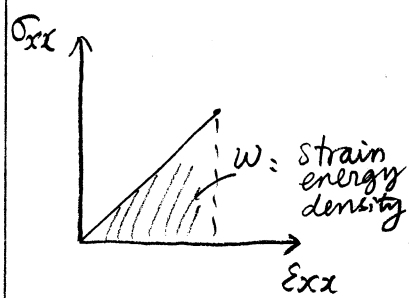
In this section, we will use the <u>strain energy</u> to show why the $\sigma \sim \frac{1}{r}$ singularity is not allowed for a crack tip while the $\sigma \sim \frac{1}{\sqrt{r}}$ singularity is allowed.

§1. Elastic Strain Energy



Consider a uni-axial tensile test of an elastic bar.

Let F be the applied force and Δx be the elongation



The elastic energy stored in the bar equals to the work done by the applied force (shade area on the F-Δx plot)

$$U = \frac{1}{2} F \cdot \Delta x$$

Recall that $F = \sigma_{xx} \cdot A$
$\Delta x = \varepsilon_{xx} \cdot L$

$$U = \frac{1}{2} \sigma_{xx} \cdot A \cdot \varepsilon_{xx} \cdot L = \frac{1}{2} \sigma_{xx} \varepsilon_{xx} \cdot V$$

where V is the total volume of the sample.

Define strain energy density

$$w \equiv \frac{U}{V} = \frac{1}{2} \sigma_{xx} \varepsilon_{xx}$$

In this example, the strain energy density is uniform inside the sample. In general, w is a (scalar) field quantity

$$w(\underset{\sim}{x}) = \frac{1}{2} \sigma_{ij}(\underset{\sim}{x}) \, \varepsilon_{ij}(\underset{\sim}{x})$$

The strain energy density w at point $\underset{\sim}{x}$ is a function of the local strain ε_{ij} at $\underset{\sim}{x}$.

$$w(\varepsilon_{ij}) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

Hence:
$$\sigma_{ij} = \frac{\partial w(\varepsilon_{ij})}{\partial \varepsilon_{ij}}$$

$$C_{ijkl} = \frac{\partial^2 w}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = \frac{\partial^2 w}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}} = C_{klij}$$

— this is the cause of the major symmetry of C_{ijkl} tensor.

* We can also write w as a function of local stress σ_{ij}

$w(\sigma_{ij}) = \frac{1}{2}S_{ijkl}\sigma_{ij}\sigma_{kl}$, where S_{ijkl} is the elastic compliance tensor.

$$\varepsilon_{ij} = \frac{\partial w(\sigma_{ij})}{\partial \sigma_{ij}}$$

$$S_{ijkl} = \frac{\partial^2 w}{\partial \sigma_{ij}\partial \sigma_{kl}} = S_{klij}$$

* Notice that $w \geq 0$ always. ($w=0$ when $\varepsilon_{ij}=0$)

i.e. any elastic deformation must increase elastic strain energy.

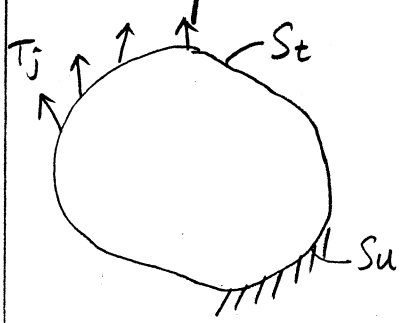
This put some constraints on elastic constants.

For isotropic material, $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

we must have

$E > 0$	Young's modulus
$-1 < \nu < \frac{1}{2}$	Poisson's ratio
$\mu > 0$	Shear modulus

Example 1:



consider an elastic medium subjected to arbitrary surface traction $T_j(\underset{\sim}{x})$

The elastic energy stored in the medium must equal the work done at the surface:

$$U = \frac{1}{2} \oint_S T_j(\underset{\sim}{x})\, u_j(\underset{\sim}{x})\, dS(\underset{\sim}{x})$$

$$= \frac{1}{2} \oint_S \sigma_{ij}\, u_j\, n_i\, dS$$

apply Gauss's theorem $\longrightarrow$

$$= \frac{1}{2} \int_V (\sigma_{ij} u_j)_{,i}\, dV$$

$$= \frac{1}{2} \int_V (\cancel{\sigma_{ij,i} u_i} + \sigma_{ij} u_{j,i})\, dV$$

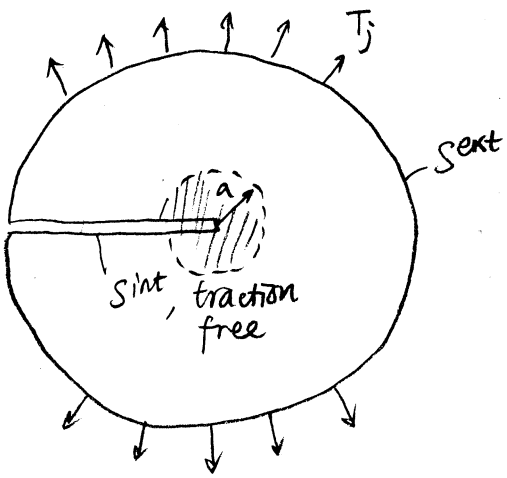
equilibrium condition $\sigma_{ij,i} = 0$ $\longrightarrow$

$$= \frac{1}{2} \int_V \sigma_{ij}\, \varepsilon_{ij}\, dV$$

$$= \int_V w(\underset{\sim}{x})\, dV(\underset{\sim}{x})$$

This confirms that the volume integral of $w(\underset{\sim}{x})$ is the total elastic strain energy U.

§2. Strain Energy of a Crack Tip. (or notch)



consider an elastic medium with a crack (or notch) that is loaded by T_j on external surfaces S^{ext} (away from the crack tip)

We expect the surface displacement u_j to be finite. hence the total stored elastic energy to be finite

$$U = \frac{1}{2}\int_{S^{ext}} T_j\, u_j \, dS \longrightarrow \text{finite}$$

Also notice that

$$U = \int_V w(\underset{\sim}{x})\, dV \quad \text{and} \quad w(\underset{\sim}{x}) \geq 0 \text{ everywhere}$$

i.e. there are no "cancellations" of elastic energy.

This means the elastic energy in any sub volume of the medium also has to be finite.

$$U_{sub} = \int_{V_{sub}} w(\underset{\sim}{x})\, dV \longrightarrow \text{finite.}$$

Now imagine that the stress field at the crack tip is

$$\sigma_{ij} \sim \frac{1}{r^p}$$

$$\varepsilon_{ij} \sim \frac{1}{r^p}$$

then

$$w \sim \frac{1}{r^{2p}}$$

The strain energy stored inside the circle of radius a around the crack tip

$$U_a = \int_0^a \int_0^{2\pi} \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, r \, dr \, d\theta$$

$$\sim \int_0^a r^{1-2p} \, dr$$

when $p < 1$, $1-2p > -1$, $U_a \to$ finite

when $p > 1$, $1-2p < -1$, $U_a \to$ infinite

when $p = 1$, $U_a \sim \int_0^a r^{-1} dr = \ln a - \ln 0 \to$ infinite

Hence $U_a \to \infty$ for $p \geq 1$.

But from previous analysis, the strain energy stored in any sub volume has to be finite.

Therefore, for a crack (or notch) loaded from far away, the stress singularity $\sigma_{ij} \sim \frac{1}{r^p}$ must satisfy $p < 1$.

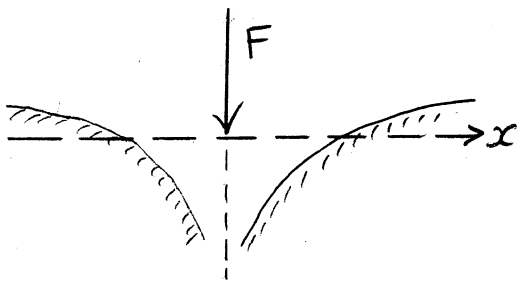
Thus the leading singularity of crack tip is

$$\sigma_{ij} \sim \frac{1}{\sqrt{r}}$$ (see notes "Wedge". P.3)

* However, $\sigma \sim \frac{1}{r}$ singularity will be allowed if this is required by the Boundary Condition, i.e. in order to satisfy B.C., we need to do work that goes to infinity. Part of this work goes to the singularity that has infinite energy.

(See notes "dislocations")

For example, recall our solution of point force on an elastic half-space



surface displacement

$$\tilde{u}_y \sim \log|x|$$

$$\therefore \varepsilon_{xy} \sim \frac{1}{x} \quad (\text{for } y=0)$$

$$\varepsilon_{xy} \sim \frac{1}{r} \quad \text{in general}$$

Notice that the displacement at origin (where F exerts) also diverges

$$\tilde{u}_y(x=0) \to \infty$$

$\therefore$ the work done by F is ∞.

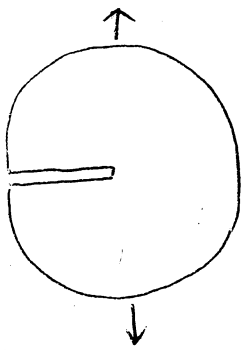
So the solution is self-consistent. (work, energy $\to \infty$)

In practice, the singularities (either $\frac{1}{\sqrt{r}}$, or $\frac{1}{r}$) do not exist. The crack cannot be infinitely sharp and the force cannot be infinitely concentrated.

* But this is not the reason why the crack tip singularity cannot be $\sigma \sim \frac{1}{r}$.

Knowing that $\sigma, \varepsilon \sim \frac{1}{\sqrt{r}}$ for a crack tip helps us to design more efficient numerical methods to handle cracks. (faster convergence).

$\lambda = \frac{1}{2}$.

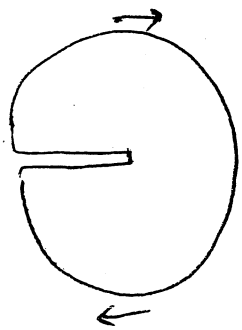


$$A_1 = A(\lambda-1)\sin(\lambda-1)\alpha$$
$$A_2 = -A(\lambda+1)\sin(\lambda+1)\alpha.$$

A is an arbitray constant.

Let $K_I = 3A\sqrt{\frac{\pi}{2}}$. — mode I stress intensity factor

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- even with } \theta$$
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- even with } \theta$$
$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- odd with } \theta$$



K_{II} — mode II stress intensity factor.

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{-\frac{5}{4}\sin\left(\frac{\theta}{2}\right) + \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- odd with } \theta$$
$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{-\frac{3}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- odd with } \theta$$
$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{\frac{1}{4}\cos\left(\frac{\theta}{2}\right) + \frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- even with } \theta.$$

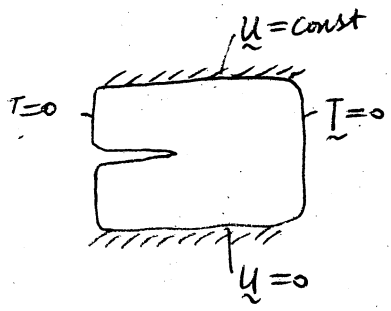
Stress intensity factor is linked to the driving force for crack tip extension.

$$J = \frac{1-\nu}{2\mu}K_I^2 \qquad \text{for mode I loading}$$

$$J = \frac{1-\nu}{2\mu}K_{II}^2 \qquad \text{for mode II loading}$$

---

* calculation of J is a subject of micromechanics.

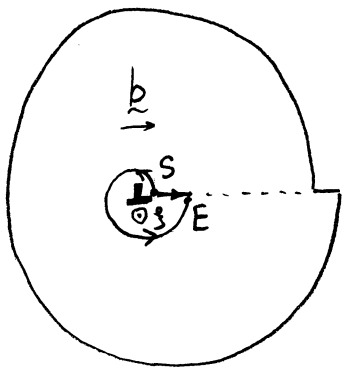


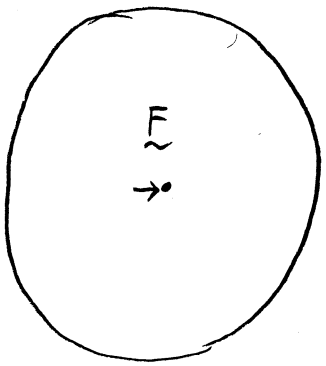
If loading condition is such that external force does no work when crack tip moves, then

$$J = -\frac{\Delta U^{tot}}{\Delta x}$$

ΔU^{tot} is the change of elastic energy as crack tip moves forward by Δx.

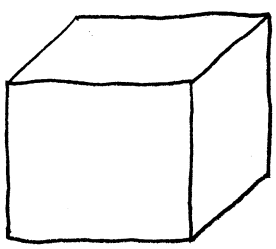
In this section, we discuss two important solutions, both having $\sigma \sim \frac{1}{r}$ singularity. They are the stress field of a dislocation line and that of a line force.



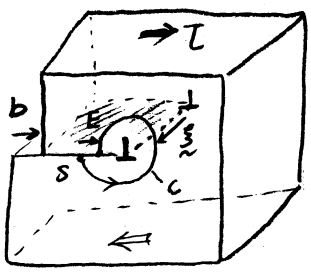


a line force is a body force concentrated at a point
i.e. $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{F}\,\delta(\underset{\sim}{x})$

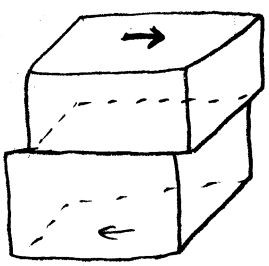
§1. Dislocation and Burgers Vector



starting with a stress-free elastic medium



a dislocation is introduced by making a cut on an internal surface and introducing a displacement jump across that surface.
The dislocation ⊥ is the boundary line of the surface.



