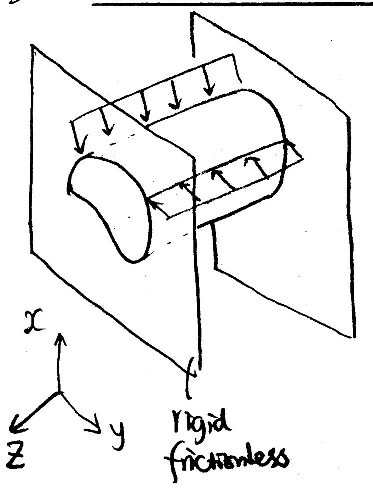
The equations of elasticity can be greatly simplified if we restrict the solution to be 2-Dimensional, i.e. does not depend on one coordinate (e.g. z).

There are several types of 2D elasticity problems:

- plane strain
- plane stress
- anti-plane strain (e.g.* straight screw dislocations)

etc.

§1. Plane Strain



consider a long bar with ends constrained by rigid-frictionless plates

Let the loads applied on the side of the bar be independent of z.

Then $u_z = 0$ every where

u_x, u_y independent of z.

i.e. the unknowns of this problem is $u_x(x,y)$ and $u_y(x,y)$

We can easily show

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0, \qquad \varepsilon_{yz} = 0, \qquad \varepsilon_{zz} = 0$$

So the only non-zero strain components are

ε_{xx}, ε_{yy}, ε_{xy} — hence the name plane strain.

In isotropic elasticity, $\sigma_{xz} = \sigma_{yz} = 0$.

but in general $\sigma_{zz} \neq 0$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} \rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

equilibrium condition

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \cancel{\frac{\partial \sigma_{zx}}{\partial z}} + F_x = 0 \qquad \left(\frac{\partial}{\partial z} = 0\right)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \cancel{\frac{\partial \sigma_{zy}}{\partial z}} + F_y = 0$$

$$\cancel{\frac{\partial \sigma_{xz}}{\partial x}} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + \cancel{F_z} = 0 \qquad (\sigma_{xz} = \sigma_{yz} = 0, \ F_z = 0)$$

so the only non-trival equations are

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + F_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \end{aligned}}, \qquad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

compatibility condition

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

only one non-trivial equation when $i,j,k,l = x,y$

$$\boxed{\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0}$$

Generalized Hooke's Law

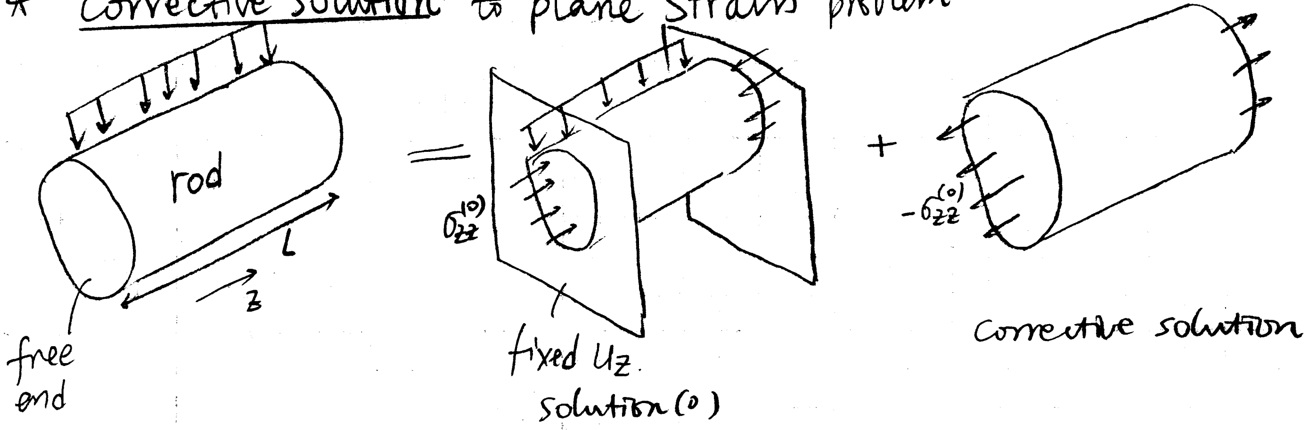
$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} = 0 \rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\therefore \boxed{\begin{aligned} \varepsilon_{xx} &= \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} \\ \varepsilon_{yy} &= -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy} \end{aligned}}$$

* corrective solution to plane strain problem



In reality, we seldom have rods with both ends constrained by rigid, frictionless plates.

