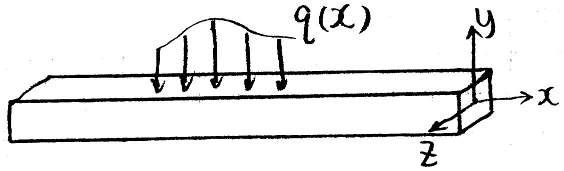
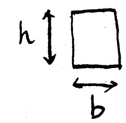
§1 Review of Euler-Bernoulli Beam Theory

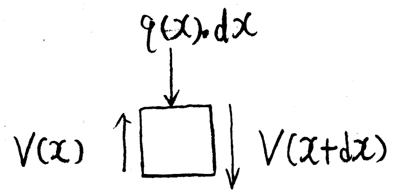
(Mechanics of Materials)





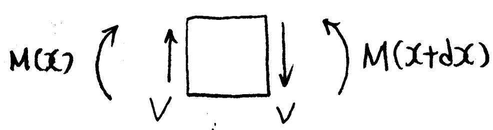
cross section

moment of inertia $I_z = \frac{bh^3}{12}$



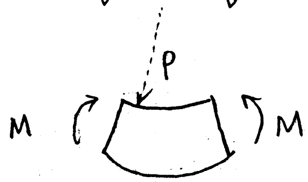
$$\frac{d}{dx}V(x) = -q(x) \quad \cdots ①$$

V(x): shear force



$$\frac{d}{dx}M(x) = V(x) \quad \cdots ②$$

M(x): bending moment

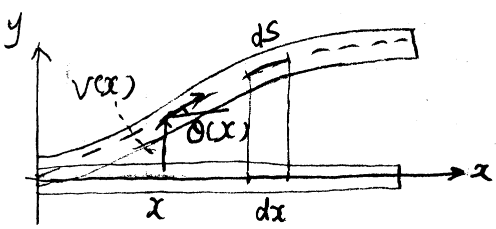


$$K(x) = \frac{M(x)}{EI} \quad \cdots ③$$

$K = \frac{1}{\rho}$: curvature

ρ: radius of curvature

E: Young's modulus



from geometry

$$K(x) = \frac{d\theta}{dS}$$

$$\tan\theta(x) = \frac{dv(x)}{dx}$$

v(x): vertical displacement

θ(x): local beam orientation

in the limit of $\theta(x) \ll 1$

$$K(x) \cong \frac{d^2}{dx^2}v(x) = v''(x) \simeq \theta'(x) \quad \cdots ④$$

combining ①, ②, ③, ④ we have

4th order ODE, needs 4 B.C.

$$EI\,v''''(x) = -q(x)$$

($v(x) = u_y(x)$) ← our notation

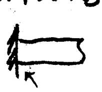
$$EI\,v''(x) = M(x)$$

$$\sigma_{xx}(x, y) = -\frac{M(x)\cdot y}{I_z}$$

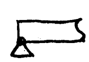
2 B.C. at each end

↓

Boundary conditions:

a) cantilever 

$v=0, \quad \theta=0$

b) simple (pin) support 

$v=0, \quad M=0$

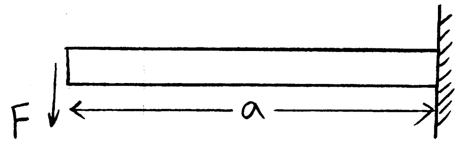
c) free end 

$V=0, \quad M=0$

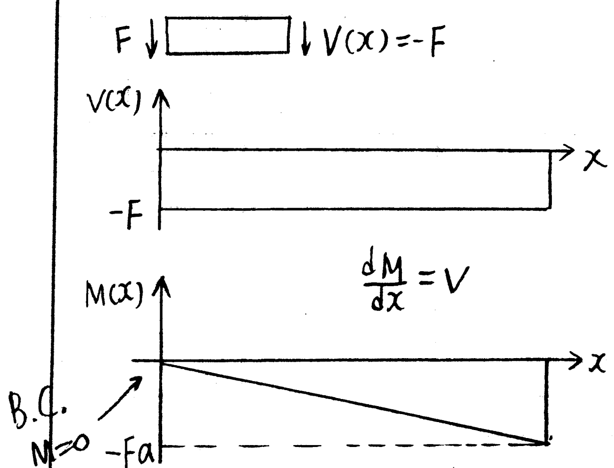
d) 

$\theta=0, \quad V=0$

Example 1.



