• We have seen that a plane strain problem can be represented by a PDE for the stress function: \( \nabla^4 \phi = 0 \) and the corresponding boundary conditions.

• We have used polynomials with unknown coefficients as trial solutions to the biharmonic equation.

• Another widely used set of solutions is Fourier series.

• Compared with polynomials, solution by Fourier series has the advantage of forward and backward Fourier transform (convenience) and the easiness to generalize to unbounded domains (e.g., half-space).

\[ \nabla^4 \phi = 0 \]

suppose \( \phi(x, y) = g(x) f(y) \) — separation of variables

furthermore let \( g(x) = e^{\alpha x} \), \( f(y) = e^{\beta y} \)

\[ \nabla^4 \phi = (\alpha^4 + 2\alpha^2 \beta^2 + \beta^4) e^{\alpha x} e^{\beta y} = 0 \]

\[ (\alpha^2 + \beta^2)^2 = 0 \rightarrow \alpha^2 + \beta^2 = 0 \rightarrow \alpha = \pm i \beta \]

This suggests that

\[ \phi(x, y) = e^{i\lambda x} e^{\pm i\lambda y} \]

is a solution to \( \nabla^4 \phi = 0 \)

— this can be easily verified.

* Notice that for each \( \lambda \), this only corresponds to \( 2 \) independent solutions. \( e^{i\lambda x} e^{\lambda y}, e^{i\lambda x} e^{-\lambda y} \)

since \( \nabla^4 \phi = 0 \) is 4th order, we expect two more solutions of the similar form.

It can be easily verified that

\[ \phi(x, y) = e^{i\lambda x} y e^{\pm i\lambda y} \]

is also a solution to \( \nabla^4 \phi = 0 \)
A linear combination of these 4 solutions is still a solution to the biharmonic equation

\[ \phi(x,y) = e^{i\lambda x} \left[ (C_1 + C_2 y) e^{\lambda y} + (C_3 + C_4 y) e^{-\lambda y} \right] \]

* The stress function corresponding to a physical solution must be real. Hence only the real part of the above solution will be considered.

* Notice that  
\[ e^{i\lambda x} = \cos \lambda x + i \sin \lambda x \]
and \( C_1, C_2, C_3, C_4 \) are (in general) complex numbers.

* Therefore, the above general solution can also be rewritten (in real numbers) as

\[ \phi(x,y) = \cos \lambda x \cdot f(y) \quad \text{an even function of } x \]

or

\[ \phi(x,y) = \sin \lambda x \cdot f(y) \quad \text{an odd function of } x \]

where  
\[ f(y) = (A + B y) e^{\lambda y} + (C + D y) e^{-\lambda y} \]

* Using the definitions

\[ \cosh \lambda y = \frac{e^{\lambda y} + e^{-\lambda y}}{2} \quad \text{even function of } y \]

\[ \sinh \lambda y = \frac{e^{\lambda y} - e^{-\lambda y}}{2} \quad \text{odd function of } y \]

\( f(y) \) can also be written into even and odd functions of \( y \)

\[ f(y) = (A' + B' y) \cosh \lambda y + (C' + D' y) \sinh \lambda y \]

* \( A' \cosh \lambda y + B' y \sinh \lambda y \quad \text{an even function of } y \)

\[ B' y \cosh \lambda y + C' \sinh \lambda y \quad \text{an odd function of } y \]
Summary: Using sin, cos, sinh, cosh, we have obtained a series of general solutions to the biharmonic equation that can be either even (symmetric) or odd (anti-symmetric) in \( x \) and/or \( y \). For example,

\[
\phi(x, y) = \cos \lambda x (A \cosh \lambda y + D \sinh \lambda y)
\]
satisfies \( \nabla^4 \phi = 0 \) and is an even function for both \( x \) and \( y \).

82. Example 1

Let \( \lambda = \frac{\pi}{2a} \).

\[
\therefore \quad p(x) = p_0 \cos \frac{\pi x}{2a}
\]

\( \phi_{yy} \) motivates a Fourier trial solution.

