- We have seen that a plane strain problem can be represented by a PDE for the stress function: $\nabla^4\phi = 0$ and the corresponding Boundary Conditions.

- We have used polynomials with unknown coefficients as trial solutions to the biharmonic equation
- Another widely used set of solutions is Fourier Series,
- Compared with polynomials, solution by Fourier series has the advantage of forward and backward Fourier transform. (convenience) and the easiness to generalize to unbounded domains (e.g. half-space)

§1. Trial Solution

$$\nabla^4\phi = 0$$

suppose $\phi(x,y) = g(x) f(y)$ —— separation of variables

furthermore let $g(x) = e^{\alpha x}$, $f(y) = e^{\beta y}$

$$\nabla^4\phi = (\alpha^4 + 2\alpha^2\beta^2 + \beta^4)\, e^{\alpha x} e^{\beta y} = 0$$

$$(\alpha^2+\beta^2)^2 = 0 \rightarrow \alpha^2+\beta^2 = 0 \rightarrow \alpha = \pm i\beta$$

This suggest that

$\boxed{\phi(x,y) = e^{i\lambda x}\, e^{\pm\lambda y}}$ is a solution to $\nabla^4\phi = 0$

—— this can be easily verified.

* Notice that for each λ, this only corresponds to 2-independent solutions. $e^{i\lambda x} e^{\lambda y}$, $e^{i\lambda x} e^{-\lambda y}$

Since $\nabla^4\phi = 0$ is 4th order, we expect two more solutions of the similar form.

It can be easily verified that $\boxed{\phi(x,y) = e^{i\lambda x} \cdot y \cdot e^{\pm\lambda y}}$ is also a solution to $\nabla^4\phi = 0$

A linear combination of these 4 solutions is still a solution to the biharmonic equation

$$\phi(x,y) = e^{i\lambda x}\left[(C_1 + C_2 y)e^{\lambda y} + (C_3 + C_4 y)e^{-\lambda y}\right]$$

* The stress function corresponding to a physical solution must be real. Hence only the <u>real part</u> of the above solution will be considered.

* Notice that $e^{i\lambda x} = \cos\lambda x + i\sin\lambda x$
and C_1, C_2, C_3, C_4 are (in general) complex numbers.

* Therefore, the above general solution can also be rewritten (in real numbers) as

$\phi(x,y) = \cos\lambda x \cdot f(y)$ ⟵ an even function of x

or

$\phi(x,y) = \sin\lambda x \cdot f(y)$ ⟵ an odd function of x

where $f(y) = (A + By)e^{\lambda y} + (C + Dy)e^{-\lambda y}$

* Using the definitions $\cosh\lambda y \equiv \dfrac{e^{\lambda y} + e^{-\lambda y}}{2}$ ← even func. of y

$\sinh\lambda y \equiv \dfrac{e^{\lambda y} - e^{-\lambda y}}{2}$ ← odd func. of y

$f(y)$ can also be written into even and odd functions of y

$$f(y) = (A' + B'y)\cosh\lambda y + (C' + D'y)\sinh\lambda y$$

* $A'\cosh\lambda y + D'y\sinh\lambda y$ ⟵ an even function of y

$B'y\cosh\lambda y + C'\sinh\lambda y$ ⟵ an odd function of y

Summary: using sin, cos, sinh, cosh, we have obtained a series of general solutions to the biharmonic equation that can be either even (symmetric) or odd (antisymmetric) in x and/or y. For example,

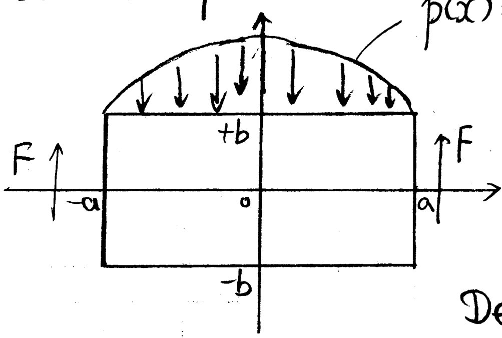
$$\phi(x,y) = \cos\lambda x\,(A'\cosh\lambda y + D' y \sinh\lambda y)$$

satisfies $\nabla^4\phi = 0$ and is an even function for both x and y.