Shear and moment diagram



$$\frac{dM}{dx} = V$$

$$M(x) = -F \cdot x$$

$$\sigma_{xx}(x,y) = -\frac{M(x)\cdot y}{I} = \frac{Fxy}{I}$$

$$\frac{d\theta(x)}{dx} = \frac{M(x)}{EI} = -\frac{F}{EI}(x)$$

B.C. $\theta(x=a) = 0$

$$\therefore \theta(x) = \frac{F}{2EI}(a^2 - x^2)$$

$$\frac{d u_y(x)}{dx} = \theta(x) = \frac{F}{2EI}(a^2 - x^2)$$

B.C. $u_y(x=a) = 0$

$$\therefore u_y(x) = \frac{F}{2EI}\left(a^2 x - \frac{x^3}{3} - \frac{2a^3}{3}\right)$$

Reading. Barber. Chap 5

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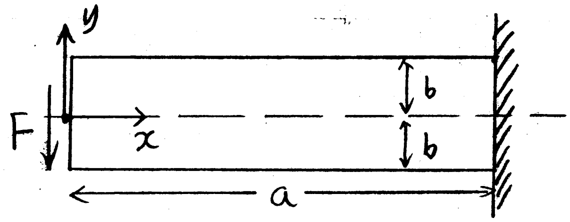
§2. Solve the above problem using Airy Stress function



assume unit thickness in z. $I = \frac{(2b)^3}{12} = \frac{2b^3}{3}$

* can be a thick beam
Mechanics of Materials only applies to thin beam
* we can also obtain all stress components which is difficult in Mech. of Mater.

Boundary conditions,

Strong B.C.

$$\begin{cases} \sigma_{xy} = 0 & y = \pm b \\ \sigma_{yy} = 0 & y = \pm b \\ \sigma_{xx} = 0 & x = 0 \end{cases}$$

(first two: top, bottom surfaces)

Weak B.C.

$$\begin{cases} \int_{-b}^{b} \sigma_{xy}\, dy = F & x = 0 \\ \int_{-b}^{b} \sigma_{xx}\, y\, dy = 0 & x = 0 \\ \int_{-b}^{b} \sigma_{xx}\, dy = 0 & x = 0 \end{cases} \quad \text{left end}$$

$$\begin{cases} \int_{-b}^{b} \sigma_{xy}\, dy = F & x = a \\ \int_{-b}^{b} \sigma_{xx}\, y\, dy = Fa & x = a \\ \int_{-b}^{b} \sigma_{xx}\, dy = 0 & x = a \end{cases} \quad \text{right end}$$

From Mechanics of Materials, we expect

$$M = -Fx \quad , \quad \sigma_{xx} = \frac{Fxy}{I}$$

but σ_{xx} alone may not satisfy all the equilibrium and compatibility conditions.

Let stress function be

$$\phi = C_1 x y^3 \qquad \text{(trial solution)}$$

$$\begin{cases} \sigma_{xx} = 6C_1 xy & \leftarrow \text{Strong B.C. } \sigma_{xx}=0 \ (x=0) \text{ is satisfied (left end)} \\ \sigma_{xy} = -3C_1 \; y^2 & \\ \sigma_{yy} = 0 & \leftarrow \text{Strong B.C. } \sigma_{yy}=0 \ (y=\pm b) \text{ is satisfied (top, bottom)} \end{cases}$$

But B.C. $\sigma_{xy}=0$, $(y=\pm b)$ is violated!

*This is the long side of the beam. we should try to satisfy the strong B.C.

Modify the trial solution

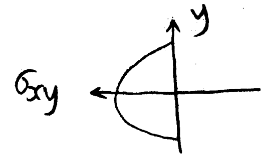
$$\phi = C_1 x y^3 + C_2 xy$$

$$\begin{cases} \sigma_{xx} = 6C_1 xy \\ \sigma_{xy} = -3C_1 y^2 - C_2 \\ \sigma_{yy} = 0 \end{cases}$$

$\leftarrow$ adding a constant to σ_{xy} without changing other stress components

$y=\pm b$. $\quad \sigma_{xy} = -3C_1 b^2 - C_2 = 0$, $\qquad C_2 = -3C_1 b^2$

$\therefore \quad \sigma_{xy} = 3C_1 (b^2 - y^2)$



parabolic distribution of shear stress

* it was difficult to derive in Mech. of Mater.