Decompose the problem into different symmetries:

**Symmetries:**
- \( \phi_{yy} \) — odd in \( y \), even in \( x \)
- \( \phi_{yy} \) — even in \( y \), even in \( x \)
- \( \phi_{xx} \) —
- \( \phi_{xx} \) —

**Trial solutions:**
- \( \phi = \cos \lambda x (B \cosh \lambda y + C \sinh \lambda y) \)
- \( \phi = \cos \lambda x (A \cosh \lambda y + D \sinh \lambda y) \)
Let's first find solution (a).

\[ \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} = -\lambda^2 \cos \lambda x \ (\text{by } \cosh \lambda y + \sinh \lambda y) \]

B.C. \quad \sigma_{yy} = -\frac{1}{2} p(x) = -\frac{1}{2} p_0 \cos \lambda x \quad y = b

\[ \sigma_{yy} = \frac{1}{2} p(x) = \frac{1}{2} p_0 \cos \lambda x \quad y = -b. \]

\[ -\lambda^2 (B_0 \cosh \lambda b + C \sinh \lambda b) = -\frac{1}{2} p_0 \]

\[ \sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} = \cos \lambda x \left[ B_0 \lambda^2 y \cosh \lambda y + (2B + C \lambda^4) \sinh \lambda y \right] \]

B.C. \quad \sigma_{xx} = 0 \quad x = \pm a. \quad \text{(strong B.C.)}

automatically satisfied by the trial solution!

\[ \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = \lambda \sin \lambda x \left[ (B + C \lambda) \cosh \lambda y + B \lambda y \sinh \lambda y \right] \]

B.C. \quad \sigma_{xy} = 0 \quad y = b \quad \text{(another strong B.C.)}

\[ (B + C \lambda) \cosh \lambda b + B \lambda b \sinh \lambda b = 0 \]

The two equations in the box can be solved together to find coefficients \( B \) and \( C \).

Introduce short hand notation: \( \cosh \lambda b = c, \quad \sinh \lambda b = s \).

\[ \begin{align*}
B_0 \cdot c + C \cdot s &= \frac{p_0}{2A^2} \\
(B + C \lambda) \cdot c + B \lambda b \cdot s &= 0
\end{align*} \]

also notice \( \cosh \lambda b - \sinh \lambda b = 1 \)

solve by Matlab

\[ \begin{align*}
B &= \frac{p_0}{2A^2} \cosh \lambda b \\
C &= -\frac{p_0}{\lambda b - \cosh \lambda b \sinh \lambda b} \lambda b \sinh \lambda b + \cosh \lambda b
\end{align*} \]
Solution (b) can be found similarly.

B.C. $y = b$

$y = -b$

$x = \pm a$

$x = \pm a$

The solution to the original problem is obtained by superimposing solution (a) and solution (b).
§3. Fourier series for \( f(x) \) on \( -a \leq x \leq a \)

\[
f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x + b_n \sin \lambda_n x
\]

\[
\lambda_n = n \frac{\pi}{a}.
\]

\[
a_0 = \frac{1}{a} \int_{-a}^{a} f(x) \, dx
\]

\[
a_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos(\lambda_n x) \, dx
\]

\[
b_n = \frac{1}{a} \int_{-a}^{a} f(x) \sin(\lambda_n x) \, dx
\]

Here we represent an arbitrary function \( f(x) \) over the domain \( x \in [-a, a] \) in terms of basis functions, \( 1, \cos \lambda_n x, \sin \lambda_n x \).

The constant term is needed here because all other basis functions have zero average.

\[
\int_{-a}^{a} \cos \lambda_n x \, dx = \frac{1}{\lambda_n} \sin \lambda_n x \bigg|_{-a}^{a} = 0 \quad \sin \lambda_n a = 0
\]

\[
\int_{-a}^{a} \sin \lambda_n x \, dx = -\frac{1}{\lambda_n} \cos \lambda_n x \bigg|_{-a}^{a} = 0 \quad \cos \lambda_n a = \cos \lambda_n a
\]

The properties of the Fourier series stem from the orthogonal conditions among basis functions e.g.

\[
\frac{1}{a} \int_{-a}^{a} \cos \lambda_n x \cos \lambda_m x \, dx = \frac{1}{2a} \int_{-a}^{a} \cos((\lambda_n + \lambda_m) x + \cos((\lambda_n - \lambda_m) x \, dx
\]

\[
= \delta_{nm}
\]

We say that \( \{1, \cos \lambda_n x, \sin \lambda_n x, n=1,2,\ldots\} \) form a complete basis on the domain \( x \in [-a,a] \), i.e. it can represent any function on this domain.
However, the Fourier series above is not the only way to represent an arbitrary function in the domain of \([a, a]\).

The multiplicity of representation is related to the finiteness of the domain, over which \(f(x)\) is defined.

In the above Fourier series, the represented function extends beyond the original domain \([a, a]\) in a periodic manner.

