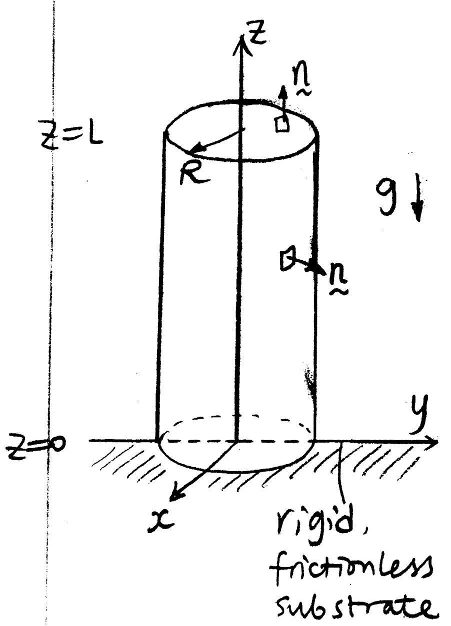

Example: Elastic rod standing vertically in gravitational field on a rigid, frictionless substrate

<u>Formulation of the B.V.P.</u>

body force $F_z = -\rho g$

equilibrium: $\sigma_{ij,i} + F_j = 0$

i.e. $\begin{cases} \sigma_{ix,i} = 0 \\ \sigma_{iy,i} = 0 \\ \sigma_{iz,i} = \rho g \end{cases}$

Boundary Condition:

top surface: $(z=L)$ zero traction force $(\underset{\sim}{n} = \underset{\sim}{e}_z)$

$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$

side surface: $\sqrt{x^2+y^2} = R$ zero traction force

$\sigma_{xi} n_i = \sigma_{yi} n_i = \sigma_{zi} n_i = 0$

(top and side surfaces: S_t)

bottom surface: $z=0$ $u_z = 0$

$\sigma_{xz} = \sigma_{yz} = 0$

mixed B.C.

How to solve this B.V.P.?

By trial and error.

Step ①

Let's <u>try</u> a solution with

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0$$

the only non-zero stress component is σ_{zz}

equilibrium condition reduces to: $\frac{\partial}{\partial z} \sigma_{zz} = \rho g$

$\therefore \quad \sigma_{zz} = \rho g z + C \qquad$ (C is a constant)

Boundary Condition: $\sigma_{zz} = 0$ at $z = L$

$$\therefore \quad C = -\rho g L$$

$$\sigma_{zz} = \rho g (z - L)$$

(This is a solution that satisfies the equilibrium condition and traction boundary conditions.)

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step② Let's find all strain components from the generalized Hooke's Law

$$\varepsilon_{xx} = -\frac{\nu}{E}\sigma_{zz} = -\frac{\nu \rho g}{E}(z-L)$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{zz} = -\frac{\nu \rho g}{E}(z-L)$$

$$\varepsilon_{zz} = \frac{1}{E}\sigma_{zz} = \frac{\rho g}{E}(z-L)$$

$$\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \qquad \text{(all shear strains components zero)}$$

(So we find all strain components as well. Does the strain satisfy compatibility condition? We will find out by trying to find the displacement solution.)

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step③ Try to find displacement u_x, u_y, u_z.

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

arbitrary function

$$\varepsilon_{zz} = \frac{\partial}{\partial z}u_z = \frac{\rho g}{E}(z-L) \xrightarrow{\text{integrate}} u_z = \frac{\rho g}{E}\left(\frac{z^2}{2} - Lz\right) + f(x,y)$$

$$\varepsilon_{xx} = \frac{\partial}{\partial x}u_x = -\frac{\nu \rho g}{E}(z-L) \longrightarrow u_x = -\frac{\nu \rho g}{E}x(z-L) + g(y,z)$$

$$\varepsilon_{yy} = \frac{\partial}{\partial y}u_y = -\frac{\nu \rho g}{E}(z-L) \longrightarrow u_y = -\frac{\nu \rho g}{E}y(z-L) + h(x,z)$$

Q: How do we determine the unknown functions $f(x,y)$, $g(y,z)$, $h(x,z)$?

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0 \rightarrow \frac{\partial g(y,z)}{\partial y} + \frac{\partial h(x,z)}{\partial x} = 0 \quad \text{---(1)}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0 \rightarrow -\frac{\nu \rho g}{E} y + \frac{\partial h(x,z)}{\partial z} + \frac{\partial f(x,y)}{\partial y} = 0 \quad \text{---(2)}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \rightarrow -\frac{\nu \rho g}{E} x + \frac{\partial g(y,z)}{\partial z} + \frac{\partial f(x,y)}{\partial x} = 0 \quad \text{---(3)}$$

From Eq (2). $-\frac{\nu \rho g}{E} y + \frac{\partial f(x,y)}{\partial y} = -\frac{\partial h(x,z)}{\partial z} = m(x)$ ---(4)

Left hand side independent of z

Right hand side independent of y

$\therefore$ both sides must be independent of both y and z, i.e. only a function of (x)

$$\frac{\partial f(x,y)}{\partial y} = \frac{\nu \rho g}{E} y + m(x)$$

unknown functions of x but independent of y

$$f(x,y) = \frac{\nu \rho g}{2E} y^2 + m(x) y + p(x) \quad \text{---(5)}$$

Similarly from Eq (3).

$$-\frac{\nu \rho g}{E} x + \frac{\partial f(x,y)}{\partial x} = -\frac{\partial g(y,z)}{\partial z} = n(y) \quad \text{---(6)}$$

independent of z

independent of x

unknown function of y

$$\frac{\partial f(x,y)}{\partial x} = \frac{\nu \rho g}{E} x + n(y)$$

unknown functions of y but independent of x

$$f(x,y) = \frac{\nu \rho g}{2E} x^2 + n(y)\, x + q(y) \quad \text{---(7)}$$

Combine Eq (5) and (7). $f(x,y)$ must have the following form:

$$\boxed{f(x,y) = \frac{\nu \rho g}{2E}(x^2+y^2) + C\,xy + D}$$

i.e. $p(x) = \frac{\nu \rho g}{2E} x^2 + D$

$q(y) = \frac{\nu \rho g}{2E} y^2 + D$

$m(x) = C \cdot x$

$n(y) = C \cdot y$

We still need to determine the unknown constants C and D.