Next, we need to determine the constant C_1.

use weak B.C. at left end $(x=0)$

$$F = \int_{-b}^{b} \sigma_{xy}\, dy = \int_{-b}^{b} 3C_1 (b^2 - y^2)\, dy = 4C_1 b^3$$

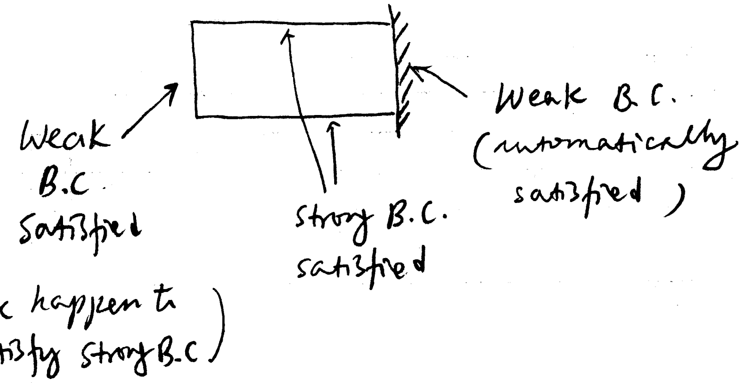
$$C_1 = \frac{F}{4b^3}$$

$$\boxed{\phi = \frac{F}{4b^3}\left(xy^3 - 3b^2 xy\right)}$$

The (weak) B.C. at right end is automatically satisfied,

because the stress function approach automatically satisfies equilibrium

$$\begin{cases} \sigma_{xx} = \frac{F}{2b^3} xy \\ \sigma_{xy} = \frac{3F}{4b^3}(b^2-y^2) \\ \sigma_{yy} = 0 \end{cases}$$



§3. Displament field. (Barber. P.111–115 Chap 9)

First obtain the strain field

$$\begin{cases} \varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} = \frac{3F}{2Eb^3}xy \\ \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{xx}}{E} = -\frac{3F\nu}{2Eb^3}xy \\ \varepsilon_{xy} = \frac{\sigma_{xy}}{E}(1+\nu) = \frac{3F(1+\nu)}{4Eb^3}(b^2-y^2) \end{cases}$$

* assume plain stress here, but solution can be easily transformed to plain strain condition

$E = 2\mu(1+\nu)$, $2\mu = \frac{E}{1+\nu}$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \rightarrow u_x = \frac{3F}{4Eb^3}x^2y + f(y)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \rightarrow u_y = -\frac{3F\nu}{4Eb^3}xy^2 + g(x)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{3F}{8Eb^3}x^2 + \frac{1}{2}f'(y) - \frac{3F}{8Eb^3}y^2 + \frac{1}{2}g'(x) = \frac{3F(1+\nu)}{4Eb^3}(b^2-y^2)$$

$$\frac{3F}{8Eb^3}x^2 + \frac{1}{2}g'(x) = \frac{3F\nu}{8Eb^3}y^2 - \frac{1}{2}f'(y) + \frac{3F(1+\nu)}{4Eb^2}(b^2-y^2) = C$$

↑ only a function of x ↑ only a function of y ↑ so they must be a constant.

$$\begin{cases} g'(x) = -\dfrac{3Fx^2}{4Eb^3} + C \\ f'(y) = \dfrac{3F\nu}{4Eb^3} y^2 + \dfrac{3F(1+\nu)(b^2-y^2)}{2Eb^3} - C \end{cases}$$

$$\begin{cases} g(x) = -\dfrac{Fx^3}{4Eb^3} + Cx + B \\ f(y) = \dfrac{F\nu y^3}{4Eb^3} + \dfrac{F(1+\nu)(3b^2y - y^3)}{2E} - Cy + A \end{cases}$$

$$\begin{cases} u_x = \dfrac{3Fx^2y}{4Eb^3} + \dfrac{3F(1+\nu)y}{2E} - \dfrac{F(2+\nu)y^3}{4Eb^3} + A - Cy \\ u_y = -\dfrac{3F\nu xy^2}{4Eb^3} - \dfrac{Fx^3}{4Eb^3} + B + Cx \end{cases}$$

constants : A, B — rigid-body translation
C — rigid-body rotation.

* These constants are not necessarily zero.
They should be determined by Boundary Conditions.

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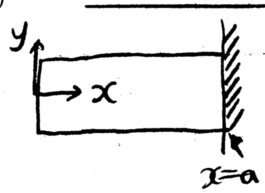
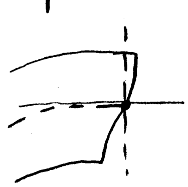
§4. End condition



strong boundary condition would be
$u_x = 0, \quad u_y = 0 \quad$ on $x = a$.