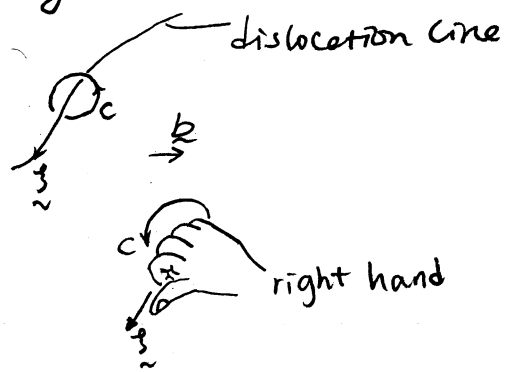
In this case, the dislocation experience a (Peach-Koehler) force to the right from the applied stress τ
By the time the dislocation escapes to the right, the entire upper half is displaced by $\underset{\sim}{b}$ wrt the lower half

* Dislocations are important defects of crystals.
They are the fundamental carriers of plastic deformation in crystals. (see "Computer Simulations of Dislocations", Sect. 1.2-1.3)

Therefore, a medium containing a dislocation must have a displacement jump somewhere.

This can be written mathematically as

$$\oint_c \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}} d\underset{\sim}{x} = \underset{\sim}{b}$$



where c is some closed loop (called the <u>Burgers circuit</u>) around the dislocation line

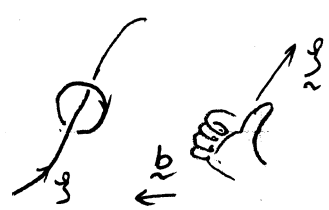
$\underset{\sim}{b}$ is called the <u>Burgers vector</u>

Notice that the direction of $\underset{\sim}{b}$ depend on the direction of c.

As a convention, let us define a line sense $\underset{\sim}{\xi}$ along the dislocation, and let the direction of c follow $\underset{\sim}{\xi}$ from the right-hand rule.

Thus, the orientation of $\underset{\sim}{b}$ depends on the choice of line sense $\underset{\sim}{\xi}$

If we reverse the choice of $\underset{\sim}{\xi}$ for the same dislocation, the orientation of $\underset{\sim}{b}$ will reverse as well.



A visual way to identify the Burgers vector $\underset{\sim}{b}$ is to draw the Burgers circuit around the dislocation starting from the cut-plane. (see figure on page 1).

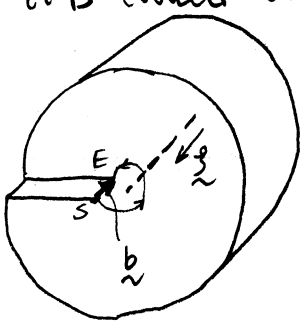
The vector connecting the starting point S and the end point E is the Burgers vector.

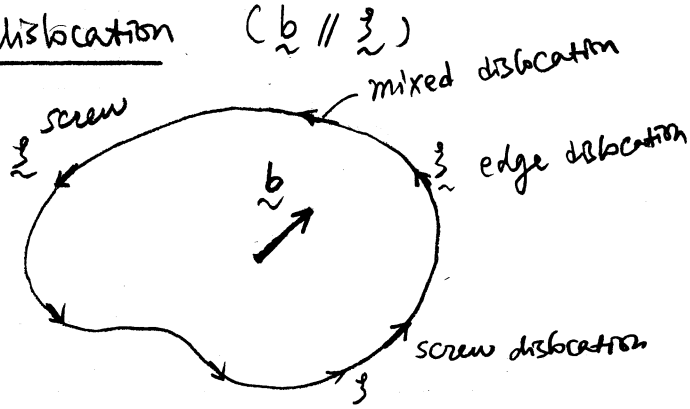
* Show that for the same dislocation, if we reverse $\underset{\sim}{\xi}$ the Burgers vector reverse as well

In the Figure on page 1, the Burgers vector $\underset{\sim}{b}$ is perpendicular to the line direction $\underset{\sim}{\xi}$

This is called an <u>edge dislocation</u> ($\underset{\sim}{b} \perp \underset{\sim}{\xi}$)

When $\underset{\sim}{b}$ is parallel to the line direction $\underset{\sim}{\xi}$ it is called a <u>screw dislocation</u> ($\underset{\sim}{b} \parallel \underset{\sim}{\xi}$)





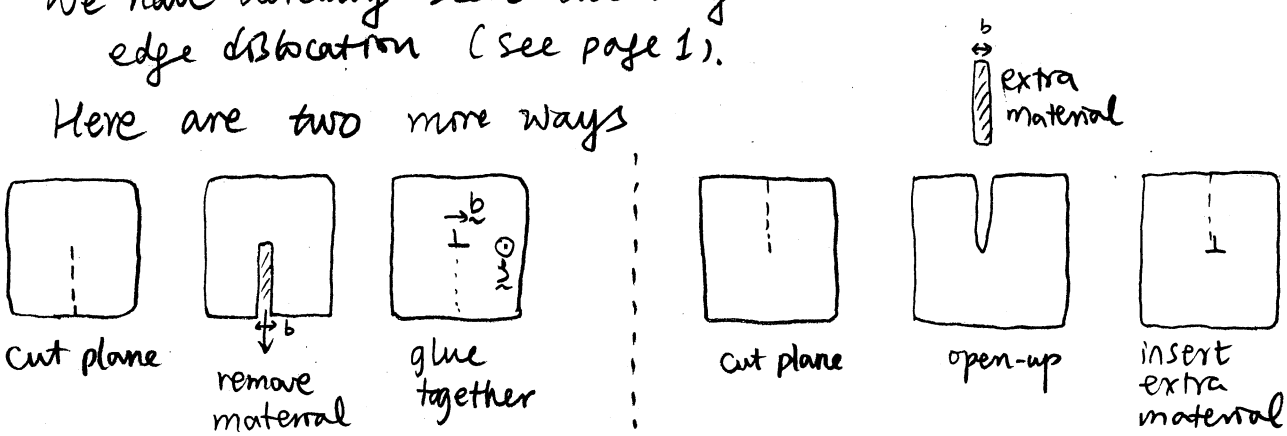
In general, the dislocation line can be curved but the Burgers vector stays constant along the line.

When $\underset{\sim}{b}$ is neither perpendicular nor parallel to $\underset{\sim}{\xi}$, it is called a <u>mixed dislocation</u>.

The same dislocation can be created by many different ways (with different choice of cut-planes).

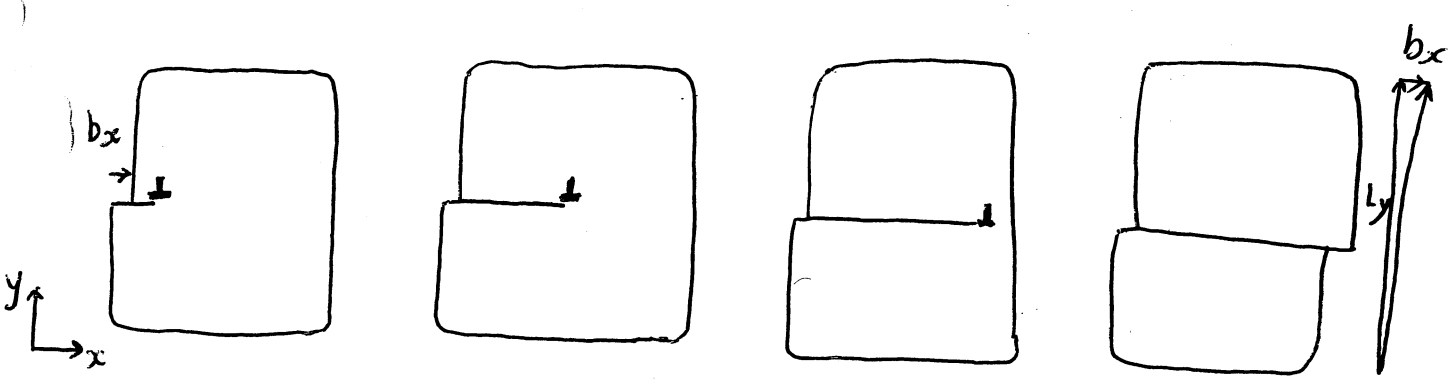
We have already seen two ways to create the same edge dislocation (see page 1).

Here are two more ways



* show that in both cases, we get the same $\underset{\sim}{k}$ as before.

§2. Dislocation motion and plastic strain

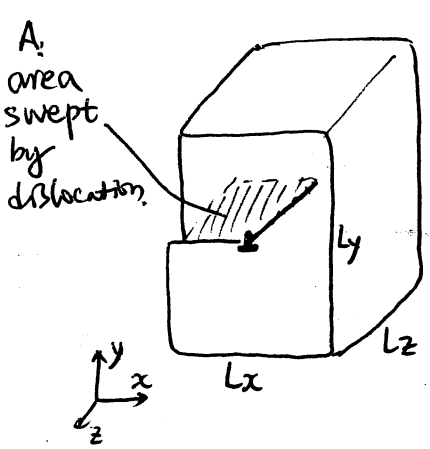


dislocation nucleates from left surface → dislocation travels to the right → dislocation exits from right surface

net result: upper half of the material (crystal) slips by b_x with respect to lower half

plastic deformation:

$$\varepsilon_{xy}^{pl} = \frac{1}{2}\cdot\frac{b_x}{L_y} = \frac{b_x}{2}\,\frac{L_x L_z}{L_x L_y L_z} = \frac{1}{2}\,\frac{b_x \cdot A^{tot}}{V}$$



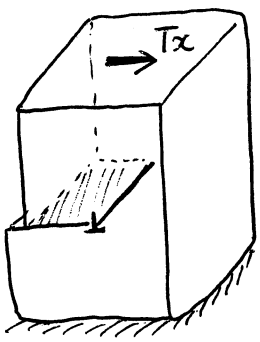
A^{tot}: total area swept by dislocation

V: material volume.

It can be shown that, in general, when a dislocation line swept an area A on a plane with normal vector $\underset{\sim}{n}$, it produces a plastic strain of

$$\varepsilon_{ij}^{pl} = \frac{1}{2}\cdot\frac{b_i n_j + b_j n_i}{V}\cdot A$$

§3. Force on dislocation line from stress field.



Assuming a uniform traction force T_x is applied to the top surface.

This leads to an applied (external) stress fiel

$$\sigma_{xy}^{ext} = T_x$$

The total stress field in the medium is the superposition of σ_{ij}^{ext} and the internal stress field σ_{ij}^{int} of the dislocation.

$$\sigma_{ij}^{tot} = \sigma_{ij}^{ext} + \sigma_{ij}^{int}$$

The total work done by the applied traction force as the dislocation moves from left end to right end is

$$\Delta W = \underbrace{T_x (L_x \cdot L_z)}_{\text{total force}} \cdot \underbrace{b_x}_{\text{distance}} > 0 \quad (\text{when } T_x > 0)$$

This means it is energetically favorable for the dislocation to move from left to right when $T_x > 0$.

We can interprete ΔW as the work done by a generalized force f_x (per unit length) exerted on the dislocation line.

$$\Delta W = \underbrace{f_x \cdot L_z}_{\text{total force}} \cdot \underbrace{L_x}_{\text{distance}}$$

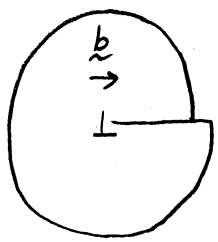
In general, the peach-Koehler force is

$$\boxed{\underset{\sim}{f} = (\underset{\sim}{b} \cdot \underset{\approx}{\sigma}) \times \underset{\sim}{\xi}}$$

The Peach-Koehler formula

$$f_x = \frac{\Delta W}{L_z L_x} = b_x T_x = b_x \cdot \sigma_{xy}$$

§4. Stress field of a dislocation line and that of a line force in an infinite medium

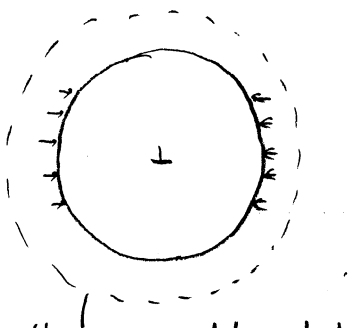


Look for ϕ whose u_r contains θ (for discontinuity)

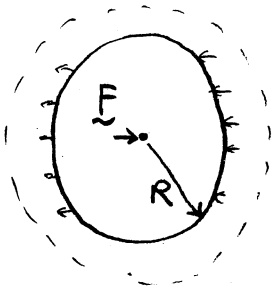
$$\phi = A r\theta \sin\theta + B r\theta\cos\theta + C r\ln r\cos\theta + D r\ln r\sin\theta$$

A, C terms — even with θ

B, D terms — odd with θ



there should not be any net force integrated over the entire surface



We expect $\sigma(r,\theta) = f(r)\,g(\theta)$

The total traction force on any circle with radius R must balance F.

$$\int_0^{2\pi} \sigma_{ij}(R,\theta)\cdot n_j(\theta)\, R\,d\theta = F$$

$$f(R)\cdot R = \text{const}$$

$$f(R) \sim \frac{1}{R}.$$

or $f(r) \sim \frac{1}{r}$.

Look for ϕ whose $\sigma \sim \frac{1}{r}$

$$\phi = A r\theta\sin\theta + B r\theta\cos\theta + C r\ln r\cos\theta + D r\ln r\sin\theta$$

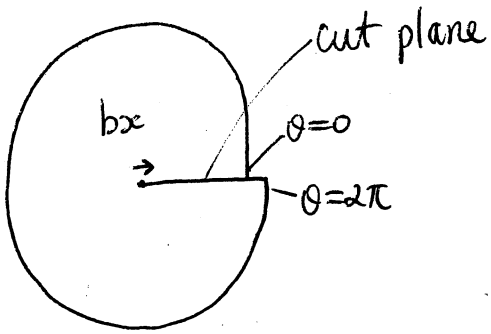
Same as trial solution for dislocations!