From Eq. (4) $-\frac{\partial h(x,z)}{\partial z} = m(x) = Cx \rightarrow h(x,z) = -Cxz + r(x)$

From Eq. (6) $-\frac{\partial g(y,z)}{\partial z} = n(y) = Cy \rightarrow g(y,z) = -Cyz + S(y)$

Insert these results into Eq. (1) : $\frac{\partial g(y,z)}{\partial y} + \frac{\partial h(x,z)}{\partial x} = 0$

$$-Cz + S'(y) - Cz + r'(x) = 0$$

$$S'(y) + r'(x) = 2Cz$$

left hand side independent of z Right hand side independent of x, y

$\therefore$ $\boxed{C = 0}$

$S'(y) = -r'(x) = E \leftarrow \therefore$ both sides must be a constant E.

independent of x independent of y

$$S(y) = Ey + G$$
$$r(x) = -Ex + H$$

$$\therefore \boxed{\begin{aligned} h(x,z) &= -Ex + H \\ g(y,z) &= Ey + G \end{aligned}}$$

$$\begin{cases} u_z = \frac{\rho g}{E}\left(\frac{z^2}{2} - Lz\right) + \underline{\frac{\nu \rho g}{2E}(x^2+y^2) + D} \\ u_x = -\frac{\nu \rho g}{E} x(z-L) + \underline{Ey + G} \\ u_y = -\frac{\nu \rho g}{E} y(z-L) \underline{- Ex + H} \end{cases}$$

We still need to determine the unknown constants

$\underline{D, E, G, H}$

Constants D, G, H corresponds to rigid-body translation

e.g. consider $\begin{cases} u_z = D \\ u_x = G \\ u_y = H \end{cases}$

They can be determined by fixing one point in the Rod as a point of reference — otherwise we get infinite number of solutions

For convenience, let's choose the origin as the reference point

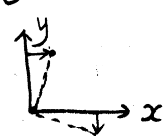
i.e. $u_x(0,0,0) = u_y(0,0,0) = u_z(0,0,0) = 0$.

$\therefore \quad D = G = H = 0.$

↑ also consistent with our B.C.

Constant E corresponds to a rigid-body rotation around z-axis.

e.g. consider $\begin{cases} u_x = Ey \\ u_y = -Ex \end{cases}$



Let's assume the rod does not undergo any rigid-body rotation — otherwise we get infinite number of solutions.

$\therefore \quad E = 0.$

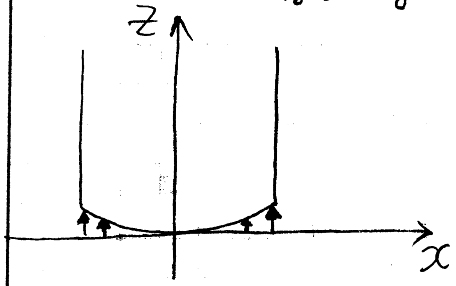
Here is the displacement field of our trial solution:

$$\begin{cases} u_z = \dfrac{\rho g}{E}\left(\dfrac{z^2}{2} - Lz\right) + \dfrac{\nu \rho g}{2E}(x^2+y^2) \\ u_x = -\dfrac{\nu \rho g}{E}\, x\,(z-L) \\ u_y = -\dfrac{\nu \rho g}{E}\, y\,(z-L) \end{cases}$$

- no more undecided constants
- satisfies both equilibrium and compatibility conditions.
- satisfies all traction boundary conditions.
- does it satisfy the displacement boundary condition $u_z(x,y,0) = 0$?

on the plane $z=0$. violates our B.C.

$$u_z(x,y) = \frac{\nu \rho g}{2E}(x^2+y^2) \neq 0 \ !$$



The bottom of the elastic rod curls up!
How did that happen?

Of course in reality, the bottom of the rod <u>does not</u> curl up.
This only shows that the trial solution is not the "true solution".
That's too bad ... especially after so much work...

Wait a minute. In elasticity theory, we would say the solution is still useful, in the following sense(s).

First, we can always modify our original problem and let the rod sit on a curved substrate with exactly a shape of

$$u_z(x,y) = \frac{\nu \rho g}{2E}(x^2+y^2) \quad \text{---- i.e. a parabola.}$$

Then our trial solution is the exact solution of this problem.

Second, intuitively we expect our trial solution is pretty good except near the bottom. Because elasticity equations are linear, we can imagine a "correction solution" that can be added to our solution to get the "true solution".
From the Saint Venant's principle, we expect the "correction solution" to be significant only near the rod bottom.

Our trial solution satisfies the displacement B.C. only at a single point $x=0, y=0, z=0$, instead of on the entire plane $z=0$.

Hence we say the displacement B.C. is only satisfied in the <u>weak sense</u>.