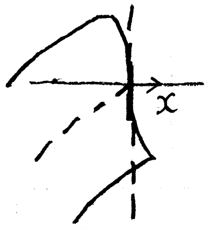
Since in the above solution both u_x and u_y depend on y, the strong B.C. cannot be satisfied.
We need to modified it to a weak B.C.
There are several options:

Option a: reduce B.C. to a point



$$\begin{cases} u_x = 0 \\ u_y = 0 \\ \dfrac{\partial u_y}{\partial x} = 0 \end{cases} \quad \text{on } x = a,\ y = 0. \quad \text{(angle of neutral axis = 0)} \quad \rightarrow \quad \begin{cases} A = 0 \\ B = -\dfrac{Fa^3}{2Eb^3} \\ C = \dfrac{3Fa^2}{4Eb^3} \end{cases}$$

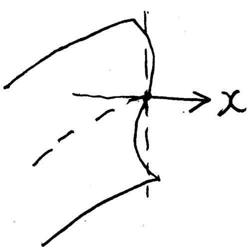
Option b: reduce B.C. to a point



$$\begin{cases} u_x = 0 \\ u_y = 0 \\ \frac{\partial u_x}{\partial y} = 0 \end{cases}$$ on $x=a$, $y=0$ (surface normal remain along x-axis)

$$\rightarrow \begin{cases} A = 0 \\ B = -\frac{Fa^3}{2Eb^3}\left(1+3(1+\nu)\frac{b^2}{a^2}\right) \\ C = \frac{3Fa^2}{4Eb^3}\left(1+2(1+\nu)\frac{b^2}{a^2}\right) \end{cases}$$

☆ Option c: integral over surface
— probably closest to reality



$$\begin{cases} \int_{-b}^{b} u_x \, dy = 0 \\ \int_{-b}^{b} u_y \, dy = 0 \\ \int_{-b}^{b} u_x \, y \, dy = 0 \end{cases}$$ (averaged rotation = 0)

$$\rightarrow \begin{cases} A = 0 \\ B = -\frac{Fa^3}{2Eb^3}\left(1+\frac{(12+11\nu)}{5}\frac{b^2}{a^2}\right) \\ C = \frac{3Fa^2}{4Eb^3}\left(1+\frac{8+9\nu}{5}\frac{b^2}{a^2}\right) \end{cases}$$

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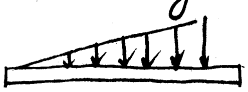
§5. General Solution Strategy

— using higher order polynomials

Step 1: determine the maximum order of polynomial using mechanics of materials arguments.



suppose: normal loading $q(x) \sim x^n$

↓

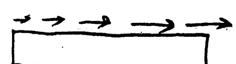
shear force $V(x) \sim x^{n+1}$

bending moment $M(x) \sim x^{n+2}$

$\sigma_{xx} \sim x^{n+2} \cdot y$

$\phi \sim x^{n+2} \cdot y^3$

maximum order = $n+5$



suppose: shear loading $\sim x^m$

↓

Shear force $V(x) \sim x^m$

bending moment $M(x) \sim x^{m+1}$

$\sigma_{xx} \sim x^{m+1} \cdot y$

$\phi \sim x^{m+1} y^3$

maximum order = $m+4$

(step 2) write down a polynomial function $\phi(x,y)$ that contains all terms up to order

max(n+5, m+4)

$\phi(x,y) = c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x^3 + \cdots$

1

x y ← no physical meaning

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x^2 xy y^2 — coefficients: c_1 c_2 c_3

x^3 x^2y xy^2 y^3 — c_4 c_5 c_6 c_7

x^4 x^3y x^2y^2 xy^3 y^4 — c_8 c_9 c_{10} c_{11} c_{12}

x^5 x^4y x^3y^2 x^2y^3 xy^4 y^5 — c_{13} c_{14} c_{15} c_{16} c_{17} c_{18}

...

(step 3) $\nabla^4\phi = 0$ (compatibility condition)

This leads to a set of algebraic equation for c_i

* You'd better use a computer program, e.g. Matlab to avoid making mistakes.

(step 4) Boundary conditions (strong & weak)

Leads to another set of algebraic equations for c_i

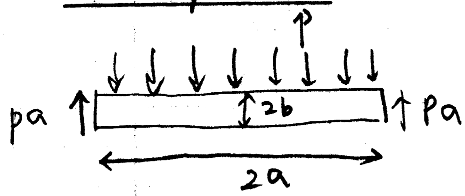
(step 5) Solve all equations and determine c_i

* Some of the equations can be redundant, but Matlab can handle that

* If Matlab fails to give a solution, then perhaps a strong B.C. needs to be reduced to a week B.C.