---

§2. Example 1

$$p(x) = p_0 \cos\frac{\pi x}{2a}$$

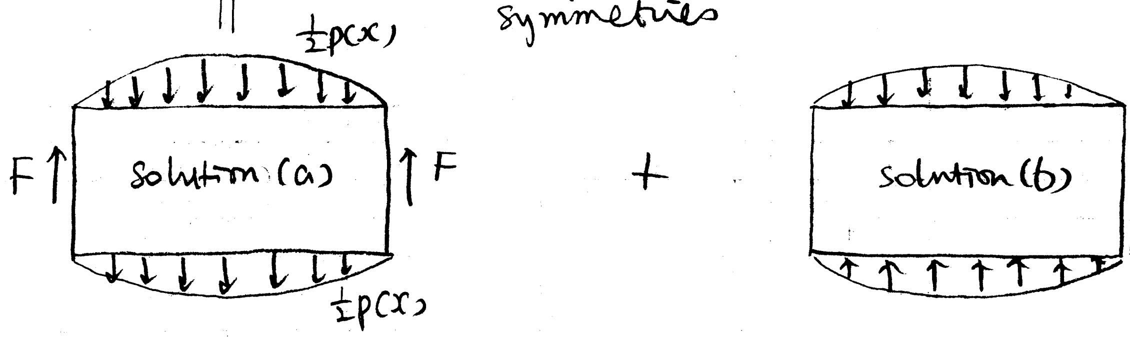


Let $\lambda = \frac{\pi}{2a}$.

$\therefore\ p(x) = p_0 \cos\lambda x = \sigma_{xx}(x, y=b)$

* B.C. motivates a Fourier trial solution

Decompose the problem into different symmetries



Symmetries:

σ_{yy} — odd in y, even in x	σ_{yy} — even in y, even in x
σ_{xy} — odd in x	σ_{xy} —
σ_{xx} — ______	σ_{xx} —
$\phi(x,y)$ — ______	$\phi(x,y)$ — ______

trial solution:

$$\phi = \cos\lambda x\,(B y\cosh\lambda y + C\sinh\lambda y) \qquad \phi = \cos\lambda x\,(A\cosh\lambda y + D y\sinh\lambda y)$$

Let's first find solution (a).

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = -\lambda^2 \cos\lambda x\,(By\cosh\lambda y + C\sinh\lambda y)$$

B.C. $\quad \sigma_{yy} = -\frac{1}{2}p(x) = -\frac{1}{2}p_0\cos\lambda x \qquad y=b$

$\quad \sigma_{yy} = \frac{1}{2}p(x) = \frac{1}{2}p_0\cos\lambda x \qquad y=-b.$

$$\boxed{-\lambda^2(Bb\cosh\lambda b + C\sinh\lambda b) = -\frac{1}{2}p_0}$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \cos\lambda x\,[B\lambda^2 y\cosh\lambda y + (2B\lambda + C\lambda^2)\sinh\lambda y]$$

B.C. $\quad \sigma_{xx}=0 \qquad x=\pm a.$ (strong B.C.)

automatically satisfied by the trial solution!

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \lambda\sin\lambda x\,[(B+C\lambda)\cosh\lambda y + B\lambda y\sinh\lambda y]$$

B.C. $\quad \sigma_{xy}=0 \qquad y=b \qquad$ (another strong B.C.)

$$\boxed{(B+C\lambda)\cosh\lambda b + B\lambda b\sinh\lambda b = 0}$$

The two equations in the box can be solved together to find coefficients B and C.

