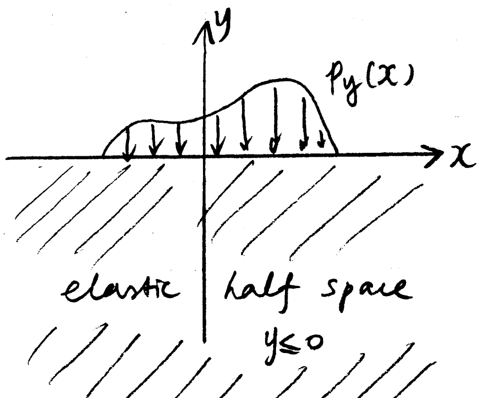
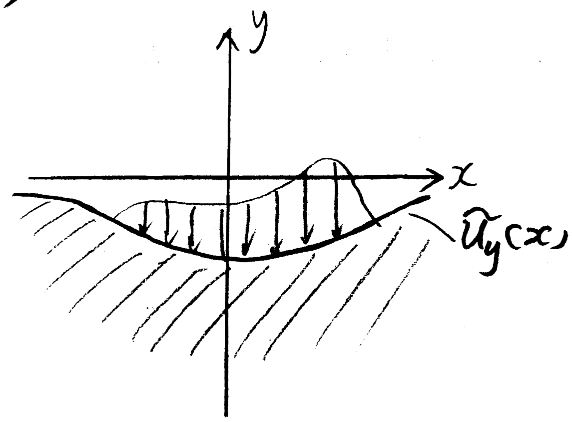
§1. Problem Statement

Suppose the top surface ($y=0$) of an elastic half-space is subjected to an arbitrary load distribution $p_y(x)$,
What will be the displacement field (shape change) of the top surface? i.e. $u_y(x, y=0)$?





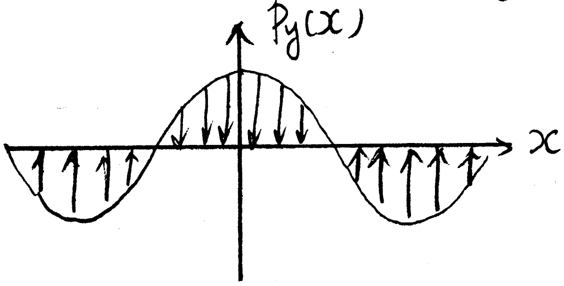
introduce new notation $\tilde{u}_y(x) \equiv u_y(x, y=0)$
for surface displacement.

We can consider the half-space as a linear system.

Input: or perterbation surface load $p_y(x)$ $\rightarrow$ "system" elastic half-space $\rightarrow$ Output: or response surface displacement $\tilde{u}_y(x)$

* Barber solved this problem following the solution he obtained in polar coordinates
Here we will solve it using the Fourier method we developed before.

§2 Sinusoidal Loading



Consider the special case of

$$P_y(x) = P_0 \cos\lambda x$$

for some arbitrary λ.

(alternating compressive and tensile loading)

B.C. $\sigma_{yy}(x, y=0) = -P_y(x) = -P_0 \cos\lambda x$

$\sigma_{xy}(x, y=0) = 0$ (frictionless, no tangential force)

Trial Solution:

$$\boxed{\phi(x,y) = \cos\lambda x \cdot (A + By) \cdot e^{\lambda y}} \qquad (\lambda > 0)$$

* We do not include the $e^{-\lambda y}$ term because we want the solution to remain finite when $y \to -\infty$

$$\sigma_{xx} = \phi_{,yy} = \cos\lambda x \left[(A\lambda^2 + 2B\lambda) + B\lambda^2 y \right] e^{\lambda y}$$

$$\sigma_{yy} = \phi_{,xx} = -\cos\lambda x \cdot \lambda^2 \cdot (A + By)\, e^{\lambda y}$$

$$\sigma_{xy} = -\phi_{,xy} = \sin\lambda x \cdot \lambda \cdot \left[(A\lambda + B) + B\lambda y \right] e^{\lambda y}$$

at $y = 0$, $\sigma_{xy} = (A\lambda + B)\cdot\lambda \cdot \sin\lambda x = 0$

$\therefore A\lambda + B = 0, \quad B = -A\lambda$

$$\sigma_{yy} = -\cos\lambda x \cdot \lambda^2 \cdot A = -P_0 \cos\lambda x$$

$\therefore A = \dfrac{P_0}{\lambda^2}$

$$\boxed{\begin{aligned} \sigma_{xx} &= -P_0 \cos\lambda x\,(1 + \lambda y)\, e^{\lambda y} \\ \sigma_{yy} &= -P_0 \cos\lambda x\,(1 - \lambda y)\, e^{\lambda y} \\ \sigma_{xy} &= -P_0 \lambda \sin\lambda x \cdot y \cdot e^{\lambda y} \end{aligned}}$$

$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$

(plane strain)

The next step is to find the displacement field.