§6. Example 2

(Barber P.57)



$q(x) = p \cdot x^0$

$n = 0.$

maximum order = 5

$\phi = C_1 x^2 + C_2 xy + C_3 y^2 + \cdots + C_{18} y^5$

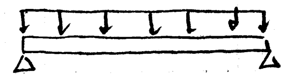
use matlab program

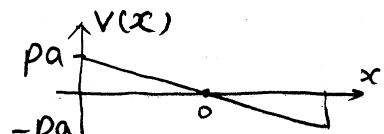
S522a.m

(next page)

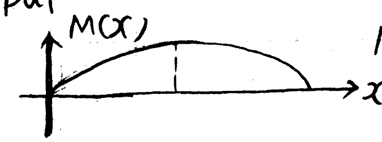
* on coursework – Homeworks folder.

* Mechanics of Materials Solution





$V(x) = -px$



$M(x) = \frac{1}{2} p (a^2 - x^2)$

$\sigma_{xx} = -\frac{My}{I} = \frac{3p}{4b^3}(x^2 - a^2) \cdot y$

S522a.m

define $C_1 \cdots C_{18}$, x, y, a, b, p as symbolic variables

$$\phi = C_1 x^2 + C_2 xy + C_3 y^2 + \cdots + C_{18} y^5$$

Sxx : $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$

Syy : $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$ ← they are also polynomial of x, y.

Sxy : $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

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t_1 : $\sigma_{yy}(y=b)$ ← surface tractions, also polynomials.

t_2 : $\sigma_{xy}(y=b)$

t_3 : $\sigma_{yy}(y=-b)$

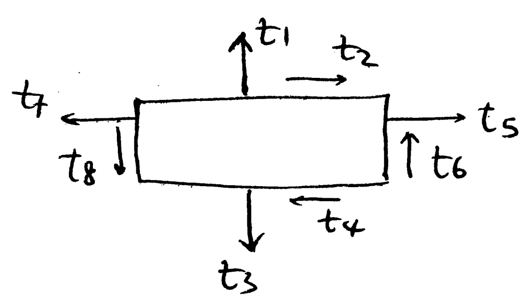
t_4 : $\sigma_{xy}(y=-b)$

t_5 : $\sigma_{xx}(x=a)$

t_6 : $\sigma_{xy}(x=a)$

t_7 : $\sigma_{xx}(x=-a)$

t_8 : $\sigma_{xy}(x=-a)$



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introduce polynomial coefficients: $S_1 \cdots S_{16}$, $b_1 \cdots b_3$, ...

Let
$t_1 = S_1 x^3 + S_2 x^2 + S_3 x + S_4$
$t_2 = S_5 x^3 + S_6 x^2 + S_7 x + S_8$
$t_3 = S_9 x^3 + S_{10} x^2 + S_{11} x + S_{12}$
$t_4 = S_{13} x^3 + S_{14} x^2 + S_{15} x + S_{16}$

↑ these coefficients can be found by Matlab.

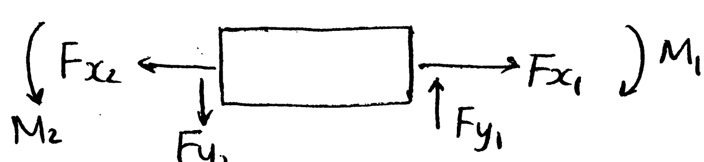
$$\nabla^4 \phi = b_1 x + b_2 y + b_3$$

total resultant force and moment at ends.

$F_{x1} = \int_{-b}^{b} t_5 \, dy$

$F_{y1} = \int_{-b}^{b} t_6 \, dy$

$M_1 = \int_{-b}^{b} t_5 \, y \, dy$



$F_{x2} = \int_{-b}^{b} t_7 \, dy$

$F_{y2} = \int_{-b}^{b} t_8 \, dy$

$S_1 \cdots S_{16}$, b_1, b_2, b_3, Fx_1, Fy_1, M_1, Fx_2, Fy_2, M_2

are all linear combinations of $C_1 \cdots C_{18}$

Equations to solve:

$S_1=0$, $S_2=0$, $S_3=0$, $S_4+p=0$

$S_5=0$, $S_6=0$, $S_7=0$, $S_8=0$

...

$S_{16}=0$

$b_1=0$, $b_2=0$, $b_3=0$

$Fx_1=0$, $M_1=0$, $Fy_1-pa=0$

$Fx_2=0$, $M_2=0$, $Fy_2+pa=0$

more equations than unknowns.

equations contain redundancy

but Matlab can handle it.