However, the plane strain solution is a good approximation to a long rod subjected to lateral loading that is z-independent

To better describe the physical problem (rods with free end), a corrective solution should be added, in which axial forces are applied to the ends.

Because there may be a net force F at the end, the corrective solution is not necessary limited to the neighborhood of the ends.

$$F = -\int_A \sigma_{zz}^{(0)}\, dA.$$

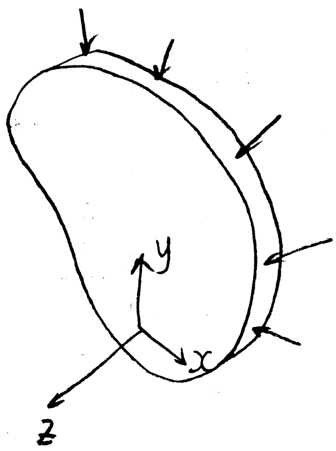
The corrective solution can be approximated by a simple solution

$$\sigma_{zz}^{(1)} = \frac{F}{A} = -\frac{1}{A}\int_A \sigma_{zz}^{(0)}\, dA.$$

The rod lengths with change by $\Delta L = \frac{FL}{EA} = -\frac{L}{EA}\int_A \sigma_{zz}^{(0)}\, dA$

* Of course, the "true" corrective solution is going to be more complicated by this. So we can imagine a correction solution to this correction solution, which should then be localized near the ends.

§2. Plane Stress



The opposite limit of plane strain problem (thick rods) is a thin film — plane stress problem.

Because $\sigma_{xz} = \sigma_{yz} = \sigma_{zz}$ on the surface, if the film is sufficiently thin, we can expect

$$\sigma_{xz} \approx 0, \quad \sigma_{yz} \approx 0, \quad \sigma_{zz} \approx 0$$

everywhere <u>inside</u> the film.

Plane Stress condition:

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \qquad , \qquad \frac{\partial}{\partial z} = 0.$$

Generalized Hookes Law:

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\ \varepsilon_{yy} &= \frac{1}{E}(-\nu\sigma_{xx} + \sigma_{yy}) \end{aligned}} \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \, . \qquad \varepsilon_{zz} \neq 0 .$$

equilibrium condition:

$$\boxed{\begin{aligned} \sigma_{xx,x} + \sigma_{yx,y} + F_x &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + F_y &= 0 \end{aligned}}$$

Same as plane strain

compatibility condition:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

$$\boxed{\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0} \qquad i=j=x, \quad k=l=y$$

Same as plane strain

$$* \quad \begin{cases} \varepsilon_{zz,xx} = 0 \\ \varepsilon_{zz,yy} = 0 \\ \varepsilon_{zz,xy} = 0 \end{cases} \quad \text{(cannot be satisfied all at the same time)} \qquad \begin{aligned} &i=j=z, \quad k=l=x \\ &i=j=z, \quad k=l=y \\ &i=j=z, \quad k=x, \ l=y \end{aligned}$$

Because $\varepsilon_{zz}(x,y) \neq 0$, we have more compatibility conditions.

§3. Equivalence between Plane Strain and Plane Stress

Both Plane strain and Plane Stress Problems seek solutions for

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ and $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$

as functions of x and y.

They satisfy the same equilibrium and compatibility conditions.

The only difference is in the Generalized Hooke's Law.

Plane Strain	Plane Stress
$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}$	$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}$
$\varepsilon_{yy} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}$	$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy}$
$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$	$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$

introduce Kolosov's constant κ

$$\boxed{\kappa = 3-4\nu} \qquad\qquad \boxed{\kappa = \frac{3-\nu}{1+\nu}}$$

Then both Plane stress and Plane Strain Problems satisfy the same condition

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} \\ \varepsilon_{yy} &= -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{xx} \end{aligned}}$$

Hence we can solve each 2D problem in either plane strain or in plane stress condition and find the other solution by replacing the Kolosov's constant.

§4. Airy Stress function

- Introduce a scalar function $\phi(x,y)$, and write different components of the stress tensor as derivatives of ϕ.
- We need to use 2nd derivatives, so that they transform as rank-2 tensor.
- Assume zero body force: $F_x = F_y = 0$

  then the equilibrium condition becomes

  $$\sigma_{xx,x} + \sigma_{xy,y} = 0 \qquad \text{and} \qquad \sigma_{yx,x} + \sigma_{yy,y} = 0$$

- These can be <u>automatically satisfied</u> by defining

$$\boxed{\sigma_{xx} = \phi_{,yy}} \qquad \boxed{\sigma_{yy} = \phi_{,xx}} \qquad \boxed{\sigma_{xy} = -\phi_{,xy}}$$

notice the minus sign!