Start with strain field (Generalized Hooke's Law in Plane Strain)

$$
\begin{cases}
\varepsilon_{xx} = \dfrac{1-\nu^2}{E}\sigma_{xx} - \dfrac{\nu(1+\nu)}{E}\sigma_{yy} \\
\varepsilon_{yy} = -\dfrac{\nu(1+\nu)}{E}\sigma_{xx} + \dfrac{1-\nu^2}{E}\sigma_{yy} \\
\varepsilon_{xy} = \dfrac{1}{2\mu}\sigma_{xy} = \dfrac{1+\nu}{E}\sigma_{xy}
\end{cases}
$$

$$
\boxed{
\begin{aligned}
\varepsilon_{xx} &= -\frac{p_0}{E}\cos\lambda x\,\left[(1-\nu-2\nu^2) + (1+\nu)\lambda y\right] e^{\lambda y} \\
\varepsilon_{yy} &= -\frac{p_0}{E}\cos\lambda x\,\left[(1-\nu-2\nu^2) - (1+\nu)\lambda y\right] e^{\lambda y} \\
\varepsilon_{xy} &= -\frac{p_0\lambda}{E}\sin\lambda x \cdot (1+\nu)\cdot y \cdot e^{\lambda y}
\end{aligned}
}
$$

Next, integrate to find displacement field

$$
\boxed{
\begin{aligned}
u_x &= -\frac{p_0}{\lambda E}\sin\lambda x\,\left[(1-\nu-2\nu^2) + (1+\nu)\lambda y\right] e^{\lambda y} + C \\
u_y &= -\frac{p_0}{\lambda E}\cos\lambda x\,\left[(2-2\nu^2) - (1+\nu)\lambda y\right] e^{\lambda y} + D
\end{aligned}
}
$$

the displacement field on the top surface ($y=0$) is:

$$
\boxed{
\begin{aligned}
\tilde{u}_x(x) &= -\frac{p_0}{\lambda E}\sin\lambda x\,(1-\nu-2\nu^2) + C \\
\tilde{u}_y(x) &= -\frac{p_0}{\lambda E}\cos\lambda x\,(2-2\nu^2) + D
\end{aligned}
}
$$

rigid-body translation, set to zero

Recall that the input (perturbation) is $p_y(x) = p_0 \cos\lambda x$,

the output (response) is

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{\lambda E}\cdot p_y(x)$$

* Notice that the response is inversely proportional to the wave vector λ of the perturbation

§3. Fourier Transform

Just as <u>Fourier Series</u> can be applied to a function defined on a finite domain, the <u>Fourier Transform</u> can be applied to a function $f(x)$ defined on the infinite domain $-\infty < x < \infty$.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k)\, e^{ikx}\, dk \qquad \leftarrow \text{inverse F.T.}$$

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx \qquad \leftarrow \text{Fourier transform}$$

common Fourier Transform Pairs

$f(x)$	$\hat{f}(k)$
$\delta(x)$	1
1	$2\pi\,\delta(k)$
$\frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{x^2}{2\sigma^2}}$	$e^{-\frac{\sigma^2 k^2}{2}}$
$e^{-\lambda\lvert x\rvert}$	$\frac{2\lambda}{k^2+\lambda^2}$
$\frac{1}{x}$	$-\pi i\,\mathrm{sgn}(k)$
$\log x$	$-\frac{\pi}{\lvert k\rvert}$
$f'(x)$	$ik\cdot\hat{f}(k)$

* $\delta(x)$ is Dirac-delta function defined as $\delta(x) = \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases}$ and $\int_{-\infty}^{+\infty} \delta(x)\, f(x)\, dx = f(0)$

* $\mathrm{sgn}(k) = \begin{cases} +1 & k>0 \\ 0 & k=0 \\ -1 & k<0 \end{cases}$

$$\hat{f}(k) = \mathcal{F}[f(x)]$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}(k)]$$