Solution:

$$\phi = -\frac{p}{40b^3}\left(10x^2b^3+15x^2yb^2-2y^3b^2+5y^3a^2-5x^2y^3+y^5\right) \quad \text{---} (*)$$

$$\sigma_{xx} = -\frac{py}{20b^3}\left(-6b^2+15a^2-15x^2+10y^2\right)$$

$$\sigma_{yy} = \frac{p}{4b^3}\left(-2b^3-3yb^2+y^3\right)$$

$$\sigma_{xy} = -\frac{3px}{4b^3}\left(-b^2+y^2\right)$$

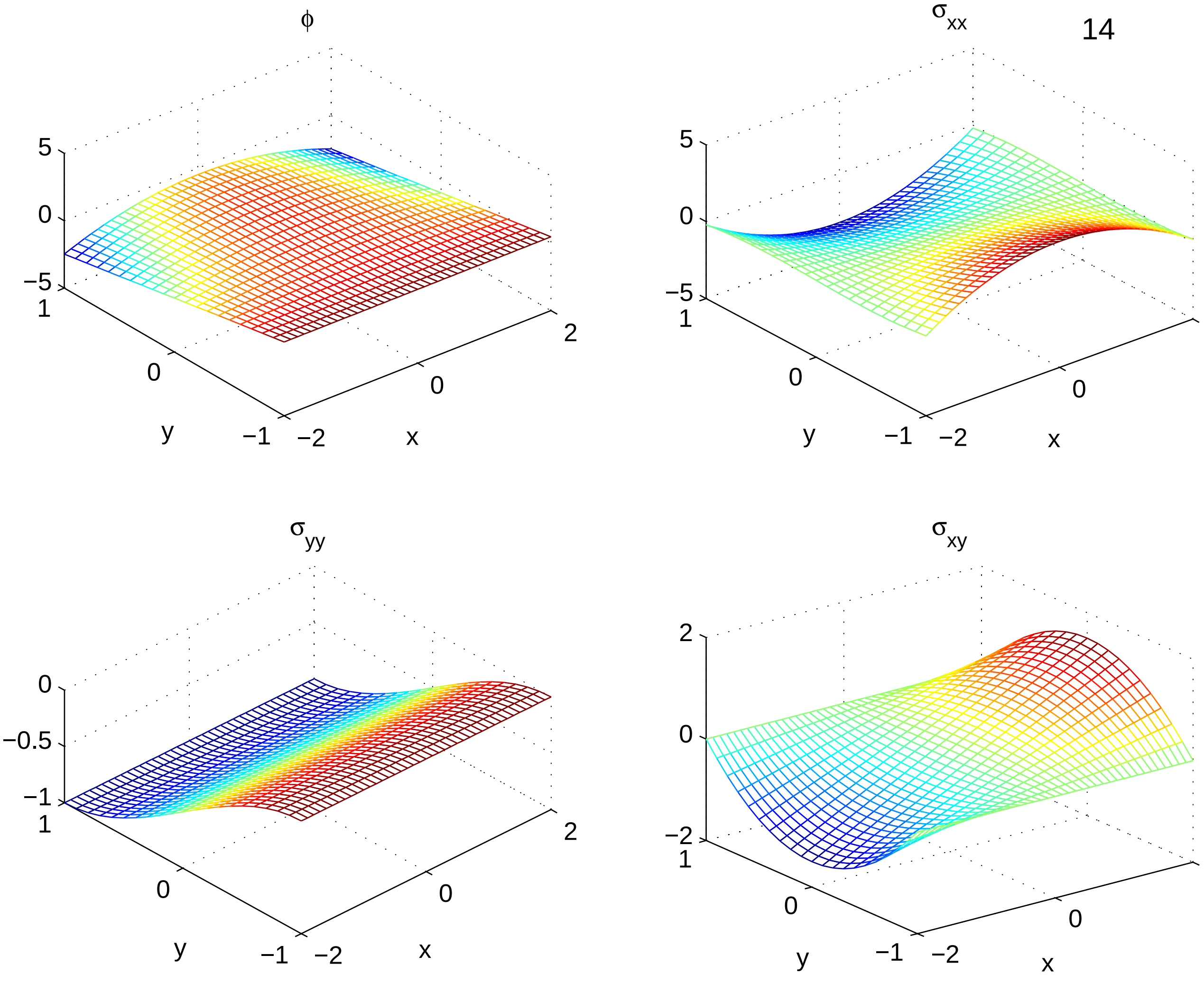
plot S522a-plot.pdf

inspect whether B.C. are satisfied.