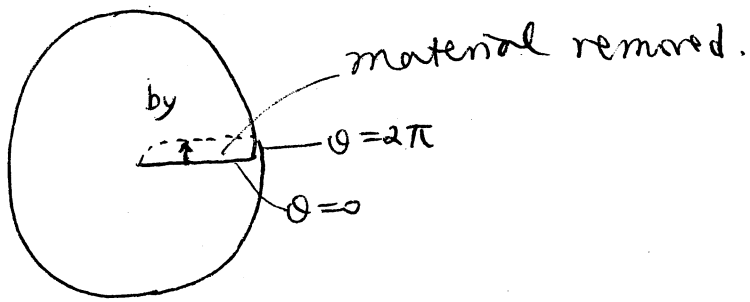
there should not be any displacement jump on any surface

Solution strategy:

find coefficients to give

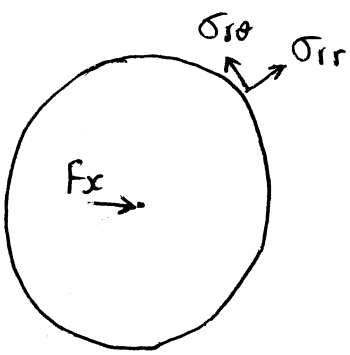
displacement jump but no net force	net force but no displacement jump.

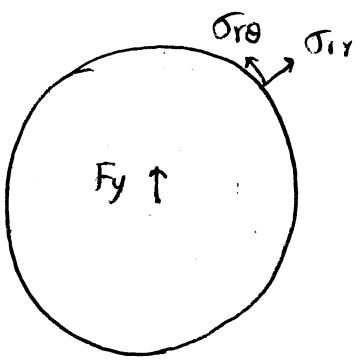




$$b_x = u_r\big|_{\theta=2\pi} - u_r\big|_{\theta=0} = \frac{\pi}{2\mu}\left[B(\kappa-1) - D(\kappa+1)\right]$$

$$b_y = u_\theta\big|_{\theta=2\pi} - u_\theta\big|_{\theta=0} = \frac{\pi}{2\mu}\left[A(\kappa-1) + C(\kappa+1)\right]$$





$$F_x + \int_0^{2\pi}(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\,r\,d\theta = 0 \Rightarrow F_x = -2\pi A$$

$$F_y + \int_0^{2\pi}(\sigma_{rr}\sin\theta + \sigma_{r\theta}\cos\theta)\,r\,d\theta = 0 \Rightarrow F_y = 2\pi B$$

---

dislocation: b_x, $b_y = 0$, $F_x = 0$, $F_y = 0$.

$\kappa = 3-4\nu$ plane strain

$\kappa+1 = 4(1-\nu)$

$$D = -\frac{2\mu}{\pi(\kappa+1)}\,b_x = -\frac{\mu}{2\pi(1-\nu)}\,b_x$$

$(A = B = C = 0)$

$$\Rightarrow \begin{cases} u_r \\ u_\theta \end{cases}, \quad \begin{matrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{matrix} = \frac{\mu b_x}{2\pi(1-\nu)r}(\cdots)$$

angular dependence

$b_x = 0$, b_y, $F_x = 0$, $F_y = 0$.

$$C = \frac{2\mu}{\pi(\kappa+1)}\,b_y = \frac{\mu}{2\pi(1-\nu)}\,b_y \quad (A = B = D = 0)$$

Line force. (Kelvin Solution)

F_x, $F_y = 0$, $b_x = 0$, $b_y = 0$.

plane strain

$\kappa = 3 - 4\nu$

$$\frac{\kappa-1}{\kappa+1} = \frac{2-4\nu}{4-4\nu} = \frac{1-2\nu}{2(1-\nu)}$$

$$A = -\frac{F_x}{2\pi}$$

$$C = -\frac{\kappa-1}{\kappa+1} \cdot A = \frac{1-2\nu}{4\pi(1-\nu)} F_x$$

$$B = D = 0.$$

$$\Rightarrow \begin{cases} \sigma_{rr} = \frac{F_x}{r}(\cdots) \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{cases}$$

angular dependence.

$F_x = 0$. F_y. $b_x = 0$. $b_y = 0$.

$$B = \frac{F_y}{2\pi}$$

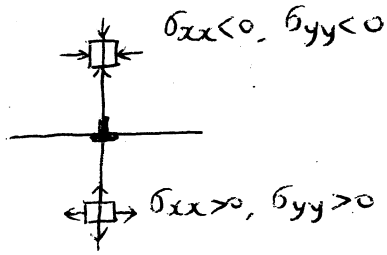
$$D = \frac{\kappa-1}{\kappa+1} B = \frac{1-2\nu}{4\pi(1-\nu)} F_y$$

$$\Rightarrow \begin{cases} \sigma_{rr} = \frac{F_y}{r}(\cdots) \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{cases}$$

Stress field of dislocation. b_x, $b_y = 0$.

$$\phi = -\frac{\mu b_x}{2\pi(1-\nu)} r \ln r \sin\theta \qquad r\sin\theta = y, \quad r = \sqrt{x^2+y^2}$$

$$\phi(x,y) = -\frac{\mu b_x}{4\pi(1-\nu)} y \ln(x^2+y^2)$$

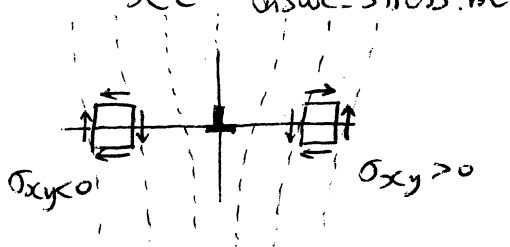


$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = -\frac{\mu b_x}{2\pi(1-\nu)} \frac{y(3x^2+y^2)}{(x^2+y^2)^2}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{\mu b_x}{2\pi(1-\nu)} \frac{y(x^2-y^2)}{(x^2+y^2)^2}$$

see disloc_stress.m

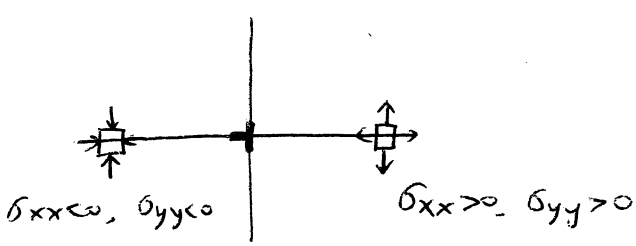
$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \frac{\mu b_x}{2\pi(1-\nu)} \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$



Stress field of dislocation $b_x = 0$, b_y

$$\phi = \frac{\mu b_y}{2\pi(1-\nu)} r \ln r \cos\theta$$

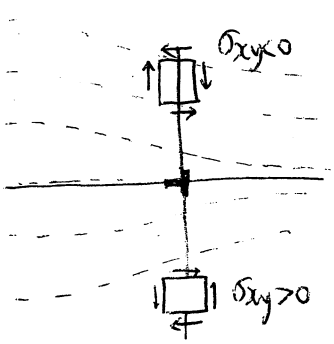
$$\phi(x,y) = \frac{\mu b_y}{4\pi(1-\nu)} x \ln(x^2+y^2)$$



$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \frac{\mu b_y}{2\pi(1-\nu)} \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

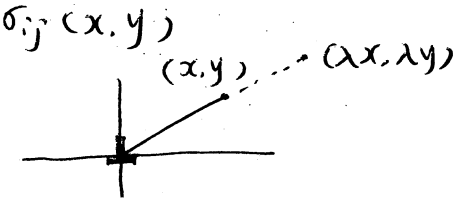
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{\mu b_y}{2\pi(1-\nu)} \frac{x(x^2+3y^2)}{(x^2+y^2)^2}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \frac{\mu b_y}{2\pi(1-\nu)} \frac{y(x^2-y^2)}{(x^2+y^2)^2}$$



Notice that $\sigma_{ij}(\lambda x, \lambda y) = \frac{1}{\lambda}\sigma_{ij}(x,y)$

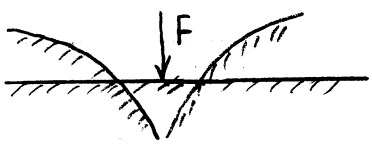
$$\sigma_{ij} \sim \frac{1}{r}$$



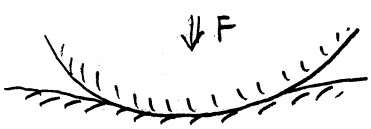
§1. Compare 2D and 3D problems

elastic half-space subjected to plane-wave loading

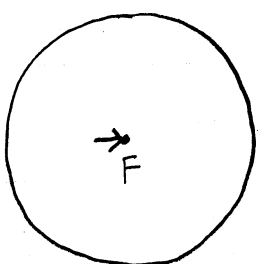
⇓ Fourier transform



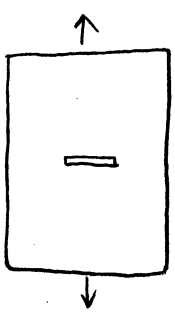
elastic half-space subjected to point force loading



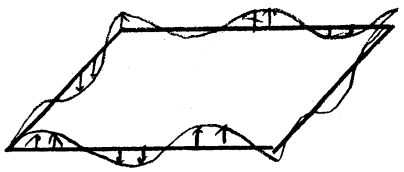
cylindrical contact

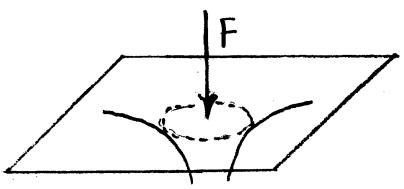


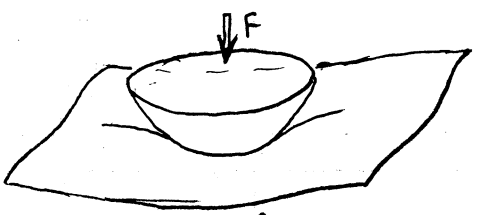
concentrated line force



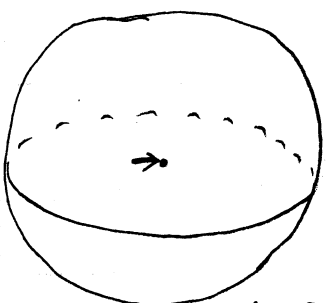
slit-like crack



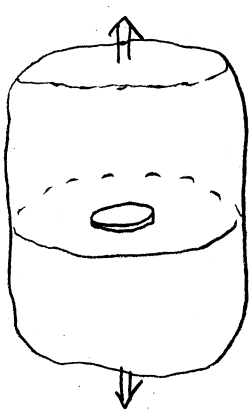




spherical contact



concentrated point force



penny-shaped crack

§2. Elasticity equations in 3D

equilibrium: $\sigma_{ij,i} + F_j = 0$

compatibility: (avoided by write strain in terms of displacements)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Generalized Hooke's Law:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$
$$= \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} \quad \text{(isotropic)}$$

write everything in terms of u_i

$$\begin{aligned}
0 &= \sigma_{ij,i} + F_j \\
&= (\lambda\,\varepsilon_{kk}\,\delta_{ij})_{,i} + 2\mu\,\varepsilon_{ij,i} + F_j \\
&= \lambda\,\varepsilon_{kk,j} + 2\mu\,\varepsilon_{ij,i} + F_j \\
&= \lambda\,u_{k,kj} + \mu(u_{i,j} + u_{j,i})_{,i} + F_j \\
&= \lambda\,u_{k,kj} + \mu\,u_{i,ji} + u_{j,ii} + F_j
\end{aligned}$$

$$\boxed{(\lambda+\mu)\,u_{i,ij} + \mu\,u_{j,ii} + F_j = 0}$$

recall $\lambda = \dfrac{2\mu\nu}{1-2\nu}$ $\qquad \lambda+\mu = \dfrac{\mu}{1-2\nu}$

$$\boxed{\frac{\mu}{1-2\nu}\,u_{i,ij} + \mu\,u_{j,ii} + F_j = 0}$$

In tensor notation

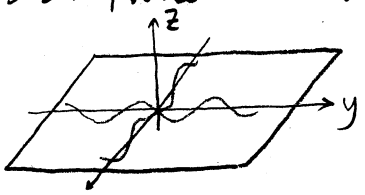
$$\boxed{\frac{\mu}{1-2\nu}\,\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) + \mu\,\nabla^2\underset{\sim}{u} + \underset{\sim}{F} = 0}$$

In the absense of body force ($\underset{\sim}{F}=0$)

$$\boxed{\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) + (1-2\nu)\,\nabla^2\underset{\sim}{u} = 0}$$

independent of μ!

§3. Plane-wave loading - Trial solution



$z \le 0$
half-space

B.C.

$$T_x(x,y) = \sigma_{xz}\big|_{z=0} = 0$$
$$T_y(x,y) = \sigma_{yz}\big|_{z=0} = 0$$
$$T_z(x,y) = \sigma_{zz}\big|_{z=0} = S \cdot e^{i(k_x x + k_y y)}$$

Trial solution:

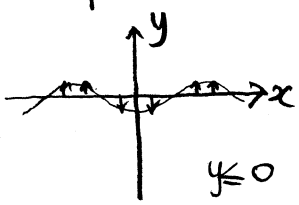
$$u_x = (A_1 + B_1 z)\, e^{i(k_x x + k_y y)}\, e^{k_z z}$$
$$u_y = (A_2 + B_2 z)\, e^{i(k_x x + k_y y)}\, e^{k_z z}$$
$$u_z = (A_3 + B_3 z)\, e^{i(k_x x + k_y y)}\, e^{k_z z}$$
$$k_z = \sqrt{k_x^2 + k_y^2}$$

six unknowns: $A_1\ B_1\ A_2\ B_2\ A_3\ B_3$

3 equations from equilibrium cond.