Introduce short hand notation: $\cosh\lambda b \equiv c$, $\sinh\lambda b \equiv s$.

$$\begin{cases} Bb\cdot c + C\cdot s = \dfrac{p_0}{2\lambda^2} \\ (B+C\lambda)\cdot c + B\lambda b\cdot s = 0 \end{cases}$$

also notice $\cosh^2\lambda b - \sinh^2\lambda b = 1$

solve by Matlab $\longrightarrow$

$$\begin{cases} B = \dfrac{\frac{p_0}{2\lambda}\cosh\lambda b}{\lambda b - \cosh\lambda b\sinh\lambda b} \\ C = -\dfrac{p_0}{}\ \dfrac{\lambda b\sinh\lambda b + \cosh\lambda b}{} \end{cases}$$

Solution (b) can be found similarly.

B.C.

________ $y=b$

________ $y=-b$

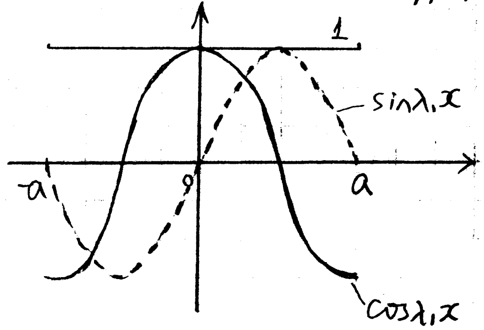
________ $x=\pm a$

________ $x=\pm a$.

The solution to the original problem is obtained by superimposing solution (a) and solution (b).

§3. Fourier Series for $f(x)$ on $-a \leq x \leq a$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\lambda_n x + b_n \sin\lambda_n x$$



$$\lambda_n = n\frac{\pi}{a}.$$

$$a_0 = \frac{1}{a}\int_{-a}^{a} f(x)\,dx$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos(\lambda_n x)\,dx$$

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin(\lambda_n x)\,dx$$

Here we represent an arbitrary function $f(x)$ over the domain $x \in [-a, a]$ in terms of basis functions,

$1, \ \cos\lambda_n x, \ \sin\lambda_n x.$

The constant term is needed here because all other basis functions have zero average.

$$\int_{-a}^{a} \cos\lambda_n x\,dx = \frac{1}{\lambda_n}\sin\lambda_n x\Big|_{-a}^{a} = 0 \qquad \sin\lambda_n a = 0$$

$$\int_{-a}^{a} \sin\lambda_n x\,dx = -\frac{1}{\lambda_n}\cos\lambda_n x\Big|_{-a}^{a} = 0 \qquad \cos(-\lambda_n a) = \cos\lambda_n a$$

The properties of the Fourier Series stem from the orthogonal conditions among basis functions e.g.

$$\frac{1}{a}\int_{-a}^{a} \cos\lambda_n x \cos\lambda_m x\,dx = \frac{1}{2a}\int_{-a}^{a} \cos(\lambda_n+\lambda_m)x + \cos(\lambda_n-\lambda_m)x\,dx$$

$$= \delta_{nm}.$$

We say that $\{1, \cos\lambda_n x, \sin\lambda_n x, n=1,2,\cdots\}$ form a complete basis on the domain $x \in [a, a]$, i.e. it can represent any function on this domain.

However, the Fourier Series above is not the only way to represent an arbitrary function in the domain of $[-a, a]$

The multiplicity of representation is related to the finiteness of the domain, over which $f(x)$ is defined.

In the above Fourier series, the represented function extends beyond the original domain $[-a, a]$ in a periodic manner.

If we use a different representation, the function may look different beyond the domain $[-a, a]$, but that is not important.