If we use a different representation, the function may look different beyond the domain \([a, a]\), but that is not important.

In general, we are seeking a set of basis functions \(g_i(x)\) \(i = 1, 2, \ldots\) such that

\[
\frac{1}{a} \int_{-a}^{a} g_i(x) g_j(x) \, dx = \delta_{ij}
\]

and that \(\{g_i(x)\}\) form a complete (which is more difficult to prove) basis over the domain \([a, a]\).

Then

\[
f(x) = \sum_{i=1}^{\infty} c_i \; g_i(x)
\]

\[
\frac{1}{a} \int_{-a}^{a} f(x) \; g_j(x) \, dx = \sum_{i=1}^{\infty} c_i \; \frac{1}{a} \int_{-a}^{a} g_i(x) \; g_j(x) \, dx
\]

\[
= \sum_{i=1}^{\infty} c_i \; \delta_{ij} = c_j
\]

\[
c_j = \frac{1}{a} \int_{-a}^{a} f(x) \; g_j(x) \, dx
\]

or equivalently

\[
c_i = \frac{1}{a} \int_{-a}^{a} f(x) \; g_i(x) \, dx
\]

In the above, we have seen that \(\{g_i(x)\} = \{1, \cos \lambda x, \sin \lambda x, \ldots\}\) qualifies as such a basis set.

To derive a theoretical basis set, we consider the eigenfunctions of the associated differential operator

\[
\lambda_i = n \frac{\pi}{a}, \quad n = 1, 2, \ldots
\]
§ 4. Another Basis Set

- Notice that in § 2 Example 1, the traction \( p(x) = P \cos \frac{\pi x}{2a} \) does not belong to the basis set described in § 3.

This means that if we were to represent \( p(x) \) in this basis set, it will be a linear superposition of more than one (in fact, infinite) basis functions — that would be inconvenient.

- If we use the basis set in § 3, the boundary condition \( \sigma_{xx} = 0, \ x = \pm a \) wouldn't be automatically satisfied.

\[
\cos \lambda_n a = \cos n \pi x = (-1)^n
\]

- This motivates the search for a different basis set.

\[
\phi_n(x) = \cos \lambda_n x \quad n = 1, 2, 3, \ldots
\]

\[
f(x) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x \quad (\text{neglect sine terms if we constrain } f(x) \text{ to be an even function of } x)
\]

\[
\lambda_n = \frac{(2n-1) \pi}{2a}
\]

\[
\lambda_1 = \frac{\pi}{2a}, \quad \lambda_2 = \frac{3\pi}{2a}, \quad \lambda_3 = \frac{5\pi}{2a}, \ldots
\]

Notice the constant term \((\frac{a}{2})\) is gone.

Again, one can show that

\[
\int_{-a}^{a} \cos \lambda_n x \, \cos \lambda_m x \, dx = \delta_{mn}
\]

(orthogonal cond.)

and that \([\cos \lambda_n x]\) — notice the different definition of \(\lambda_n\) — forms a complete basis for even functions on domain \([-a, a]\) (more difficult to prove)
To show that \( \{ \varphi_n(x) = \cos \lambda_n x \} \quad \lambda_n = \frac{(2n-1)\pi}{2a}, \quad n = 1, 2, 3, \ldots \)
form a complete basis set for even functions in \([-a, a]\)

Let's consider an arbitrary even function \( f(x) \).

We now extend the domain of \( f(x) \) to \(-2a \leq x \leq 2a\), as illustrated above.

\[
\begin{align*}
f(x+2a) &= -f(x) \\
f(x+4a) &= -f(x+2a) = f(x)
\end{align*}
\]

We can apply the Fourier series expansion over the extended domain \( x \in [-2a, 2a] \)

\[
f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2a}
\]

\[
a_n = \frac{1}{2a} \int_{-2a}^{2a} f(x) \cos \frac{n\pi x}{2a} \, dx
\]

Because of the anti-symmetry of \( f(x) \) in domain \([-2a, 2a]\),

\[ a_n = 0 \text{ if } n \text{ is an even number.} \]

\[ a_0 = \frac{1}{2a} \int_{-2a}^{2a} f(x) \, dx = 0. \]

\[
\therefore f(x) = \sum_{n=1,3,5,\ldots} a_n \cos \frac{n\pi x}{2a} = \sum_{k=1}^{\infty} a_k \cos \frac{(2k-1)\pi x}{2a}
\]
where
\[ a_k = \frac{1}{2a} \int_{-a}^{a} f(x) \cos \frac{(2k-1)\pi x}{2a} \, dx \]
\[ = \frac{1}{a} \int_{-a}^{a} f(x) \cos \frac{2k\pi x}{2a} \, dx \]
(also by symmetry)

Therefore, any even function \( f(x) \) in \(-a \leq x \leq a\)
can be represented by
\[ f(x) = \sum_{k=1}^{\infty} a_k \cos \frac{(2k-1)\pi x}{2a} \]

Hence \( \{g_k(x) = \cos \frac{(2k-1)\pi x}{2a}\} \) is a complete basis set.