3 equations from B.C.

compare with 2D problem



$y \le 0$

B.C.

$$T_x(x) = \sigma_{xy}\big|_{y=0} = 0$$
$$T_y(x) = \sigma_{yy}\big|_{y=0} = -A\cos kx$$

Trial solution:

$$\phi = \cos kx\ (A + By)\, e^{ky}$$

Solution: $B = -Ak$

$$\sigma_{xx} = -Ak^2 \cos kx\,(1+ky)\, e^{ky}$$
$$\sigma_{yy} = -Ak^2 \cos kx\,(1-ky)\, e^{ky}$$
$$\sigma_{xy} = -Ak^3 \sin kx\ y\, e^{ky}$$
$$u_x = -\frac{Ak}{E}\sin kx\left[(1-\nu-2\nu^2) + (1+\nu)ky\right]e^{ky}$$
$$u_y = -\frac{Ak}{E}\cos kx\left[(2-2\nu^2) - (1+\nu)ky\right]e^{ky}$$

Equilibrium equation: $\nabla(\nabla\cdot\underset{\sim}{u}) + (1-2\nu)\nabla^2\underset{\sim}{u} = 0$

$$\nabla^2 u_x = 2B_1 k_z\, e^{i(k_x x + k_y y)}\, e^{k_z z}$$
$$\nabla^2 u_y = 2B_2 k_z\, e^{i(k_x x + k_y y)}\, e^{k_z z}$$
$$\nabla^2 u_z = 2B_3 k_z\, e^{i(k_x x + k_y y)}\, e^{k_z z}$$

$$\nabla\cdot\underset{\sim}{u} = \left[(A_1 + B_1 z)\, i k_x + (A_2 + B_2 z)\, i k_y + (A_3 + B_3 z)\, i k_z + B_3\right] \cdot e^{i(k_x x + k_y y)}\, e^{k_z z}$$

$$\nabla(\nabla\cdot\underset{\sim}{u}) + (1-2\nu)\nabla^2\underset{\sim}{u} = 0$$

$$\rightarrow \begin{cases} -B_1 k_x - B_2 k_y + iB_3 k_z = 0 \\ k_x(-k_x A_1 - k_y A_2 + iB_3 + ik_z A_3) + 2(1-2\nu)B_1 k_z = 0 \\ k_y(-k_x A_1 - k_y A_2 + iB_3 + ik_z A_3) + 2(1-2\nu)B_2 k_z = 0 \end{cases}$$

(these are the only three independent equations)

$$\rightarrow \begin{cases} -B_1 k_x - B_2 k_y + iB_3 k_z = 0 \\ B_1 k_y - B_2 k_x = 0 \\ k_x A_1 + k_y A_2 - ik_z A_3 = (3-4\nu)iB_3 \end{cases}$$

B.C.

$$\sigma_{xz} = 2\mu\varepsilon_{xz} = \mu(u_{x,z} + u_{z,x})$$

$$\sigma_{yz} = 2\mu\varepsilon_{yz} = \mu(u_{y,z} + u_{z,y})$$

$$\sigma_{zz} = (\lambda+2\mu)\varepsilon_{zz} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{xx}$$

$$= (\lambda+2\mu)u_{z,z} + \lambda u_{y,y} + \lambda u_{x,x}$$

at $z=0$:

$$u_{x,z} + u_{z,x} = 0$$

$$u_{y,z} + u_{z,y} = 0$$

$$(\lambda+2\mu)u_{z,z} + \lambda u_{y,y} + \lambda u_{x,x} = S\cdot e^{i(k_x x + k_y y)}\cdot e^{k_z z}$$

$$\rightarrow \begin{cases} \text{three more algebraic equations} \\ \text{for } A_1, A_2, A_3, B_1, B_2, B_3 \end{cases}$$

[half-space_3d.m] on coursework

solution

$$\begin{cases} A_1 = -\dfrac{i K_x (1-2\nu)}{2\mu K_z^2} \cdot S \\ A_2 = -\dfrac{i K_y (1-2\nu)}{2\mu K_z^2} \cdot S \\ A_3 = \dfrac{1-\nu}{\mu K_z} \cdot S \end{cases} \qquad \begin{cases} B_1 = -\dfrac{i K_x}{2\mu K_z} \cdot S \\ B_2 = -\dfrac{i K_y}{2\mu K_z} \cdot S \\ B_3 = -\dfrac{1}{2\mu} \cdot S \end{cases}$$

define $\varphi = e^{i(K_x x + K_y y)} e^{K_z z}$

$$u_x = -\frac{i K_x}{2\mu K_z^2} \cdot \left[(1-2\nu) + K_z z\right] \cdot \varphi \cdot S$$

$$u_y = -\frac{i K_y}{2\mu K_z^2} \left[(1-2\nu) + K_z z\right] \cdot \varphi \cdot S$$

$$u_z = \frac{1}{2\mu K_z} \left[2(1-\nu) - K_z z\right] \cdot \varphi \cdot S$$

normal displacement on the surface

$$\tilde{u}_z = u_z(z=0) = \frac{1-\nu}{\mu} \cdot \frac{e^{i(K_x x + K_y y)}}{K_z} \cdot S$$

$$K_z = \sqrt{K_x^2 + K_y^2}$$

Summary:

Elastic half space subjected to traction force:

$$T_x = T_y = 0, \quad T_z = S\, e^{i(K_x x + K_y y)}$$

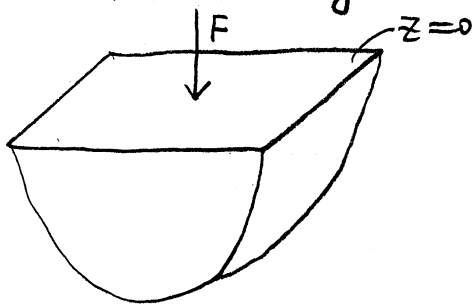
The displacement on the surface is

$$\tilde{u}_z = S \cdot \frac{1-\nu}{\mu} \cdot \frac{e^{i(K_x x + K_y y)}}{\sqrt{K_x^2 + K_y^2}}$$

$$\tilde{u}_z = \frac{1-\nu}{\mu} \frac{T_z}{\sqrt{K_x^2 + K_y^2}}$$

* Let $K_y \to 0$ reduce to 2D solution

§4. Point loading on half-space



B.C. on the surface $z=0$

$$T_x(x,y)=0,$$
$$T_y(x,y)=0,$$
$$T_z(x,y)=-F\,\delta(x,y)$$

Solution can be obtained by inverse Fourier transform.

Define 2D Fourier transform

$$\hat{f}(k_x,k_y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,e^{-i(k_x x+k_y y)}\,dx\,dy$$

inverse transform

$$f(x,y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{f}(k_x,k_y)\,e^{i(k_x x+k_y y)}\,dk_x\,dk_y$$

Fourier transform pairs

real space		Fourier space
$\delta(x,y)$		1
$T_z(x,y)=-F\,\delta(x,y)$	$\rightarrow$	$\hat{T}_z(k_x,k_y)=-F$
		$\downarrow$
$\tilde{u}_z(x,y)=-\frac{F(1-\nu)}{2\pi\mu}\frac{1}{\sqrt{x^2+y^2}}$	$\leftarrow$	$\hat{u}_z(k_x,k_y)=-F\cdot\frac{1-\nu}{\mu}\cdot\frac{1}{\sqrt{k_x^2+k_y^2}}$

$$=-\frac{F(1-\nu)}{2\pi\mu}\cdot\frac{1}{r}$$

$$r=\sqrt{x^2+y^2}$$

notice that $\frac{1}{r}\to 0$ when $r\to\infty$.

$\frac{1}{r}$ only has singularity at $r\to 0$.

* compare with the 2D solution $\ln r$, which has singularity at both $r\to\infty$ and $r\to 0$.

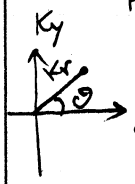
* Proof of inverse transform

$$\tilde{u}_z(x,y) \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(-F)\frac{1-\nu}{\mu}\frac{e^{i(k_x x+k_y y)}}{\sqrt{k_x^2+k_y^2}}\,dk_x\,dk_y = -\frac{F(1-\nu)}{\mu}\cdot\frac{1}{2\pi r}$$

by symmetry we expect $\tilde{u}_z(x,y)$ to be a function of $r \equiv \sqrt{x^2+y^2}$

Hence, without loss of generality, we can consider the point

$x = r, \quad y = 0.$



$k_x x + k_y y = k_r \cdot r \cdot \cos\theta \qquad k_r = \sqrt{k_x^2+k_y^2}$

$$\tilde{u}_z(r) = \frac{1}{(2\pi)^2}\frac{(-F)(1-\nu)}{\mu}\int_0^{\infty}\int_0^{2\pi}\frac{e^{ik_r\cdot r\cdot\cos\theta}}{k_r}\cdot k_r\,dk_r\,d\theta$$

$$= -\frac{F(1-\nu)}{2(2\pi)^2\mu}\int_{-\infty}^{+\infty}\int_0^{2\pi}e^{ik_r r\cos\theta}\,dk_r\,d\theta$$

$$= -\frac{F(1-\nu)}{4\pi\mu}\int_0^{2\pi}\delta(r\cos\theta)\,d\theta$$

because $\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikx}\,dk = \delta(x)$

$$= -\frac{F(1-\nu)}{4\pi\mu}\cdot\frac{1}{r}\int_0^{2\pi}\delta(\cos\theta)\,d\theta$$

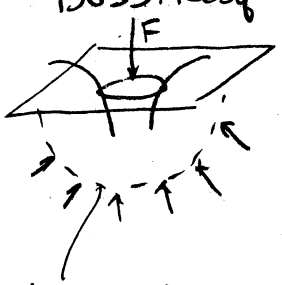
because $\delta(ax) = \frac{1}{a}\delta(x)$

$$= -\frac{F(1-\nu)}{2\pi\mu}\cdot\frac{1}{r}$$

because $\int_0^{2\pi}\delta(\cos\theta)\,d\theta = 2\int_0^{\pi}\delta(\cos\theta)\,d\theta = 2\int_{-1}^{1}\frac{\delta(x)}{\sqrt{1-x^2}}\,dx = 2$

$$= -\frac{F(1-\nu^2)}{\pi E}\frac{1}{r}$$

Bossinesq solution:



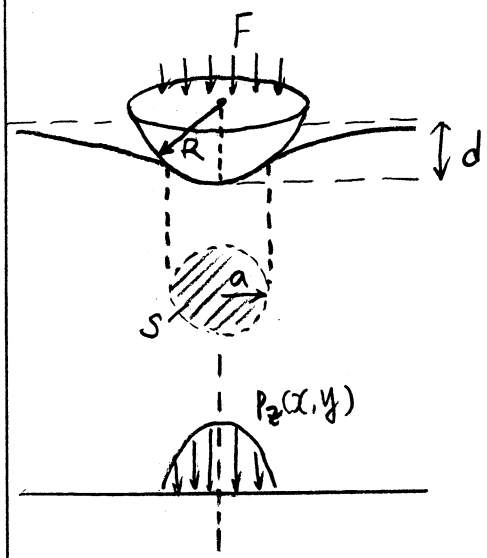
$$\boxed{\begin{aligned} u_z\big|_{z=0} &= -\frac{F(1-\nu)}{2\pi\mu}\frac{1}{r} \\ u_r\big|_{z=0} &= -\frac{F(1-2\nu)}{4\pi\mu}\frac{1}{r}\end{aligned}}$$

from dimensional analysis, we expect

$$\sigma \sim \frac{1}{r^2}$$

$$\therefore\ u \sim \frac{1}{r}$$

§5. Hertz contact problem



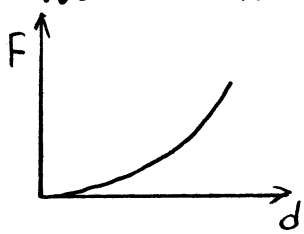
For simplicity, consider a rigid sphere of radius R indenting an elastic half-space.

(in general, the indentor can be a deformable ellipsoid)

By symmetry, we expect the contact area to be a circle of radius a.

Let F be the magnitude of indenting force and d be the indentation depth.

We want to know the F-d relationship and F-a relationship.



We also want to know the pressure distribution on the surface

$$p_z(x,y) = -\sigma_{zz}\big|_{z=0}$$

$$\begin{cases} p_z(x,y) \geq 0 & x^2+y^2 \leq a^2 \quad \text{(inside contact area } S\text{)} \\ p_z(x,y) = 0 & x^2+y^2 > a^2 \end{cases}$$

The shape of the indentor can be described by

$$u_0(x,y) = \frac{x^2}{2R} + \frac{y^2}{2R}$$

(see lecture notes "Contact", p.5)

Let $\tilde{u}_z(x,y)$ be the surface displacement of the half-space.

then

$$\begin{cases} \tilde{u}_z(x,y) = u_0(x,y) - d & x^2+y^2 \leq a^2 \\ \tilde{u}_z(x,y) < u_0(x,y) - d & x^2+y^2 > a^2 \end{cases}$$

using the Boussinesq solution, $\tilde{u}_z(x,y)$ and $p_z(x,y)$ are related to each other

$$-\frac{1-\nu^2}{\pi E}\iint_{S:\ x^2+y^2\le a^2} \frac{p_z(x',y')}{\sqrt{(x-x')^2+(y-y')^2}}\,dx'dy' = \tilde{u}_z(x,y)$$

For (x,y) inside area S: $(x^2+y^2\le a^2)$

$$\iint_S \frac{p_z(x',y')}{\sqrt{(x-x')^2+(y-y')^2}}\,dx'dy' = -\frac{\pi E}{1-\nu^2}\left(-d+\frac{x^2}{2R}+\frac{y^2}{2R}\right)$$

What distribution of $p_z(x',y')$ solves the above equation?