In general, we are seeking a set of basis functions $\varphi_i(x)$ $\quad i = 1, 2, \ldots$ such that

$$\boxed{\frac{1}{a}\int_{-a}^{a} \varphi_i(x)\,\varphi_j(x)\,dx = \delta_{ij}}$$

and that $\{\varphi_i(x)\}$ form a complete (which is more difficult to prove) basis over the domain $[-a, a]$

Then

$$\boxed{f(x) = \sum_{i=1}^{\infty} c_i\,\varphi_i(x)}$$

$$\frac{1}{a}\int_{-a}^{a} f(x)\,\varphi_j(x)\,dx = \sum_{i=1}^{\infty} c_i \frac{1}{a}\int_{-a}^{a} \varphi_i(x)\,\varphi_j(x)\,dx$$

$$= \sum_{i=1}^{\infty} c_i\,\delta_{ij} = c_j$$

$$c_j = \frac{1}{a}\int_{-a}^{a} f(x)\,\varphi_j(x)\,dx$$

or equivalently

$$\boxed{c_i = \frac{1}{a}\int_{-a}^{a} f(x)\,\varphi_i(x)\,dx}$$

In the above, we have seen that $\{\varphi_i(x)\} = \{1, \cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, \ldots\}$ qualifies as such a basis set

$\lambda_n = n\frac{\pi}{a}, \quad n = 1, 2, \ldots$

But there are other basis sets as well

§4. Another Basis Set

- Notice that in §2. Example 1, the traction $p(x) = p_0 \cos \frac{\pi x}{2a}$ does not belong to the basis set described in §3.

  This means that if we were to represent $p(x)$ in this basis set, it will be a linear superposition of more than one (in fact, infinite) basis functions. — that would be inconvenient

- If we use the basis set in §3, the boundary condition $\sigma_{xx} = 0$, $x = \pm a$ wouldn't be automatically satisfied

  $$\cos \lambda_n a = \cos n\pi = (-1)^n$$

- This motivates the search for a different basis set.

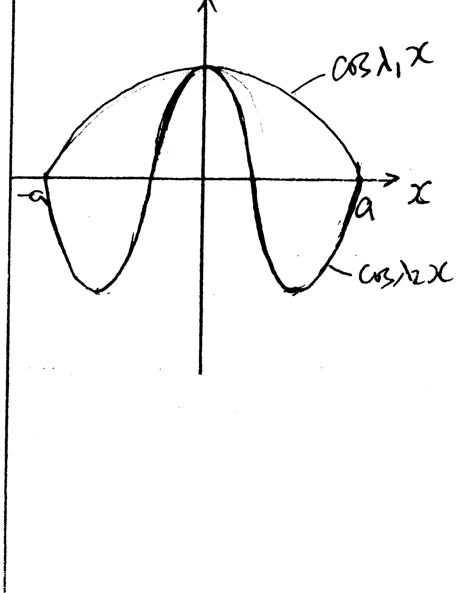
  $$\varphi_n(x) = \cos \lambda_n x \qquad n = 1, 2, 3, \cdots$$

  $$f(x) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x$$

  (neglet sine terms if we constrain $f(x)$ to be an even function of x)

  $$\lambda_n = \frac{(2n-1)\pi}{2a}$$

  $$\lambda_1 = \frac{\pi}{2a}, \quad \lambda_2 = \frac{3\pi}{2a}, \quad \lambda_3 = \frac{5\pi}{2a}, \cdots$$



notice the constant term $\left(\frac{a_0}{2}\right)$ is gone.

Again, one can show that

$$\frac{1}{a}\int_{-a}^{a} \cos \lambda_n x \cos \lambda_m x \, dx = \delta_{mn}$$

(orthongonal cond.)

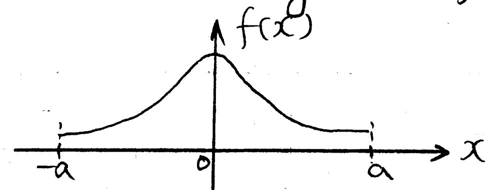
and that $\{\cos \lambda_n x\}$ — notice the different definition of λ_n