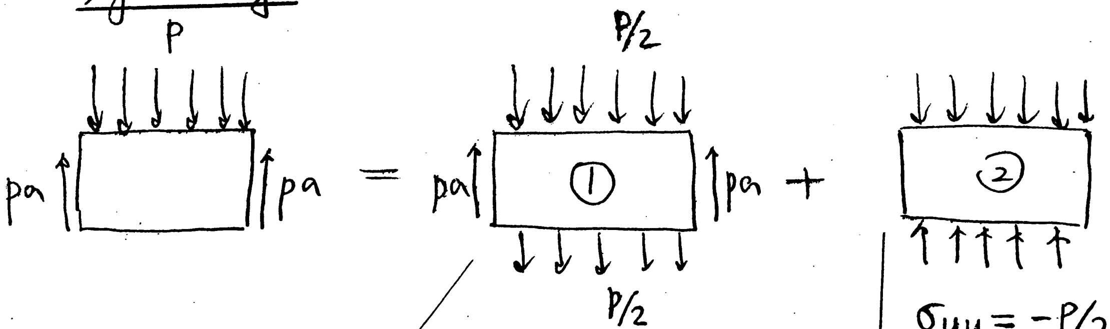
```
%adpated from maple file S522 from J. R. Barber, Elasticity
% http://www-personal.engin.umich.edu/~jbarber/elasticity/maple/S522
%
% adapted by Wei Cai, caiwei@stanford.edu, for ME 340 Elasticity
% Spring 2006, Stanford University
%
%This file gives the solution of the simply-supported beam problem,
%following the strategy of Section 5.2.2.
%

clear all;

syms C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 C11 C12 C13 C14 C15 C16 C17 C18
syms x y a b p

%stress function
phi = C1*x^2+C2*x*y+C3*y^2+C4*x^3+C5*x^2*y+C6*x*y^2+C7*y^3+C8*x^4+ ...
      C9*x^3*y+C10*x^2*y^2+C11*x*y^3+C12*y^4+C13*x^5+C14*x^4*y+ ...
      C15*x^3*y^2+C16*x^2*y^3+C17*x*y^4+C18*y^5;

%stress field
sxx =  diff(diff(phi,y),y);
syy =  diff(diff(phi,x),x);
sxy = -diff(diff(phi,x),y);

%traction force on y=b
t1 = subs(syy,y,b);   %Ty on y=b
t2 = subs(sxy,y,b);   %Tx on y=b

%traction force on y=-b
t3 = subs(syy,y,-b);  %Ty on y=-b
t4 = subs(sxy,y,-b);  %Tx on y=-b

%traction force on x=a
t5 = subs(sxx,x,a);   %Tx on x=a
t6 = subs(sxy,x,a);   %Ty on x=a

%traction force on x=-a
t7 = subs(sxx,x,-a);  %Tx on x=-a
t8 = subs(sxy,x,-a);  %Ty on x=-a

%find coefficients of polynomials t1,t2,t3,t4
s1 = subs(diff(t1,x,3),x,0)/factorial(3);
s2 = subs(diff(t1,x,2),x,0)/factorial(2);
s3 = subs(diff(t1,x,1),x,0)/factorial(1);
s4 = subs(t1,x,0);

s5 = subs(diff(t2,x,3),x,0)/factorial(3);
s6 = subs(diff(t2,x,2),x,0)/factorial(2);
s7 = subs(diff(t2,x,1),x,0)/factorial(1);
s8 = subs(t2,x,0);
```

```
s9  = subs(diff(t3,x,3),x,0)/factorial(3);
s10 = subs(diff(t3,x,2),x,0)/factorial(2);
s11 = subs(diff(t3,x,1),x,0)/factorial(1);
s12 = subs(t3,x,0);

s13 = subs(diff(t4,x,3),x,0)/factorial(3);
s14 = subs(diff(t4,x,2),x,0)/factorial(2);
s15 = subs(diff(t4,x,1),x,0)/factorial(1);
s16 = subs(t4,x,0);

%The biharmonic equation is 4th order, so applying it to a 5th order polynomial
%generates a first order polynomial.
%
biharm = diff(phi,x,4)+diff(phi,y,4)+2*diff(diff(phi,x,2),y,2);

%coefficients of biharm
b1 = subs(diff(biharm,x,1),{x,y},{0,0});
b2 = subs(diff(biharm,y,1),{x,y},{0,0});
b3 = subs(biharm,{x,y},{0,0});

%integrated force and torque on x=a
Fx1 = simplify(int(t5, y, -b, b));
Fy1 = simplify(int(t6, y, -b, b));
M1  = simplify(int(t5*y, y, -b, b));
%
%integrated force and torque on x=-a
Fx2 = simplify(int(t7, y, -b, b));
Fy2 = simplify(int(t8, y, -b, b));
M2  = simplify(int(t7*y, y, -b, b));

%Solve all these equations together
s = solve(s1, s2, s3, s4+p, s5, s6, s7, s8,  ...
          s9, s10,s11,s12,  s13,s14,s15,s16, ...
          b1, b2, b3,       Fx1,M1, Fy1-p*a, ...
          'C1','C2','C3','C4','C5','C6','C7','C8','C9',...
          'C10','C11','C12','C13','C14','C15','C16','C17','C18');

%construct cell arrays for future use
coeffs = {C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,C14,C15,C16,C17,C18};
solution = {s.C1,s.C2,s.C3,s.C4,s.C5,s.C6,s.C7,s.C8,s.C9, ...
            s.C10,s.C11,s.C12,s.C13,s.C14,s.C15,s.C16,s.C17,s.C18};