Magic formula:

$$\iint_{x'^2+y'^2\le a^2} \frac{\sqrt{1-\left(\frac{x'}{a}\right)^2-\left(\frac{y'}{a}\right)^2}}{\sqrt{(x-x')^2+(y-y')^2}}\,dx'dy' = \frac{\pi}{4a}\left(2a^2-x^2-y^2\right)$$

* For proof and more discussions, see notes "potential Field of a Uniformly Charged Ellipsoid".

* This is closely related to the fact that the potential field of a uniformly charged ellipsoid is a quadratic function inside the ellipsoid something supposedly well known to physicists familiar with electrostatics — such as Lev Landau.

* This property is also closely related to the Eshelby's inclusion (that has ellipsoidal shape) that will be discussed in ME340B "Micromechanics".

By matching the above two expressions, we get

$$p_z(x', y') = p_0 \sqrt{1-\left(\frac{x'}{a}\right)^2-\left(\frac{y'}{a}\right)^2}$$

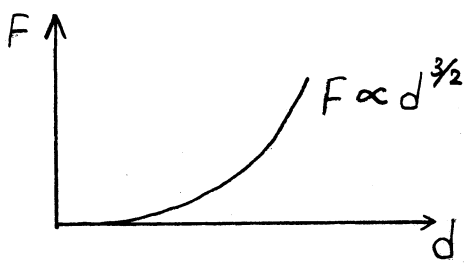
p_0 is related to the total force $F = \iint_S p_z(x', y')\,dx'dy' = p_0 \frac{2\pi a^2}{3}$

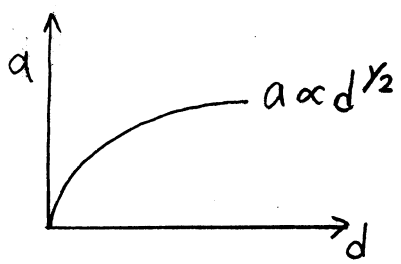
$$\boxed{p_0 = \frac{3F}{2\pi a^2}}$$

Also by matching the expression, we find

contact radius: $\boxed{a = \left(\frac{3(1-\nu^2)FR}{4E}\right)^{1/3}}$

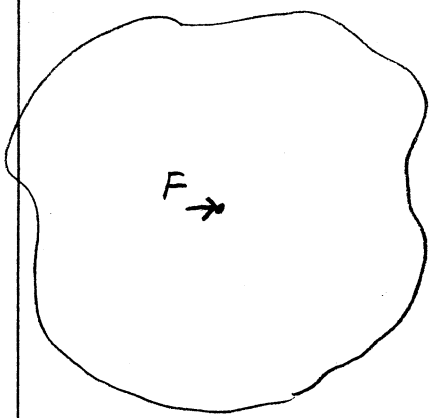
indent depth: $\boxed{d = \left(\frac{3(1-\nu^2)}{4E}\right)^{2/3} \frac{F^{2/3}}{R^{1/3}}}$





§1. Kelvin's Problem & Green function

(* Story of poor Green and Lord Kelvin on course work. "The Green of Green Functions")



Point force F acting on the interior of an infinite elastic medium.

Suppose the force F has unit magnitude, acts on the origin, and points to the j-direction, then the body force distribution can be expressed as

$$F_i(\underset{\sim}{x}) = \delta(\underset{\sim}{x})\,\delta_{ij}$$

Recall equilibrium condition: $\sigma_{ki,k} + F_i = 0$

$$\therefore \sigma_{ki,k} = -\delta(x)\,\delta_{ij}$$

Let $u_i(\underset{\sim}{x})$ be the resulting displacement in response the point force (along j-direction)

Obviously, $u_i(\underset{\sim}{x})$ depends on j

Define $G_{ij}(\underset{\sim}{x}) \equiv u_i(x)$ when point force is along j-dir.

displacement direction — point force direction.

> $G_{ij}(\underset{\sim}{x})$ is the displacement at point $\underset{\sim}{x}$ in the i-direction in response to a unit point force at origin in the j-direction. $G_{ij}(\underset{\sim}{x})$ is called the <u>Green function</u> of the infinite medium

Given $G_{ij}(\underset{\sim}{x})$, the displacement field $u_i(\underset{\sim}{x})$ in response to an arbitrary body force distribution $F_j(\underset{\sim}{x})$ is

$$\underline{u_i(\underset{\sim}{x}) = \iiint F_j(\underset{\sim}{x}')\, G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}')\, d^3\underset{\sim}{x}'}$$

§2. Solving Kelvin's Problem by Galerkin vector

Galerkin Vector Representation

define vector field $\underset{\sim}{V}$ that is related to displacement field $\underset{\sim}{u}$

through: $2\mu \underset{\sim}{u} = 2(1-\nu)\nabla^2 \underset{\sim}{V} - \nabla(\nabla\cdot\underset{\sim}{V})$

$$2\mu\, \nabla\cdot\underset{\sim}{u} = 2(1-\nu)\nabla^2(\nabla\cdot\underset{\sim}{V}) - \nabla^2(\nabla\cdot\underset{\sim}{V}) = (1-2\nu)\nabla^2(\nabla\cdot\underset{\sim}{V})$$

$$\frac{\mu}{1-2\nu}\nabla(\nabla\cdot\underset{\sim}{u}) + \mu\nabla^2\underset{\sim}{u} = (1-\nu)\nabla^4\underset{\sim}{V}$$

Equilibrium condition

$$(1-\nu)\nabla^4\underset{\sim}{V} + \underset{\sim}{F} = 0$$

$$\boxed{\nabla^4\underset{\sim}{V} = -\frac{1}{1-\nu}\underset{\sim}{F}}$$

To be specific, let $F_x = 0$, $F_y = 0$, $F_z = \delta(\underset{\sim}{x})$ (point force applied in the $j=3$ direction)

Solution: $V_x = 0$

$V_y = 0$

$$\nabla^4 V_z = -\frac{1}{1-\nu}\delta(\underset{\sim}{x})$$

Here we need to invoke another "magic" formula from electrostatics

$$\underline{\nabla^2 \frac{1}{R} = -4\pi\,\delta(\underset{\sim}{x})}\quad , \quad \text{where } R=\sqrt{x^2+y^2+z^2}$$

*Poisson's equation in electrostatics

$\Phi(\underset{\sim}{x})$ is the potential field produced by charge density field $\rho(\underset{\sim}{x})$

$$\nabla^2\Phi(\underset{\sim}{x}) = -\frac{\rho(\underset{\sim}{x})}{\epsilon_0}$$

When $\rho(\underset{\sim}{x}) = Q\,\delta(\underset{\sim}{x})$, i.e. a point charge Q at origin.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{R} \qquad R=\sqrt{x^2+y^2+z^2}$$

Therefore,

$$\nabla^2\frac{1}{R} = -4\pi\,\delta(\underset{\sim}{x})$$

It can also be verified that

$$\nabla^2 R = \frac{2}{R}$$

$$R=\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial R}{\partial x} = \frac{x}{R}$$

$$\frac{\partial^2 R}{\partial x^2} = \frac{1}{R} - \frac{x}{R^2}\cdot\frac{\partial R}{\partial x} = \frac{1}{R} - \frac{x^2}{R^3}$$

$$\nabla^2 R = \frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} = \frac{1}{R} - \frac{x^2}{R^3} + \frac{1}{R} - \frac{y^2}{R^3} + \frac{1}{R} - \frac{z^2}{R^3}$$

$$= \frac{3}{R} - \frac{x^2+y^2+z^2}{R^3} = \frac{2}{R}$$

Hence

$$\nabla^4 R = -8\pi\,\delta(\underset{\sim}{x})$$

$$\therefore\ V_z(\underset{\sim}{x}) = \frac{1}{8\pi(1-\nu)}\cdot R \qquad R=\sqrt{x^2+y^2+z^2}$$

displacement field

$$\underset{\sim}{u} = \frac{1-\nu}{\mu}\nabla^2 \underset{\sim}{V} - \frac{1}{2\mu}\nabla(\nabla\cdot\underset{\sim}{V})$$

$$\nabla\cdot\underset{\sim}{V} = \frac{\partial}{\partial z}V_z = \frac{1}{8\pi(1-\nu)}\frac{\partial R}{\partial z}$$

$$u_x = \frac{1-\nu}{\mu}\nabla^2 V_x - \frac{1}{2\mu}\frac{\partial}{\partial x}\frac{\partial}{\partial z}V_z = -\frac{1}{2\mu}\frac{\partial^2}{\partial x\partial z}V_z$$

$$u_y = -\frac{1}{2\mu}\frac{\partial^2}{\partial y\partial z}V_z$$

$$u_z = \frac{1-\nu}{\mu}\nabla^2 V_z - \frac{1}{2\mu}\frac{\partial^2}{\partial z^2}V_z$$

$$u_i = \left[\frac{1-\nu}{\mu}\nabla^2 R\,\delta_{ij} - \frac{1}{2\mu}\frac{\partial^2}{\partial x_i\partial x_j}R\right]\cdot\frac{1}{8\pi(1-\nu)}$$

in this case, $j=3$. but this result can be generalized to arbitrary j.

$$\boxed{G_{ij}(\underset{\sim}{x}) = \frac{1}{8\pi\mu}\left[\delta_{ij}R_{,kk} - \frac{1}{2(1-\nu)}R_{,ij}\right]}$$

$$G_{ij}(\underset{\sim}{x}) = \frac{1}{16\pi\mu(1-\nu)}\cdot\frac{1}{R}\left[(3-4\nu)\delta_{ij} + \frac{x_i x_j}{R^2}\right]$$

Appendix

Potential Field of a Uniformly Charged Ellipsoid

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Contents

1 Problem Statement

The purpose of this document is to discuss the derivation of a very useful result, which states that **the potential field of a uniformly charged ellipsoid is a quadratic function inside the ellipsoid** [1]. Specifically, consider an ellipsoid with uniform charge density ρ inside the following region,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

Let V_0 specify the volume of the ellipsoid. The potential field is defined as

$$\phi(\mathbf{x}) \equiv \int_{V_0} \frac{\rho}{|\mathbf{x} - \mathbf{x}'|} \, dV(\mathbf{x}') \tag{2}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{x}' = (x', y', z')$.

If point $\mathbf{x}$ is inside the ellipsoid [2],

$$\phi(x, y, z) = \pi\, abc\, \rho \int_0^\infty \left[1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{c^2 + s}\right] \frac{ds}{\sqrt{\varphi(s)}} \tag{3}$$

where

$$\varphi(s) \equiv (a^2 + s)(b^2 + s)(c^2 + s) \tag{4}$$

If point $\mathbf{x}$ is outside the ellipsoid [1],

$$\phi(x, y, z) = \pi\, abc\, \rho \int_\lambda^\infty \left[1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{c^2 + s}\right] \frac{ds}{\sqrt{\varphi(s)}} \tag{5}$$

where λ is the greatest root of the equation $f(s) = 0$, where

$$f(s) \equiv \frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} + \frac{z^2}{c^2 + s} - 1 \tag{6}$$

The physical significance of function $f(s)$ is that, for each $s > 0$, the equation $f(s) = 0$ defines an ellipsoid (larger than the original ellipsoid), which is an *isosurface* of the potential field $\phi(\mathbf{x})$ generated by the original ellipsoid (with uniform charge density). Therefore, all the value of $s \in [0, \infty)$ corresponds to a family of ellipsoids, called the *confocal family*; the potential field is a constant on each ellipsoid in the family.

Physical significance: As a consequence of the above result, the second spatial derivatives of the potential field is a uniform second order tensor inside the ellipsoid. This is closely analogous to *Eshelby's Theorem*, which states that the stress field inside an ellipsoidal inclusion is uniform [3]. Landau and Lifshitz also used the above result to show that normal stress distribution inside the contact area between two smooth elastic media (Hertzian contact problem) has the ellipsoidal shape [2].

2 Orthogonal Curvilinear Coordinates

The proof of this result is best discussed using *ellipsoidal coordinates*, which is an *orthogonal curvilinear coordinate* system. In this section, we first summarize the major results concerning orthogonal curvilinear coordinates. We will then introduce the *elliptic coordinates* (2D) and *ellipsoidal coordinates* (3D) in the following sections.