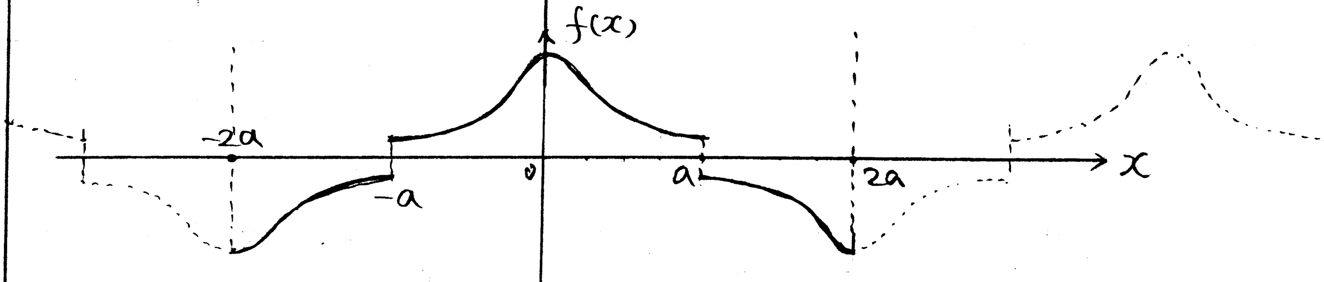
forms a <u>complete</u> basi for even functions on domain $[-a, a]$ (more difficult to prove)

To show that $\{\varphi_n(x) = \cos \lambda_n x\}$ $\lambda_n = \frac{(2n-1)}{2a}\pi$, $n = 1, 2, 3, \cdots$

form a complete basis set for even functions in $[-a, a]$

Let's consider an arbitrary even function $f(x)$.





We now extend the domain of $f(x)$ to $-2a \le x \le 2a$. as illustrated above.

(thanks to the suggestion by Chris Weinberger)

$$f(x+2a) = -f(x)$$

$$f(x+4a) = -f(x+2a) = f(x)$$

We can apply the Fourier series expansion over the extended domains $x \in [-2a, 2a]$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2a}$$

$$a_n = \frac{1}{2a}\int_{-2a}^{2a} f(x)\cos\frac{n\pi x}{2a}\,dx.$$

Because of the anti-symmetry of $f(x)$ on domains $[-2a, 2a]$,

$a_n = 0$ if n is an even number.

$$a_0 = \frac{1}{2a}\int_{-2a}^{2a} f(x)\,dx = 0.$$

$$\therefore f(x) = \sum_{n=1,3,5,\cdots} a_n \cos\frac{n\pi x}{2a} = \sum_{k=1}^{\infty} a_k \cos\frac{(2k-1)\pi x}{2a}$$

where

$$a_K = \frac{1}{2a}\int_{-2a}^{2a} f(x)\cos\frac{(2K-1)\pi x}{2a}\,dx$$

$$= \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{(2K-1)\pi x}{2a}\,dx \qquad \text{(also by Symmetry)}$$

Therefore, any even function $f(x)$ in $-a \le x \le a$ can be represented by

$$f(x) = \sum_{K=1}^{\infty} a_K \cos\frac{(2K-1)\pi x}{2a}$$

Hence $\left\{\varphi_K(x) = \cos\frac{(2K-1)\pi x}{2a}\right\}$ is a complete basis set.

In summary, an even function $f(x)$ in $-a \le x \le a$ can be represented as a Fourier series either as

①
$$\boxed{\begin{aligned} f(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \\ a_n &= \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx \\ a_0 &= \frac{1}{a}\int_{-a}^{a} f(x)\,dx \end{aligned}}$$

, i.e. basis function $\left\{\psi_n(x) = \cos\frac{n\pi x}{a}\right\}$ $n = 0, 1, 2, \cdots$

or as

②
$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2a} \\ a_n &= \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{(2n-1)\pi x}{2a} \end{aligned}}$$

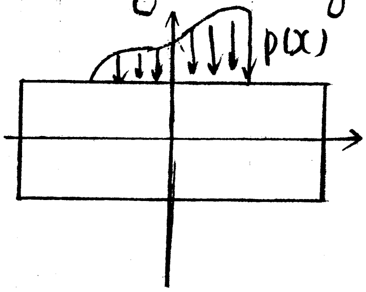
, i.e. basis function $\left\{\varphi_n(x) = \cos\frac{(2n-1)\pi x}{2a}\right\}$ $n = 1, 2, \cdots$

The two approach work equally well in their ability to represent an arbitrary even function.