Recall that when

$$p_y(x) = p_0 \cos \lambda x$$

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{\lambda} \cdot p_0 \cos \lambda x \qquad (\text{for } \lambda > 0)$$

Obviously, if λ can be either positive or negative. then

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|\lambda|} \cdot p_0 \cos \lambda x .$$

when $p_y(x) = p_0 \sin \lambda x$, we can easily show that

$$\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|\lambda|} \cdot p_0 \sin \lambda x .$$

These two expression can be written together as

$$\boxed{\begin{aligned} &\text{when } p_y(x) = p_0 e^{ik_0 x} \\ &\tilde{u}_y(x) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|k_0|} \cdot (p_0 e^{ik_0 x}) \end{aligned}}$$

Introduce the Fourier transform of $p_y(x)$ and $\tilde{u}_y(x)$

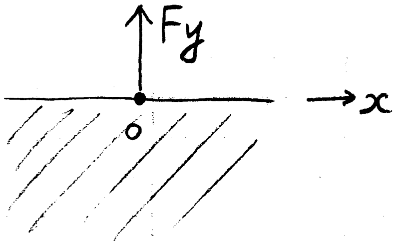
$$\hat{p}_y(k) = \int_{-\infty}^{+\infty} p_y(x) e^{-ikx} dx, \quad p_y(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{p}_y(k) e^{ikx} dk$$

$$\hat{u}_y(k) = \int_{-\infty}^{+\infty} \tilde{u}_y(x) e^{-ikx} dx, \quad \tilde{u}_y(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}_y(k) e^{ikx} dk$$

The above relation in the box can be rewritten in the Fourier space as

$$\boxed{\hat{u}_y(k) = -\frac{2(1-\nu^2)}{E} \cdot \frac{1}{|k|} \cdot \hat{p}_y(k)}$$

§4. Point Force Loading



$$p_y(x) = -F_y\,\delta(x) \quad \left(\text{concentrated load at } x=0\right)$$

In Fourier space:

$$\hat{p}_y(k) = -F_y$$

$$\hat{u}_y(k) = \frac{2(1-\nu^2)}{E}\cdot\frac{F_y}{|k|}$$

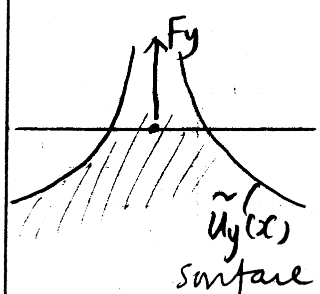
Back to real space (inverse F.T.)

$$\tilde{u}_y(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{u}_y(k)\,e^{ikx}\,dk$$

$$= \frac{2(1-\nu^2)}{E}\cdot F_y\cdot \mathrm{Re}\left[\mathcal{F}^{-1}\left[\frac{1}{|k|}\right]\right] \quad \left(\text{only take the real part}\right)$$

$$= \frac{2(1-\nu^2)}{E}\cdot F_y\cdot \mathrm{Re}\left[-\frac{1}{\pi}\log x\right]$$

$$= -\frac{2(1-\nu^2)}{\pi E}\cdot F_y\cdot\log|x| \quad \left(\begin{array}{l} * \text{ when } x<0 \\ \log x = \log|x| + \pi i\end{array}\right)$$



$$\therefore\ \tilde{u}_y(x) = -\frac{2(1-\nu^2)}{\pi E}\cdot F_y\cdot\log|x| + C$$

↑ arbitrary constant

notice that $E = 2\mu(1+\nu)$

also in plane strain $\kappa = 3-4\nu$, $\kappa+1 = 4(1-\nu)$

$$\frac{2(1-\nu^2)}{E} = \frac{2(1-\nu)(1+\nu)}{2\mu(1+\nu)} = \frac{1-\nu}{\mu} = \frac{\kappa+1}{4\mu}$$

κ: Kolosov constant

$$\boxed{\tilde{u}_y(x) = -\frac{\kappa+1}{4\pi\mu}\cdot F_y\cdot\log|x|}$$

What is the surface displacement in the x-direction?