Let the Cartesian coordinates be specified by $(x_1, x_2, x_3) = (x, y, z)$. An arbitrary differential length in space ds is specified by $(ds)^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$. Now consider a general orthogonal curvilinear coordinate system, (q_1, q_2, q_3), which are related to the Cartesian coordinates by

$$q_m = q_m(x_1, x_2, x_3), \quad x_m = x_m(q_1, q_2, q_3) \tag{7}$$

An arbitrary differential length in space can be expressed by

$$(ds)^2 = (h_1\, dq_1)^2 + (h_2\, dq_2)^2 + (h_3\, dq_3)^2 \tag{8}$$

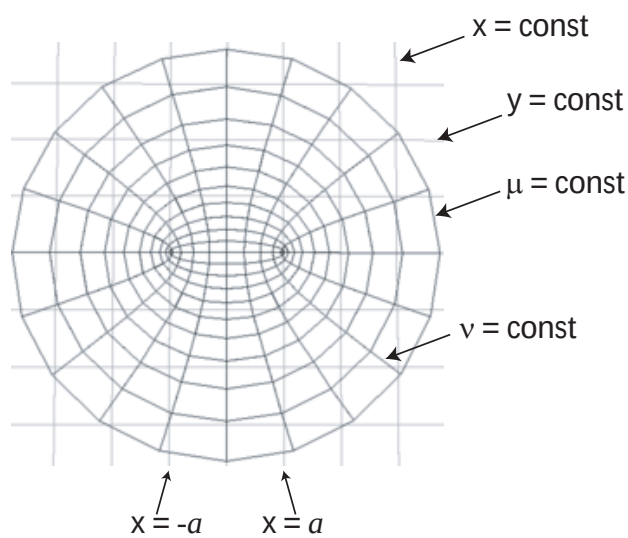

Figure 1: Contour lines in Cartesian and elliptic coordinates [5].

where h_i are called *scale factors* and have the following expressions [4].

$$\begin{aligned}(h_1)^2 &= \frac{\partial x_k}{\partial q_1}\frac{\partial x_k}{\partial q_1}\\ (h_2)^2 &= \frac{\partial x_k}{\partial q_2}\frac{\partial x_k}{\partial q_2}\\ (h_3)^2 &= \frac{\partial x_k}{\partial q_3}\frac{\partial x_k}{\partial q_3}\end{aligned} \tag{9}$$

The index notation is used here and k is a dummy index that is summed from 1 to 3. Let $\mathbf{e}_k$ be the basis vectors of the Cartesian coordinates (x_1, x_2, x_3) and let $\hat{\mathbf{e}}_k$ be the basis vectors of the curvilinear coordinates (q_1, q_2, q_3). The *gradient* of a scalar field $\phi(q_1, q_2, q_3)$ is,

$$\nabla\phi = \hat{\mathbf{e}}_1\frac{1}{h_1}\frac{\partial\phi}{\partial q_1} + \hat{\mathbf{e}}_2\frac{1}{h_2}\frac{\partial\phi}{\partial q_2} + \hat{\mathbf{e}}_3\frac{1}{h_3}\frac{\partial\phi}{\partial q_3} \tag{10}$$

The *Laplacian* of a scalar field $\phi(q_1, q_2, q_3)$ is,

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right] \tag{11}$$

In 2-dimension, the *Laplacian* of a scalar field $\phi(q_1, q_2)$ reduces to the following.

$$\nabla^2\phi = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right)\right] \tag{12}$$

3 Elliptic Coordinates

The most common definition of elliptic coordinate (μ, ν) is [5],

$$\begin{aligned}x &= a\cosh\mu\cos\nu\\ y &= a\sinh\mu\sin\nu\end{aligned}$$

With this definition, we can show that

$$\begin{aligned}
\frac{x^2}{a^2\cosh^2\mu} + \frac{x^2}{a^2\sinh^2\mu} &= \cos^2\nu + \sin^2\nu = 1 \\
\frac{x^2}{a^2\cos^2\nu} - \frac{x^2}{a^2\sin^2\nu} &= \cosh^2\mu - \sinh^2\mu = 1
\end{aligned} \tag{13}$$

Therefore, the contour lines of $\mu = \text{const}$ form a set of ellipses, and the contour lines of $\nu = \text{const}$ form a set of hyperbolas, as shown in Fig. 1. This figure also shows that in the limit of $a \to 0$, or when the distance from the origin is much greater than a, the elliptic coordinates becomes very close to polar coordinates.

The Jacobian matrix between Cartesian coordinates (x, y) and elliptic coordinates (μ, ν) is

$$J \equiv \begin{bmatrix} \frac{\partial x}{\partial \mu} & \frac{\partial x}{\partial \nu} \\ \frac{\partial x}{\partial \mu} & \frac{\partial x}{\partial \nu} \end{bmatrix} = \begin{bmatrix} a\sinh\mu\cos\nu & -a\cosh\mu\sin\nu \\ a\cosh\mu\sin\nu & a\sinh\mu\cos\nu \end{bmatrix} \tag{14}$$

The elliptic coordinates is an orthogonal coordinate system because the two columns of matrix J are orthogonal to each other. The *scale factors* are

$$\begin{aligned}
h_\mu &= \sqrt{\left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2} \\
h_\nu &= \sqrt{\left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2}
\end{aligned} \tag{15}$$

It is easy to show that

$$h_\mu = h_\nu = a\sqrt{\sinh^2\mu + \sin^2\nu} = \sqrt{\det(J)} \tag{16}$$

Therefore, the Laplacian of a scalar field ϕ is

$$\begin{aligned}
\nabla^2\phi &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x, y) \\
&= \frac{1}{a^2(\sinh^2\mu + \sin^2\nu)}\left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2}\right)\phi(\mu, \nu)
\end{aligned} \tag{17}$$

4 Alternative Definition of Elliptic Coordinates

An alternative definition of elliptic coordinates makes it more natural to generalize the concept to ellipsoidal coordinates in 3D. Consider an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{18}$$

We will assume $a > b$ without loss of generality. Now consider a family of curves defined by

$$f(s) \equiv \frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} - 1 = 0 \tag{19}$$

When $s > -b^2$, it defines an ellipse. When $-b^2 > s > -a^2$, it defines a hyperbola.

For a given (x, y), let (μ, ν) be the two largest roots of the equation $f(s) = 0$. There is a one-to-one correspondence between (x, y) and (μ, ν), if we assume $x > 0$, $y > 0$. Specifically,

$$\begin{aligned} x^2 &= \frac{(a^2+\mu)(a^2+\nu)}{a^2-b^2} \\ y^2 &= \frac{(b^2+\mu)(b^2+\nu)}{b^2-a^2} \end{aligned} \tag{20}$$

This relationship can be verified by plugging it into the definition of $f(s)$,

$$\begin{aligned} f(\mu) &= \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} - 1 = \frac{a^2+\nu}{a^2-b^2} + \frac{b^2+\nu}{b^2-a^2} - 1 = 0 \\ f(\nu) &= \frac{x^2}{a^2+\nu} + \frac{y^2}{b^2+\nu} - 1 = \frac{a^2+\mu}{a^2-b^2} + \frac{b^2+\mu}{b^2-a^2} - 1 = 0 \end{aligned} \tag{21}$$

The four components of the Jacobian matrix can be obtained.

$$\begin{aligned} \frac{\partial x}{\partial \mu} &= \frac{1}{2}\left[\frac{a^2+\nu}{(a^2+\mu)(a^2-b^2)}\right]^{1/2} \qquad \frac{\partial x}{\partial \nu} = \frac{1}{2}\left[\frac{a^2+\mu}{(a^2+\nu)(a^2-b^2)}\right]^{1/2} \\ \frac{\partial y}{\partial \mu} &= \frac{1}{2}\left[\frac{b^2+\nu}{(b^2+\mu)(b^2-a^2)}\right]^{1/2} \qquad \frac{\partial y}{\partial \nu} = \frac{1}{2}\left[\frac{b^2+\mu}{(b^2+\nu)(b^2-a^2)}\right]^{1/2} \end{aligned} \tag{22}$$

The (μ, ν) coordinate system is orthogonal because

$$\frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \nu} = 0 \tag{23}$$

Define function $\varphi(s) \equiv (a^2+s)(b^2+s)$. The *scale factors* can be expressed as

$$\begin{aligned} h_\mu &= \frac{1}{2}\sqrt{\frac{\mu-\nu}{\varphi(\mu)}} \\ h_\nu &= \frac{1}{2}\sqrt{\frac{\nu-\mu}{\varphi(\nu)}} \end{aligned} \tag{24}$$

Therefore, the Laplacian of a scalar field $\phi(\mu, \nu)$ is

$$\begin{aligned} \nabla^2\phi &= \frac{1}{h_\mu h_\nu}\left[\frac{\partial}{\partial \mu}\left(\frac{h_\nu}{h_\mu}\frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{h_\mu}{h_\nu}\frac{\partial \phi}{\partial \nu}\right)\right] \\ &= \frac{4\sqrt{\varphi(\mu)\varphi(\nu)}}{\mu-\nu}\left[\frac{\partial}{\partial \mu}\left(\sqrt{\frac{\varphi(\mu)}{\varphi(\nu)}}\frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\sqrt{\frac{\varphi(\mu)}{\varphi(\nu)}}\frac{\partial \phi}{\partial \nu}\right)\right] \\ &= \frac{4}{\mu-\nu}\left[\sqrt{\varphi(\mu)}\,\frac{\partial}{\partial \mu}\left(\sqrt{\varphi(\mu)}\,\frac{\partial \phi}{\partial \mu}\right) + \sqrt{\varphi(\nu)}\,\frac{\partial}{\partial \nu}\left(\sqrt{\varphi(\nu)}\,\frac{\partial \phi}{\partial \nu}\right)\right] \end{aligned} \tag{25}$$

For derivation details see `elliptic_coord.m`.

5 Ellipsoidal Coordinates

Generalizing the elliptic coordinates defined above, we obtain the ellipsoidal coordinates [6]. Consider an ellipsoid, Consider an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{26}$$

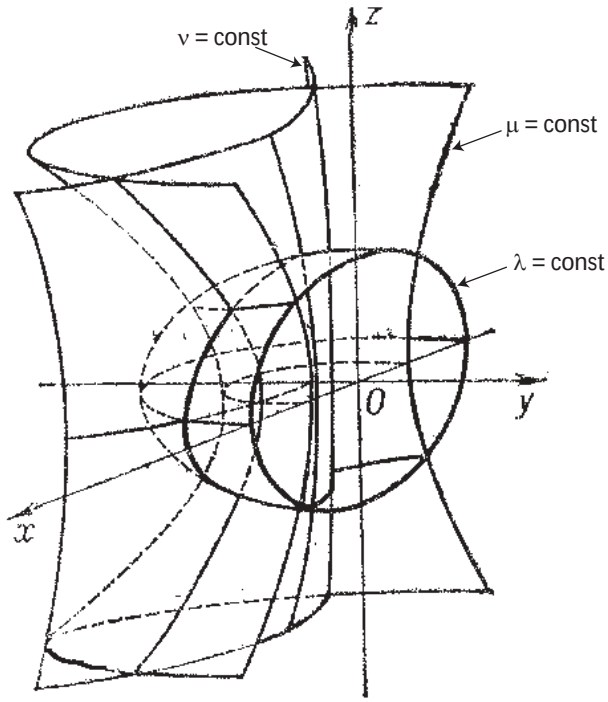

Figure 2: Isosurfaces of ellipsoidal coordinates [7].

We will assume $a > b > c$ without loss of generality. Now consider a family of curves defined by

$$f(s) \equiv \frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s} - 1 = 0 \tag{27}$$

For $\lambda > -c^2$, $f(\lambda) = 0$ defines an ellipsoid. When $-c^2 > \mu > -b^2$, $f(\mu) = 0$ defines a one-sheet hyperbola. When $-b^2 > \nu > -a^2$, $f(\nu) = 0$ defines a two-sheet hyperbola, as shown in Fig. 2.

For a given (x, y, z), let (λ, μ, ν) be the three largest roots of equation $f(s) = 0$. There is a one-to-one correspondence between (x, y, z) and (λ, μ, ν), if we assume $x > 0$, $y > 0$, $z > 0$.

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}$$

$$y^2 = \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-a^2)(b^2-c^2)} \tag{28}$$

$$z^2 = \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)} \tag{29}$$

The following limit applies,

$$\lambda > -c^2 > \mu > -b^2 > \nu > -a^2 \tag{30}$$

The nine components of the Jacobian matrix can be obtained.

$$
\begin{aligned}
\frac{\partial x}{\partial \lambda} &= \frac{1}{2}\left[\frac{(a^2+\mu)(a^2+\nu)}{(a^2+\lambda)(a^2-b^2)(a^2-c^2)}\right]^{1/2} \\
\frac{\partial x}{\partial \mu} &= \frac{1}{2}\left[\frac{(a^2+\lambda)(a^2+\nu)}{(a^2+\mu)(a^2-b^2)(a^2-c^2)}\right]^{1/2} \\
\frac{\partial x}{\partial \nu} &= \frac{1}{2}\left[\frac{(a^2+\lambda)(a^2+\mu)}{(a^2+\nu)(a^2-b^2)(a^2-c^2)}\right]^{1/2}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\frac{\partial y}{\partial \lambda} &= \frac{1}{2}\left[\frac{(b^2+\mu)(b^2+\nu)}{(b^2+\lambda)(b^2-a^2)(b^2-c^2)}\right]^{1/2} \\
\frac{\partial y}{\partial \mu} &= \frac{1}{2}\left[\frac{(b^2+\lambda)(b^2+\nu)}{(b^2+\mu)(b^2-a^2)(b^2-c^2)}\right]^{1/2} \\
\frac{\partial y}{\partial \nu} &= \frac{1}{2}\left[\frac{(b^2+\lambda)(b^2+\mu)}{(b^2+\nu)(b^2-a^2)(b^2-c^2)}\right]^{1/2}
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial \lambda} &= \frac{1}{2}\left[\frac{(c^2+\mu)(c^2+\nu)}{(c^2+\lambda)(c^2-a^2)(c^2-b^2)}\right]^{1/2} \\
\frac{\partial z}{\partial \mu} &= \frac{1}{2}\left[\frac{(c^2+\lambda)(c^2+\nu)}{(c^2+\mu)(c^2-a^2)(c^2-b^2)}\right]^{1/2} \\
\frac{\partial z}{\partial \nu} &= \frac{1}{2}\left[\frac{(c^2+\lambda)(c^2+\mu)}{(c^2+\nu)(c^2-a^2)(c^2-b^2)}\right]^{1/2}
\end{aligned} \tag{33}
$$

The (λ, μ, ν) coordinate system is orthogonal because

$$
\begin{aligned}
\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \mu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \mu} &= 0 \\
\frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial \mu}\frac{\partial z}{\partial \nu} &= 0 \\
\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \nu} &= 0
\end{aligned} \tag{34}
$$

Define function $\varphi(s) \equiv (a^2+s)(b^2+s)(c^2+s)$. The *scale factors* can be expressed as

$$
\begin{aligned}
h_\lambda &= \frac{1}{2}\sqrt{\frac{(\lambda-\mu)(\lambda-\nu)}{\varphi(\lambda)}} \\
h_\mu &= \frac{1}{2}\sqrt{\frac{(\mu-\lambda)(\mu-\nu)}{\varphi(\mu)}} \\
h_\nu &= \frac{1}{2}\sqrt{\frac{(\nu-\lambda)(\nu-\mu)}{\varphi(\nu)}}
\end{aligned} \tag{35}
$$

The Laplacian of a scalar field $\phi(\lambda, \mu, \nu)$ is,

$$
\begin{aligned}
\nabla^2\phi &= \frac{1}{h_\lambda h_\mu h_\nu}\left[\frac{\partial}{\partial \lambda}\left(\frac{h_\mu h_\nu}{h_\lambda}\frac{\partial \phi}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{h_\nu h_\lambda}{h_\mu}\frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{h_\lambda h_\mu}{h_\nu}\frac{\partial \phi}{\partial \nu}\right)\right] \\
&= \frac{4\sqrt{\varphi(\lambda)}}{(\lambda-\mu)(\lambda-\nu)}\frac{\partial}{\partial \lambda}\left(\sqrt{\varphi(\lambda)}\frac{\partial \phi}{\partial \lambda}\right) + \frac{4\sqrt{\varphi(\mu)}}{(\mu-\lambda)(\mu-\nu)}\frac{\partial}{\partial \mu}\left(\sqrt{\varphi(\mu)}\frac{\partial \phi}{\partial \mu}\right) \\
&\quad + \frac{4\sqrt{\varphi(\nu)}}{(\nu-\lambda)(\nu-\mu)}\frac{\partial}{\partial \nu}\left(\sqrt{\varphi(\nu)}\frac{\partial \phi}{\partial \nu}\right)
\end{aligned} \tag{36}
$$

For derivation details see `ellipsoidal_coord.m`.