The equilibrium condition can be easily verified:

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = \phi_{,yyx} - \phi_{,xyy} = 0 \\ \sigma_{yx,x} + \sigma_{yy,y} = -\phi_{,xyx} + \phi_{,xxy} = 0 \end{cases}$$

- So the only PDE we need to worry about is the compatibility condition.

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\,\varepsilon_{xy,xy} = 0$$

$$\begin{cases} \varepsilon_{xx} = \frac{\kappa+1}{8\mu}\phi_{,yy} - \frac{3-\kappa}{8\mu}\phi_{,xx} \\ \varepsilon_{yy} = -\frac{3-\kappa}{8\mu}\phi_{,yy} + \frac{\kappa+1}{8\mu}\phi_{,xx} \\ \varepsilon_{xy} = -\frac{1}{2\mu}\phi_{,xy} \end{cases}$$

$$\frac{\kappa+1}{8\mu}(\phi_{,yyyy} + \phi_{,xxxx}) - \frac{3-\kappa}{8\mu}\cdot 2\phi_{,xxyy} + \frac{2}{2\mu}\phi_{,xyxy} = 0$$

$$\frac{\kappa+1}{8\mu}(\phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xxyy}) = 0$$

$$\therefore \quad \boxed{\nabla^4\phi = 0} \qquad \text{or} \quad \nabla^2\nabla^2\phi = 0 \qquad \text{or} \quad \phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xyxy} = 0$$

§5. Airy stress function in the presence of body forces.

When the body force can be written as derivatives of a potential

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad (\text{e.g. } V = \rho g y \text{ for gravity})$$

Then the Airy stress function can be defined as

$$\sigma_{xx} = \phi_{,yy} + V \qquad \sigma_{yy} = \phi_{,xx} + V \qquad \sigma_{xy} = -\phi_{,xy}$$

We can verify that the equilibrium condition

$$\sigma_{xx,x} + \sigma_{xy,y} + F_x = 0$$
$$\sigma_{yx,x} + \sigma_{yy,y} + F_y = 0$$

is automatically satisfied.

The compatibility condition becomes

$$\boxed{\nabla^4 \phi = -2 \cdot \frac{\kappa - 1}{\kappa + 1} \cdot \nabla^2 V}$$

in plane strain	in plane stress
$\nabla^4 \phi = -\left(\frac{1-2\nu}{1-\nu}\right) \nabla^2 V$	?

§6. Example 1

when there is no body force, we have $\nabla^4 \phi = 0$

Let's pick a trial solution

$$\phi(x, y) = \alpha x + \beta y + \gamma$$

It obviously satisfy the condition $\nabla^4 \phi = 0$.

What does this solution mean?

Let's find out the stress field.

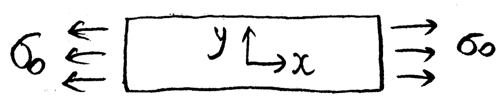
$$\sigma_{xx} = \phi_{,yy} = ?$$
$$\sigma_{yy} = \phi_{,xx} =$$
$$\sigma_{xy} = -\phi_{,xy} =$$

§7. Example 2.

Let's go to higher order polynomials

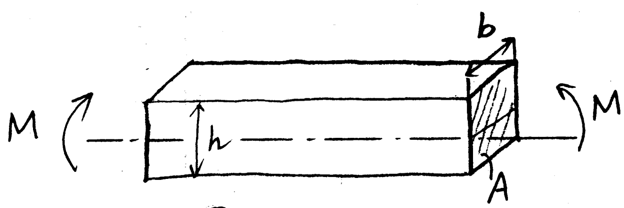
$$\phi(x,y) = \frac{1}{2}Ax^2 + \frac{1}{2}Bxy - Cxy$$

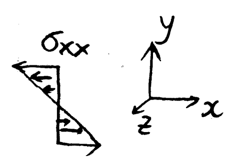
$\sigma_{xx} =$ ________ $\sigma_{yy} =$ ________ $\sigma_{xy} =$ ________



Write down the stress function for a rectangular bar under uniaxial tensile stress σ_0.

$\phi(x,y) =$ __________





$\sigma_{xx} \propto y$

$$M = -\int_{-h/2}^{h/2} b\,\sigma_{xx}(y)\cdot y\,dy$$

$$\boxed{\sigma_{xx} = -\frac{My}{I_z}}$$

$$I_z = \int_A y^2 dA = \frac{bh^3}{12}$$

↑ moment of inertia

Write down the stress function for a rectangular bar under pure bending M

$\phi(x,y) =$ __________