But the ②nd representation is prefered if we want to solve a beam under arbitrary loading using the Fourier method

* because one of the strong B.C. ($\sigma_{xx} = 0$, $x = \pm a$) can be satisfied automatically.

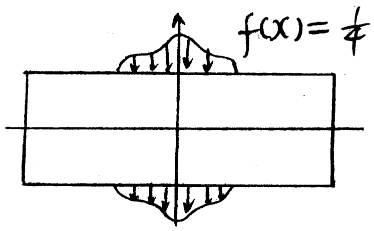
§5. Arbitrary Loading



An arbitrary loading on a symmetric structure (e.g. a rectangle) can always be decomposed into superpositions of loadings that are either even or odd in x and/or y.

For example, one of the four terms may look like this:

$f(x) = \frac{1}{2}(p(x) + p(-x))$



Symmetries:

σ_{yy} : even in x, odd in y

ϕ : even in x, odd in y.

In the following, we will solve the problem with this symmetry. Problems with a different symmetry (e.g. odd in x, even in y) can be solved in a similar way.

Decompose loading into Fourier modes (2nd approach)

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\lambda_n x \qquad \lambda_n = \frac{(2n-1)\pi}{2a}$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\lambda_n x \, dx \qquad \text{(there is no } n=0 \text{ term)}$$

Trial Solution

odd in y — even in x

$$\phi(x,y) = \sum_{n=1}^{\infty}\left(B_n y\cosh\lambda_n y + C_n\sinh\lambda_n y\right)\underline{\cos\lambda_n x}$$

$$\sigma_{yy} = \phi_{,xx} = \sum_{n=1}^{\infty}\left(B_n y\cosh\lambda_n y + C_n\sinh\lambda_n y\right)(-\lambda_n^2)\cos\lambda_n x$$

$$\sigma_{xx} = \phi_{,yy} = \sum_{n=1}^{\infty}\left(B_n\lambda_n^2 y\cosh\lambda_n y + (2B_n\lambda_n + C_n\lambda_n^2)\sinh\lambda_n y\right)\cos\lambda_n x$$

$$\sigma_{xy} = -\phi_{,xy} = \sum_{n=1}^{\infty}\left((B_n + C_n\lambda_n)\cosh\lambda_n y + B_n\lambda_n y\sinh\lambda_n y\right)\lambda_n\sin\lambda_n x$$

Boundary conditions:

$\sigma_{xx} = 0, \quad x = \pm a \leftarrow$ automatically satisfied because $\cos\lambda_n a = 0$.

$$\begin{cases} \sigma_{yy}(x, y=b) = f(x) = \sum_{n=1}^{\infty} a_n \cos\lambda_n x \\ \sigma_{xy}(x, y=b) = 0 \end{cases}$$

$$\rightarrow \begin{cases} B_n b \cosh\lambda_n b + C_n \sinh\lambda_n b = -\frac{a_n}{\lambda_n^2} \\ (B_n + C_n\lambda_n)\cosh\lambda_n b + B_n \lambda_n b \sinh\lambda_n b = 0 \end{cases} \quad \text{for } n = 1, 2, 3, \cdots$$

solve

$$\rightarrow \begin{cases} B_n = \cdots \\ C_n = \cdots \end{cases} \quad \text{for every mode } n = 1, 2, 3, \cdots$$

* Notice that this approach would not work if $f(x)$ is expanded in the "usual" Fourier series ①

Because $\cos n\pi = \pm 1 \neq 0$, each Fourier mode does not satisfy the strong B.C. $\sigma_{xx} = 0$, $x = \pm a$ individually.