6 Ellipsoidal Conductor

The discussion from here on follows Kellogg's book closely [1], with some variables renamed to follow the notation here. Suppose we would like to solve for the potential function in space produced by an ellipsoidal conductor that contains surface charges [1]. For a perfect conductor, the potential on its surface (as well as the interior) is a constant. Therefore, we are trying to solve the Poisson's equation,

$$\nabla^2\phi(\mathbf{x}) = 0 \tag{37}$$

subject to the boundary condition that $\phi(\mathbf{x}) = \phi_0$ when point $\mathbf{x}$ is on the ellipsoidal surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{38}$$

and that $\phi(\mathbf{x}) = 0$ as $|\mathbf{x}| \to \infty$.

Introducing the ellipsoidal coordinates (λ, μ, ν) as defined in the previous section. The surface of the (original) ellipsoid is simply the isosurface of $\lambda = 0$. The limit of $|\mathbf{x}| \to \infty$ corresponds to the limit of $\lambda \to \infty$. Therefore, when ϕ is expressed in term of the ellipsoidal coordinates, i.e. $\phi(\lambda, \mu, \nu)$, the boundary condition is very simple,

$$\begin{aligned} \phi(\lambda = 0, \mu, \nu) &= \phi_0 \\ \phi(\lambda \to \infty, \mu, \nu) &= 0 \end{aligned} \tag{39}$$

Notice that the Laplacian in the Possion's equation ($\nabla^2\phi = 0$) in the elliptical coordinates is defined in Eq. (36). A natural trial solution is a function $\phi(\lambda)$ that only depends on λ, but not on μ or ν. In this case,

$$\nabla^2\phi = \frac{4\sqrt{\varphi(\lambda)}}{(\lambda-\mu)(\lambda-\nu)} \frac{\partial}{\partial\lambda}\left(\sqrt{\varphi(\lambda)}\,\frac{\partial\phi}{\partial\lambda}\right) = 0 \tag{40}$$

$$\begin{aligned} \sqrt{\varphi(\lambda)}\,\frac{\partial\phi}{\partial\lambda} &= -\frac{E}{2} \\ \frac{\partial\phi}{\partial\lambda} &= -\frac{E}{2\sqrt{\varphi(\lambda)}} \\ \phi(\lambda) &= \int_\lambda^\infty \frac{E\,ds}{2\sqrt{\varphi(s)}} \end{aligned} \tag{41}$$

where E is a constant. The potential field inside the conductor is a constant and equals to the potential on the surface ($\lambda = 0$), which is,

$$\phi_0 = \int_0^\infty \frac{E\,ds}{2\sqrt{\varphi(s)}} \tag{42}$$

The surface charge of the conductor $\sigma(\mathbf{x})$ can be obtained from the following relationship.

$$\frac{\partial\phi(\mathbf{x})}{\partial n_+} = -4\pi\sigma(\mathbf{x}) \tag{43}$$

where $\frac{\partial}{\partial n_+}$ is the gradient along the surface normal.

$$\begin{aligned}\sigma &= -\frac{1}{4\pi}\frac{\partial\phi(\mathbf{x})}{\partial n_+} \\ &= -\frac{1}{4\pi}\left(\frac{1}{h_\lambda}\frac{\partial\phi(\lambda)}{\partial\lambda}\right)_{\lambda=0}\end{aligned} \tag{44}$$

Notice that at $\lambda = 0$, $h_\lambda = \sqrt{\mu\nu/\varphi(\lambda)}/2$. Therefore,

$$\sigma = \frac{1}{4\pi}\left(\frac{2\sqrt{\varphi(\lambda)}}{\sqrt{\mu\nu}}\frac{E}{2\varphi(\lambda)}\right) = \frac{E}{4\pi\sqrt{\mu\nu}} \tag{45}$$

On the surface of the ellipsoid, $\lambda = 0$,

$$\mu\nu = a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) \tag{46}$$

The equation of the plane tangent to ellipsoid at point (x, y, z) is

$$(X - x)\frac{x}{a^2} + (Y - y)\frac{y}{b^2} + (Z - z)\frac{z}{c^2} = 0 \tag{47}$$

The surface normal of the tangent plane is

$$\mathbf{n} = \left(\frac{x}{a^2}\,,\ \frac{y}{b^2}\,,\ \frac{z}{c^2}\right)\Big/\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} \tag{48}$$

The shortest distance from the origin to this plane is

$$p = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} = \frac{abc}{\sqrt{\mu\nu}} \tag{49}$$

Therefore, the surface charge density is

$$\sigma = \frac{E}{4\pi\,abc}\,p = \frac{E}{4\pi\,abc}\,\frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \tag{50}$$

This result is related to the problem of a uniformly charged ellipsoid. As will be shown in the following section, the above expression of σ is exactly the amount of charge contained in a thin shell between two similar ellipsoids, in the limit of shell thickness going to zero.

7 Ellipsoidal Shell

Consider a set of similar ellipsoids,

$$\frac{x^2}{(au)^2} + \frac{y^2}{(bu)^2} + \frac{z^2}{(cu)^2} = 1\ ,\quad \text{equivalently}\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = u^2 \tag{51}$$

whose semi-axes are au, bu and cu. They are simply the original ellipsoid scaled by a factor u in all three directions. Notice that this family of ellipsoids are different from the family of ellipsoids defined by $f(\lambda) = 0$ (whose shapes are not similar to each other). For $0 < u < 1$, these ellipsoids

are smaller than the original ellipsoid (while for $0 < \lambda < \infty$ the ellipsoids defined by $f(\lambda) = 0$ are all larger than the original ellipsoid).

Consider an ellipsoidal shell contained between two ellipsoids defined by u_1 and $u_2 = u_1 + \Delta u$. Let the volume density of the charge distribution inside the shell to be a constant ρ. In the limit of $\Delta u \to 0$, the shell reduces to a surface with a surface charge density σ. Obviously, the surface charge density is proportional to the local thickness of the shell, Δh, i.e.,

$$\sigma = \rho\, \Delta h \tag{52}$$

Let (ux, uy, uz) be a point on the surface of the ellipsoid defined by u. Let $p(x, y, z, u)$ be the shortest distance from the origin to the plane tangent to the ellipsoid at point (ux, uy, uz). Then,

$$\begin{aligned} p(x,y,z,u) &= \frac{u}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \\ \Delta h(x,y,z) &= \frac{\Delta u}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \\ \sigma &= \frac{\rho\ \Delta u}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \end{aligned} \tag{53}$$

Compare this with Eq. (50), we can conclude that the surface charge of the shell is the same as the equilibrium surface charge of a ellipsoidal conductor. The correspondence is made complete if we set $u_1 = 1$, $u_2 = 1 + \Delta u$, and

$$\begin{aligned} \frac{E}{4\pi abc} &= \rho \Delta u \\ E &= 4\pi\, abc\, \rho \Delta u \end{aligned} \tag{54}$$

This means that the potential field produced by this ellipsoidal shell ($u_1 = 1$, $u_2 = 1 + \Delta u$) is

$$\phi(\mathbf{x}) = \phi(\lambda) = 2\pi\, abc\, \rho\, \Delta u \int_\lambda^\infty \frac{ds}{\sqrt{\varphi(s)}} \tag{55}$$

The potential field inside the ellipsoidal shell is a constant and equals to the potential on the surface ($\lambda = 0$), which is,

$$\phi_0 = 2\pi\, abc\, \rho\, \Delta u \int_0^\infty \frac{ds}{\sqrt{\varphi(s)}} \tag{56}$$

In summary, Eq. (55) describes the potential generated by an ellipsoidal shell with uniform density ρ, whose boundary is the original ellipsoid and a similar ellipsoid scaled by a factor $(1 + \Delta u)$. This result can be generalized to an ellipsoidal shell between $u_1 = u$ and $u_2 = u + \Delta u$ for arbitrary u. The potential field at a point $\mathbf{x} = (x, y, z)$ outside this shell is,

$$\phi(\mathbf{x}) = 2\pi\, abc\, \rho\, u^2\, \Delta u \int_{\lambda(u)}^\infty \frac{ds}{\sqrt{\varphi(u,s)}} \tag{57}$$

where $\lambda(u)$ is the greatest root of equation $f(u, s) = 0$ for given (x, y, z) and u. $f(u, s)$ and $\varphi(u, s)$ are generalization of the previously defined functions $f(s)$ and $\varphi(s)$.

$$f(u,s) \equiv \frac{x^2}{a^2u^2 + \lambda} + \frac{y^2}{b^2u^2 + \lambda} + \frac{z^2}{c^2u^2 + \lambda} - 1 \tag{58}$$

$$\varphi(u,s) \equiv (a^2u^2 + s)(b^2u^2 + s)(c^2u^2 + s) \tag{59}$$

The factor u^2 in Eq. (57) accounts for the fact that surface area of the scaled shell and hence its total charge content is u^2 times those of the original shell ($u = 1$).

8 Uniformly Charged Ellipsoid

A uniformly charged ellipsoid can be considered as a collection of many layers of ellipsoid shells considered above. For a point $\mathbf{x}$ outside the ellipsoid, its potential value should be an integral of u from 0 to 1,

$$U_{\mathrm{e}}(\mathbf{x}) = 2\pi\, abc\, \rho \int_0^1 u^2 \int_{\lambda(u)}^{\infty} \frac{ds}{\sqrt{\varphi(u,s)}}\, du \tag{60}$$

Define new variables $v \equiv \lambda(u)/u^2$ and $t \equiv s/u^2$. Then $\phi(u,s) = u^2\phi(t)$.

$$U_{\mathrm{e}}(\mathbf{x}) = 2\pi\, abc\, \rho \int_0^1 u \int_{v}^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du \tag{61}$$

Perform integration by parts on $\int du$,

$$\int_0^1 u \int_{v}^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du = \left[\frac{u^2}{2}\int_{v}^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\right]_0^1 + \frac{1}{2}\int_0^1 u^2 \frac{1}{\sqrt{\varphi(v)}}\frac{dv}{du}\, du \tag{62}$$

Notice that v is the greatest root of equation,

$$\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} = u^2 \tag{63}$$

For $u = 1$, $v = \lambda$, while for $u \to 0$, $v \to \infty$. Hence

$$\left[\frac{u^2}{2}\int_{v}^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\right]_0^1 = \frac{1}{2}\int_{\lambda}^{\infty} \frac{dt}{\sqrt{\varphi(t)}} \tag{64}$$

$$\begin{aligned}\frac{1}{2}\int_0^1 u^2 \frac{1}{\sqrt{\varphi(v)}}\frac{dv}{du}\, du &= \frac{1}{2}\int_{\infty}^{\lambda} u^2 \frac{1}{\sqrt{\varphi(v)}}\, dv \\ &= -\frac{1}{2}\int_{\lambda}^{\infty} \left(\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v}\right)\frac{1}{\sqrt{\varphi(v)}}\, dv \end{aligned} \tag{65}$$

Therefore, the potential outside the ellipsoid is

$$U_{\mathrm{e}}(\mathbf{x}) = \pi\, abc\, \rho \int_{\lambda}^{\infty} \left(1 - \frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v}\right)\frac{1}{\sqrt{\varphi(v)}}\, dv \tag{66}$$

where λ is the greatest root of equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \tag{67}$$

as previously defined.