%stress function and stress field solution
phi2 = simplify(subs(phi, coeffs, solution));

sxx2 =  simplify(subs(sxx, coeffs, solution));
syy2 =  simplify(subs(syy, coeffs, solution));
sxy2 =  simplify(subs(sxy, coeffs, solution));

%print out results
disp('phi='); pretty(phi2)
disp('sxx='); pretty(sxx2)
```

```
disp('syy='); pretty(syy2)
disp('sxy='); pretty(sxy2)

%plot stress fields
an = 2.0; bn = 1.0; pn = 1.0;
xn = [-1:0.05:1]*an;
yn = [-1:0.1:1]*bn;
xx = ones(size(yn'))*xn;
yy = yn'*ones(size(xn));
phin = subs(phi2,{a,b,p},{an,bn,pn}); phin = subs(phin,{x,y},{xx,yy});
sxxn = subs(sxx2,{a,b,p},{an,bn,pn}); sxxn = subs(sxxn,{x,y},{xx,yy});
syyn = subs(syy2,{a,b,p},{an,bn,pn}); syyn = subs(syyn,y,yy);
sxyn = subs(sxy2,{a,b,p},{an,bn,pn}); sxyn = subs(sxyn,{x,y},{xx,yy});

figure(1);
subplot(2,2,1); mesh(xn,yn,phin);
title('\phi'); xlabel('x'); ylabel('y');
subplot(2,2,2); mesh(xn,yn,sxxn);
title('\sigma_{xx}'); xlabel('x'); ylabel('y');
subplot(2,2,3); mesh(xn,yn,syyn);
title('\sigma_{yy}'); xlabel('x'); ylabel('y');
subplot(2,2,4); mesh(xn,yn,sxyn);
title('\sigma_{xy}'); xlabel('x'); ylabel('y');
% results
%phi=
%
%                    2  3      2   2     3  2     3  2     2  3   5
%             p (10 x  b  + 15 x  y b  - 2 y  b  + 5 y  a  - 5 x  y  + y )
%      - 1/40 ------------------------------------------------------------
%                                         3
%                                        b
%sxx=
%
%                                2       2       2       2
%                        p y (-6 b  + 15 a  - 15 x  + 10 y )
%                   - 1/20 -----------------------------------
%                                        3
%                                       b
%syy=
%
%                                  3        2    3
%                            p (-2 b  - 3 y b  + y )
%                        1/4 -----------------------
%                                       3
%                                      b
%sxy=
%
%                                       2    2
%                                 p x (-b  + y )
%                            - 3/4 --------------
%                                        3
%                                       b
```


φ
σxx
σyy
σxy
x
y
5
0
−5
2
0
−2
1
0
−1
−0.5

§7. Symmetry considerations.



$\sigma_{yy} = -P/2$ (everywhere)
uniform stress field.

① σ_{yy} — odd function of y : $\frac{\partial^2 \phi}{\partial x^2}$

σ_{xy} — odd function of x : $-\frac{\partial^2 \phi}{\partial x \partial y}$

⇓

$\phi(x,y)$ must be an odd function of y (y, y^3, y^5, etc)
and even function of x. (x^0, x^2, x^4, etc)

$$\overset{①}{\phi} = C_5 x^2 y + C_7 y^3 + C_{14} x^4 y + C_{16} x^2 y^3 + C_{18} y^5$$

only 5 terms survive!

all other $C_i = 0$.

after solving this problem, add $\begin{cases} \overset{②}{\sigma_{yy}} = -\frac{P}{2} \\ \phi^{②} = -\frac{P}{4} x^2 \end{cases}$

to obtain the solution of original problem.

This is the first term in Eq. (*).