In summary, an even function \( f(x) \) in \(-a \leq x \leq a\)
can be represented as a Fourier series, either as
\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a} \]

\begin{enumerate}
\item \[ a_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos \frac{n\pi x}{a} \, dx \]
\[ a_0 = \frac{1}{a} \int_{-a}^{a} f(x) \, dx \]
\end{enumerate}

\[ \{g_n(x) = \cos \frac{n\pi x}{a}\} \quad n=0, 1, 2, \ldots \]

or as
\[ f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi (2n-1) x}{2a} \]

\[ a_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos \frac{(2n-1)\pi x}{2a} \, dx \]

\[ \{g_n(x) = \cos \frac{(2n-1)\pi x}{2a}\} \quad n=1, 2, \ldots \]

The two approaches work equally well in their ability to represent
an arbitrary even function.

But the 2nd representation is preferred if we want to solve
a beam under arbitrary loading using the Fourier method
\( x \) because one of the strong b.c. \( \phi(x=0, x=\pm a) \) can be satisfied automatically.
§5. Arbitrary Loading

An arbitrary loading on a symmetric structure (e.g., a rectangle) can always be decomposed into superpositions of loadings that are either even or odd in $x$ and/or $y$.

For example, one of the four terms may look like this:

\[ f(x) = \frac{1}{4} (p(x) + p(-x)) \]

Symmetries:
- $\phi_{xy}$: even in $x$, odd in $y$
- $\phi$: even in $x$, odd in $y$

In the following, we will solve the problem with this symmetry. Problems with a different symmetry (e.g., odd in $x$, even in $y$) can be solved in a similar way.

Decompose loading into Fourier modes (2nd approach)

\[ f(x) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x \]

\[ \lambda_n = \frac{(2n-1)\pi}{2a} \]

\[ a_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos \lambda_n x \, dx \quad \text{(there is no } n=0 \text{ term)} \]

Trial Solution

\[ \phi(x, y) = \sum_{n=1}^{\infty} \left( B_n \cosh \lambda_n y + C_n \sinh \lambda_n y \right) \cos \lambda_n x \]

\[ \sigma_{yy} = \phi_{xx} = \sum_{n=1}^{\infty} \left( B_n \cosh \lambda_n y + C_n \sinh \lambda_n y \right) (-\lambda_n^2) \cos \lambda_n x \]

\[ \sigma_{xx} = \phi_{yy} = \sum_{n=1}^{\infty} \left( B_n \cosh \lambda_n y + C_n \sinh \lambda_n y \right) (-\lambda_n^2) \cos \lambda_n x \]

\[ \sigma_{xy} = -\phi_{xy} = \sum_{n=1}^{\infty} \left( B_n \cosh \lambda_n y + C_n \sinh \lambda_n y \right) \lambda_n \sin \lambda_n x \]
Boundary conditions:

\[ \sigma_{xx} = 0, \quad x = \pm a \quad \text{automatically satisfied because } \cos \lambda a = 0. \]

\[
\begin{align*}
\sigma_{yy}(x, y=b) &= f(x) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x \\
\sigma_{xy}(x, y=b) &= 0
\end{align*}
\]

\[ \begin{align*}
B_n b \cosh \lambda_n b + C_n \sinh \lambda_n b &= -\frac{a_n}{\lambda_n} \quad \text{for } n=1, 2, 3, \ldots \\
(B_n + C_n \lambda_n) \cosh \lambda_n b + B_n \lambda_n b \sinh \lambda_n b &= 0
\end{align*} \]

solve \[ \begin{align*}
B_n &= \ldots \\
C_n &= \ldots 
\end{align*} \quad \text{for every mode } n=1, 2, 3, \ldots
\]

Notice that this approach would not work if \( f(x) \) is expanded in the "usual" Fourier series because \( \cos n\pi = \pm 1 \neq 0 \). Each Fourier mode does not satisfy the strong B.C. \( \sigma_{xx} = 0, \quad x = \pm a \) individually.