Let us now consider the potential field at a point $\mathbf{x} = (x, y, z)$ inside the ellipsoid. Here we have to cut the ellipsoid into two parts. Point $\mathbf{x}$ is on the outside of part 1, but is on the inside of part 2. Let u_0 correspond to the ellipsoid that pass through point $\mathbf{x}$, i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = u_0^2 \tag{68}$$

Therefore, part 1 contains the ellipsoidal shells with $0 < u < u_0$ and part 2 contains the ellipsoidal shells with $u_0 < u < 1$. The potential at point $\mathbf{x}$ from an ellipsoidal shell in part 1 has the same expression as the above. But potential at point $\mathbf{x}$ from an ellipsoidal shell in part 2 equals to the potential value at the shell surface (because point $\mathbf{x}$ is inside the shell). Therefore, the total potential at an interior point $\mathbf{x}$ is,

$$U_{\rm i}(\mathbf{x}) = 2\pi\, abc\,\rho \left[\int_0^{u_0} u \int_v^\infty \frac{dt}{\sqrt{\varphi(t)}}\, du + \int_{u_0}^1 u \int_0^\infty \frac{dt}{\sqrt{\varphi(t)}}\, du \right] \tag{69}$$

Perform integration by parts on the first integral,

$$\int_0^{u_0} u \int_v^\infty \frac{dt}{\sqrt{\varphi(t)}}\, du = \left[\frac{u^2}{2} \int_v^\infty \frac{dt}{\sqrt{\varphi(t)}} \right]_0^{u_0} + \frac{1}{2} \int_0^{u_0} u^2 \frac{1}{\sqrt{\varphi(v)}} \frac{dv}{du}\, du \tag{70}$$

Notice that when $u = 0$, $v = \infty$, and when $u = u_0$, $v = 0$. Hence

$$\begin{aligned} \int_0^{u_0} u \int_v^\infty \frac{dt}{\sqrt{\varphi(t)}}\, du &= \frac{u_0^2}{2} \int_0^\infty \frac{dt}{\sqrt{\varphi(t)}} - \frac{1}{2} \int_0^\infty u^2 \frac{1}{\sqrt{\varphi(v)}}\, dv \\ &= \frac{u_0^2}{2} \int_0^\infty \frac{dt}{\sqrt{\varphi(t)}} - \frac{1}{2} \int_0^\infty \left(\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} \right) \frac{1}{\sqrt{\varphi(v)}}\, dv \end{aligned}$$

The second integral in Eq. (69) can be carried out because the inner integral is a constant.

$$\int_{u_0}^1 u \int_0^\infty \frac{dt}{\sqrt{\varphi(t)}}\, du = \frac{1-u_0^2}{2} \int_0^\infty \frac{dt}{\sqrt{\varphi(t)}} \tag{71}$$

Therefore, the total potential at an interior point $\mathbf{x}$ is,

$$U_{\rm i}(\mathbf{x}) = \pi\, abc\,\rho \int_0^\infty \left(1 - \frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} \right) \frac{1}{\sqrt{\varphi(v)}}\, dv \tag{72}$$

Notice that the only difference between Eq. (66) and Eq. (72) is in the lower limit of integration (λ versus 0). Because the range of integration is constant, the potential field inside the ellipsoid, $U_{\rm i}(\mathbf{x})$, is simply a quadratic function of space.

9 Special Cases

9.1 Uniformly Charged Sphere

The situation of $a = b = c$ describes a sphere with radius a. When the sphere has a uniform charge density ρ, the potential distribution inside the sphere is,

$$\phi(x,y,z) = \pi\, a^3\, \rho\, (A - B\, x^2 - B\, y^2 - B\, z^2) = \pi\, a^3\, \rho\, (A - B\, r^2) \tag{73}$$

where

$$A = \int_0^\infty \frac{ds}{(a^2+s)^{3/2}} = \frac{2}{a} \tag{74}$$

$$B = \int_0^\infty \frac{ds}{(a^2+s)^{5/2}} = \frac{2}{3a^3} \tag{75}$$

Therefore

$$\phi(r) = \frac{2\pi\rho}{3}(3a^2 - r^2) \tag{76}$$

The potential value on the sphere surface is

$$\phi(r = a) = \frac{2\pi\rho}{3}(3a^2 - a^2) = \frac{4\pi\rho}{3}a^2 \tag{77}$$

Notice that $Q = 4\pi\, a^3\rho/3$ is the total charge contained in the sphere. Hence

$$\phi(r = a) = \frac{Q}{a} \tag{78}$$

This is equivalent to the potential produced by a point charge at origin. This confirms Newton's Theorem, which states that the potential field outside a uniformly charged sphere is equivalent to that produced by a point charge located at the center of the sphere. The potential field outside the sphere ($r > a$) is

$$\phi(r) = \frac{Q}{r} \tag{79}$$

9.2 Charged Metal Disc

Consider the case of $a = b$ and $c \to 0$. In this limit, the region inside the ellipsoid reduces to a circle of radius a in the x-y plane ($z = 0$). The potential distribution inside this area is,

$$\phi(x, y) = \iiint \frac{\rho\, dx'\, dy'\, dz'}{\sqrt{(x - x')^2 + (y - y')^2 + z'^2}}$$

Because the integration limit for z' is $\pm\, c\sqrt{1 - (x'/a)^2 - (y'/a)^2}$,

$$\phi(x, y) = 2\rho\, c \iint_{x'^2+y'^2\le a^2} \frac{dx'\, dy'}{\sqrt{(x - x')^2 + (y - y')^2}} \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \tag{80}$$

Using the results obtained above, the potential must be a quadratic function inside the area of radius a,

$$\phi(x, y) = \pi\, a^2\, c\, \rho\, (A - B\, x^2 - B\, y^2)$$

$$A = \int_0^\infty \frac{ds}{(a^2 + s)\sqrt{s}} = \frac{\pi}{a} \tag{81}$$

$$B = \int_0^\infty \frac{ds}{(a^2 + s)^2\sqrt{s}} = \frac{\pi}{2a^3} \tag{82}$$

In other words, we have obtained the following relationship,

$$\begin{aligned} \iint_{x'^2+y'^2\le a^2} \frac{dx'\, dy'}{\sqrt{(x - x')^2 + (y - y')^2}} \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} &= \frac{\pi}{2}\, a^2\, (A - B\, x^2 - B\, y^2) \\ &= \frac{\pi^2}{4\, a}(2a^2 - r^2) \end{aligned} \tag{83}$$

where $r = \sqrt{x^2 + y^2}$. The potential value is maximum at the circumference of the circle, $r = a$,

$$\phi(r = a) = 2\rho\, c\, \frac{\pi^2}{4\, a}\, a^2 = \frac{\pi\, Q}{3\, a} \tag{84}$$

where $Q = \frac{3}{4}\pi\, a^2\, c\, \rho$ is the total charge in the ellipsoid.

The results obtained here can be used to model a very thin circular metal (conductor) plate with radius a. The total charge in the metal plate is Q, whereas the equilibrium charge distribution on the surface is

$$\sigma(x,y) = \frac{3\,Q}{2\pi\, a^2}\sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \tag{85}$$

The potential on the metal surface is a constant

$$\phi_0 = \frac{\pi\, Q}{3\, a} \tag{86}$$

10 Application to Hertz Contact Problem

For simplicity, let us consider a rigid sphere of radius R indenting an elastic half space. The discussion here follows closely that in Landau and Lifshitz [2]. Choose the coordinate system such that the elastic half space occupies the $z \leq 0$ domain and the $z = 0$ plane is the surface of the half space. The Boussinesq solution tells us about the surface displacement of the elastic half space in response to a point force of magnitude F acting at the origin in the $-z$ direction.

$$u_z = -\frac{F(1-\nu^2)}{\pi\, E}\,\frac{1}{r} \tag{87}$$

where $E = 2\mu(1+\nu)$ is the Young's modulus of the elastic half space.

The shape of the indentor can be described by function

$$u_0(x,y) = \frac{x^2}{2R} + \frac{y^2}{2R} \tag{88}$$

Let d be the indentation depth. Hence inside the region of contact (S), the surface displacement of the half space is

$$u_z(x,y) = -d + u_0(x,y) = -d + \frac{x^2}{2R} + \frac{y^2}{2R} \tag{89}$$

Let $p_z(x,y)$ be the normal stress on the surface of the half space. Inside the region of contact, $p_z(x,y) < 0$; outside the region of contact, $p_z(x,y) = 0$. The total indenting force F is,

$$F = \iint_S -p_z(x,y)\, dx\, dy \tag{90}$$

Using the Boussinesq solution, the surface stress $p_z(x,y)$ and the surface displacement $u_z(x,y)$ are related by,

$$u_z(x,y) = \frac{1-\nu^2}{\pi\, E}\iint_S \frac{p_z(x',y')}{\sqrt{(x-x')^2 + (y-y')^2}}\, dx'\, dy' \tag{91}$$

Therefore, our task is to find a function $p_z(x,y)$ that satisfies the following condition,

$$\iint_S \frac{p_z(x',y')}{\sqrt{(x-x')^2 + (y-y')^2}}\, dx'\, dy' = \frac{\pi\, E}{1-\nu^2}\left(-d + \frac{x^2}{2R} + \frac{y^2}{2R}\right) \tag{92}$$

By symmetry, we expect the contact area to be a circle, and let a be its radius. Motivated by Eq. (83), we postulate the following form for $p_z(x, y)$,

$$p_z(x, y) = -p_0 \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \tag{93}$$

The constant p_0 is related with F by

$$\begin{aligned} F &= p_0 \iint_S \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2}\, dx\, dy \\ &= p_0 \frac{2\,\pi\, a^2}{3} \\ p_0 &= \frac{3\,F}{2\,\pi\, a^2} \end{aligned} \tag{94}$$

Plug the expression of $p_z(x, y)$ into Eq. (92), and using Eq. (83), we have

$$\begin{aligned} -p_0 \frac{\pi^2}{4\,a} (2a^2 - r^2) &= \frac{\pi\, E}{1 - \nu^2} \left(-d + \frac{r^2}{2R}\right) \\ -\frac{3\,F}{2\,\pi\, a^2} \frac{\pi^2}{4\,a} (2a^2 - r^2) &= \frac{\pi\, E}{1 - \nu^2} \left(-d + \frac{r^2}{2R}\right) \\ \frac{3\,(1 - \nu^2)\,F}{8\,E\,a^3} (-2a^2 + r^2) &= -d + \frac{r^2}{2R} \end{aligned} \tag{95}$$

Therefore,

$$\begin{aligned} \frac{1}{R} &= \frac{3\,(1 - \nu^2)\,F}{4\,E\,a^3} \\ a &= \left[\frac{3\,(1 - \nu^2)\,F\,R}{4\,E}\right]^{1/3} \end{aligned} \tag{96}$$

The indentation depth is

$$d = \frac{3\,(1 - \nu^2)\,F}{8\,E\,a^3}\, 2a^2 = \frac{3\,(1 - \nu^2)\,F}{4\,E\,a} = \left[\frac{3\,(1 - \nu^2)}{4\,E}\right]^{2/3} \frac{F^{2/3}}{R^{1/3}} \tag{97}$$

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A Matlab Files for Analytic Derivation

```
% File: elliptic_coord.m
% Purpose: analytic derivation of the properties of elliptic coordinates

syms x y a b mu nu

x = sqrt( (a^2+mu)*(a^2+nu)/(a^2-b^2) );
y = sqrt( (b^2+mu)*(b^2+nu)/(b^2-a^2) );

dxdmu = simplify(diff(x,mu));
dxdnu = simplify(diff(x,nu));
dydmu = simplify(diff(y,mu));
dydnu = simplify(diff(y,nu));

disp('dxdmu='); pretty(dxdmu);
disp('dxdnu='); pretty(dxdnu);
disp('dydmu='); pretty(dydmu);
disp('dydnu='); pretty(dydnu);

%check orthogonality
simplify(dxdmu*dxdnu + dydmu*dydnu)

h_mu = simplify( sqrt( (dxdmu)^2 + (dydmu)^2 ) );
h_nu = simplify( sqrt( (dxdnu)^2 + (dydnu)^2 ) );

disp('h_mu='); pretty(h_mu);
disp('h_nu='); pretty(h_nu);
```

```
% File: ellipsoidal_coord.m
% Purpose: analytic derivation of the properties of ellipsoidal coordinates

syms x y z a b c lm mu nu

x = sqrt( (a^2+lm)*(a^2+mu)*(a^2+nu)/(a^2-b^2)/(a^2-c^2) );
y = sqrt( (b^2+lm)*(b^2+mu)*(b^2+nu)/(b^2-a^2)/(b^2-c^2) );
z = sqrt( (c^2+lm)*(c^2+mu)*(c^2+nu)/(c^2-a^2)/(c^2-b^2) );

dxdlm = simplify(diff(x,lm));
dxdmu = simplify(diff(x,mu));
dxdnu = simplify(diff(x,nu));
dydlm = simplify(diff(y,lm));
dydmu = simplify(diff(y,mu));
dydnu = simplify(diff(y,nu));
dzdlm = simplify(diff(z,lm));
dzdmu = simplify(diff(z,mu));
dzdnu = simplify(diff(z,nu));

disp('dxdlm='); pretty(dxdlm);
disp('dxdmu='); pretty(dxdmu);
disp('dxdnu='); pretty(dxdnu);
disp('dydlm='); pretty(dydlm);
disp('dydmu='); pretty(dydmu);
disp('dydnu='); pretty(dydnu);
disp('dzdlm='); pretty(dzdlm);
disp('dzdmu='); pretty(dzdmu);
disp('dzdnu='); pretty(dzdnu);

%check orthogonality
[ simplify(dxdlm*dxdmu + dydlm*dydmu + dzdlm*dzdmu)
  simplify(dxdmu*dxdnu + dydmu*dydnu + dzdmu*dzdnu)
  simplify(dxdlm*dxdnu + dydlm*dydnu + dzdlm*dzdnu)
]

h_lm = simplify( sqrt( (dxdlm)^2 + (dydlm)^2 + (dzdlm)^2 ) );
h_mu = simplify( sqrt( (dxdmu)^2 + (dydmu)^2 + (dzdmu)^2 ) );
h_nu = simplify( sqrt( (dxdnu)^2 + (dydnu)^2 + (dzdnu)^2 ) );

disp('h_lm='); pretty(h_lm);
disp('h_mu='); pretty(h_mu);
disp('h_nu='); pretty(h_nu);
```

References

[1] O. D. Kellogg, *Foundations of Potential Theory*, (Dover, New York, 1953).

[2] L. D. Landau and E. M. Lifshitz, *Mechanics*, (Permagon, New York, 1959).

[3] C. R. Weinberger, W. Cai, D. M. Barnett, Micromechanics lecture notes, `http://micro.stanford.edu/∼me340b`

[4] M. H. Sadd, *Elasticity: Theory, Applications and Numerics*, (Elsevier, New York, 2005).

[5] `http://en.wikipedia.org/wiki/Elliptic_coordinates`

[6] `http://en.wikipedia.org/wiki/Ellipsoidal_coordinates`

[7] `http://eom.springer.de/E/e035420.htm`