Recall when $p_y(x) = p_0 \cos\lambda x$

$$\tilde{u}_x(x) = -\frac{1-\nu-2\nu^2}{E} \cdot \frac{1}{\lambda} \cdot p_0 \sin\lambda x$$

We can show that when $p_y(x) = p_0 e^{ik_0 x}$

$$\tilde{u}_x(x) = -\frac{1-\nu-2\nu^2}{E} \cdot \frac{-i}{|k_0|} \cdot (p_0 e^{ik_0 x})$$

Hence

$$\boxed{\hat{u}_x(k) = -\frac{1-\nu-2\nu^2}{E} \cdot \frac{-i}{|k_0|} \cdot \hat{p}_y(k)}$$

In response to a point load, $p_y(x) = -F_y\,\delta(x)$,

$$\hat{u}_x(k) = \frac{1-\nu-2\nu^2}{E} \cdot \frac{-i}{|k_0|} \cdot F_y$$

$$\tilde{u}_x(x) = \frac{1-\nu-2\nu^2}{E} \cdot F_y \cdot \mathrm{Re}\left[\mathcal{F}^{-1}\left[\frac{-i}{|k|}\right]\right]$$

$$= \frac{1-\nu-2\nu^2}{E} \cdot F_y \cdot \mathrm{Im}\left[\frac{1}{\pi}\log x\right]$$

$$= -\frac{1-\nu-2\nu^2}{E} \cdot F_y \cdot H(-x)$$

$$\left(* \ \mathrm{Im}[\log(x)] = \begin{cases} 0 & x>0 \\ \pi & x<0 \end{cases} \qquad H(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} \right)$$

step function

Remember that we can add an arbitrary constant to $\tilde{u}_y(x)$

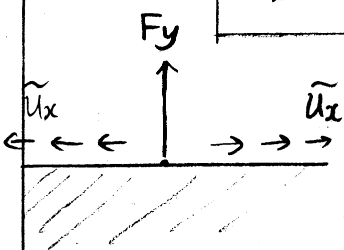
$$\therefore \quad \tilde{u}_x(x) = \frac{1-\nu-2\nu^2}{E} \cdot F_y \cdot \frac{\pi}{2} \cdot \mathrm{sgn}(x) + C'$$

arbitrary constant

notice that $\dfrac{1-\nu-2\nu^2}{E} = \dfrac{(1+\nu)(1-2\nu)}{2\mu(1+\nu)} = \dfrac{1-2\nu}{2\mu}$

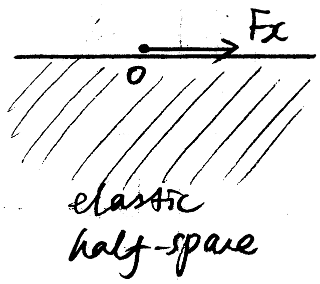
also in plane strain $\kappa = 3-4\nu$, $1-2\nu = \dfrac{\kappa-1}{2}$, $\dfrac{1-2\nu}{2\mu} = \dfrac{\kappa-1}{4\mu}$

$$\boxed{\tilde{u}_x(x) = \frac{\kappa-1}{8\mu} \cdot F_y \cdot \mathrm{sgn}(x)}$$



$(\nu < 0.5)$

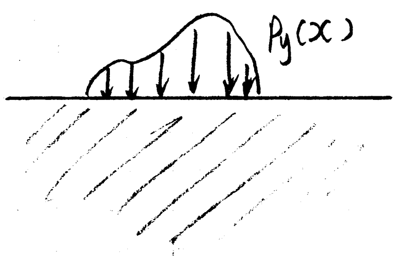
In a similar approach, we can obtain the surface displacement in response to a concentrated tangential force F_x



$$\tilde{u}_x(x) = -\frac{\kappa+1}{4\pi\mu}\cdot F_x \cdot \log|x|$$

$$\tilde{u}_y(x) = -\frac{\kappa-1}{8\mu}\cdot F_y \cdot \mathrm{sgn}(x)$$

§5. Arbitrary Load Distribution $p_y(x)$



$$\tilde{u}_y(x) = \frac{\kappa+1}{4\pi\mu}\int_{-\infty}^{+\infty} p_y(x')\cdot \log|x-x'|\cdot dx'$$

$$\tilde{u}_x(x) = -\frac{\kappa-1}{8\mu}\int_{-\infty}^{+\infty} p_y(x')\cdot \mathrm{sgn}(x-x')\cdot dx$$

(linear superposition in real space)

* A similar expression exists for distributed tangential loading.