Fundamental Equations of Elasticity:

compatibility. Equilibrium.

§1. Compatibility condition for strains

u_i has 3 degrees of freedom (at every point $\underset{\sim}{x}$)

ε_{ij} has 6 degrees of freedom

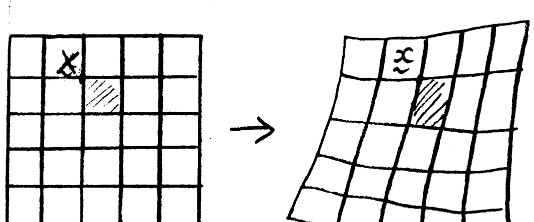
given $u_i(\underset{\sim}{x})$, we can always find $\varepsilon_{ij}(\underset{\sim}{x})$ by differentiation: $\varepsilon_{ij}(\underset{\sim}{x}) = \frac{1}{2}(u_{i,j} + u_{j,i})$

on the other hand, if we are given an arbitrary $\varepsilon_{ij}(\underset{\sim}{x})$, we may not always be able to find a single valued, continuous u_j

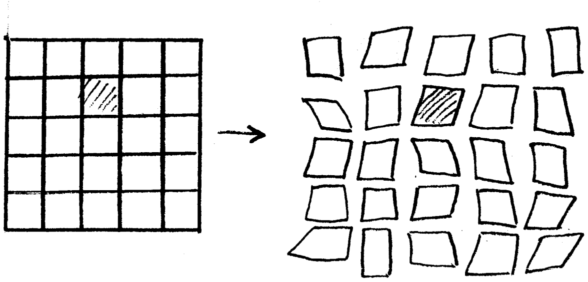
In order to be able to find a corresponding $u_j(\underset{\sim}{x})$ $\varepsilon_{ij}(\underset{\sim}{x})$ must satify some constraints.

$$\boxed{\text{compatibility condition: } \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0}$$

— This can be verified by plugging it into $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
* What is more difficult is to show this is a sufficient condition for us to find u_j corresponding to ε_{ij}. (omitted here)

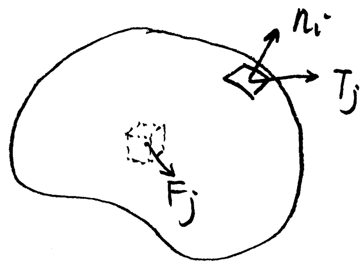


Given a displacement field $u_i(\underset{\sim}{x})$, it is straightforward to obtain the strain field.



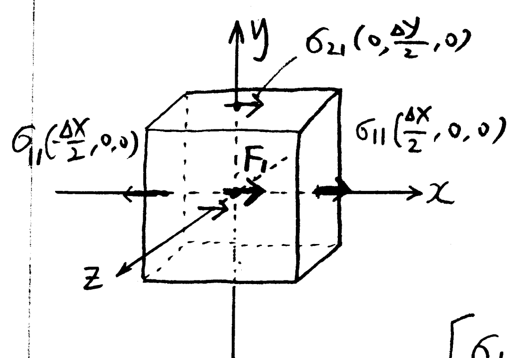
Given a strain field, $\varepsilon_{ij}(\underset{\sim}{x})$, we can imagine breaking the orginal medium into many pieces, each is deformed according to the local strain.
— but the deformed pieces may not fit together (incompatible).

§2. Equilibrium condition for Stress



consider a continuum body subjected to traction force T_j per unit area on the surface (normal vector n_i) and body force F_j per unit volume

$$\boxed{\text{equilibrium condition: } \sigma_{ij,i} + F_j = 0}$$



cube size
$\Delta x \cdot \Delta y \cdot \Delta z$

recall: σ_{ij} is force per unit area on i-th face of the stress cube in j-th direction

Total force balance in x-direction:

$$\left[\sigma_{11}(\tfrac{\Delta x}{2},0,0) - \sigma_{11}(-\tfrac{\Delta x}{2},0,0)\right]\cdot \underbrace{\Delta y\,\Delta z}_{\text{area}}$$
$$+\left[\sigma_{21}(0,\tfrac{\Delta y}{2},0) - \sigma_{21}(0,-\tfrac{\Delta y}{2},0)\right]\cdot \Delta x\,\Delta z$$
$$+\left[\sigma_{31}(0,0,\tfrac{\Delta z}{2}) - \sigma_{31}(0,0,-\tfrac{\Delta z}{2})\right]\cdot \Delta x\,\Delta y$$
$$+ F_1(0,0,0)\cdot \Delta x\,\Delta y\cdot\Delta z = 0$$

In the limit of $\Delta x, \Delta y, \Delta z \to 0$

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + F_1 = 0$$

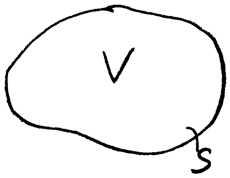
$\sigma_{i1,i} + F_1 = 0$ — total force balance in x-direction.

Similarly: $\sigma_{i2,i} + F_2 = 0$ — total force balance in y-direction

$\sigma_{i3,i} + F_3 = 0$ — total force balance in z-direction.

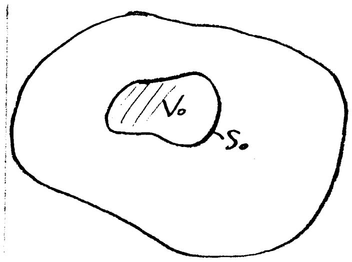
$\therefore$ $\sigma_{ij,i} + F_j = 0$. j is free index

A more formal proof can be obtained using
<u>Gauss's Theorem</u> (divergence theorem)



$$\int_S f_i n_i \, dS = \int_V f_{i,i} \, dV \quad \text{(index notation)}$$

$$\int_S \underset{\sim}{f} \cdot \underset{\sim}{n} \, dS = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{f} \, dV \quad \text{(vector notation)}$$



consider an arbitrary volume V_0 inside the elastic medium.

Force equilibrium requires

$$\underbrace{\int_{S_0} T_j \, dS}_{\substack{\text{traction force} \\ \text{on the surface } S_0}} + \underbrace{\int_{V_0} F_j \, dV}_{\substack{\text{body force} \\ \text{inside } V_0}} = 0$$

$$\int_{S_0} \sigma_{ij} n_i \, dS + \int_{V_0} F_j \, dV = 0$$

Apply Gauss's theorem, considering σ_{ij} as f_i

$$\int_{V_0} (\sigma_{ij,i} + F_j) \, dV = 0$$

Because this is true for any volume V_0,

$$\sigma_{ij,i} + F_j = 0$$ at every point inside the continuum body.

In Summary, here are all the fundamental Equations of elasticity relating the following variables

displacement	strain	stress	traction force
u_i —	ε_{ij} —	σ_{ij} —	T_j

definition of strain, stress: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$T_j = \sigma_{ij} n_i$

compatibility: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$

equilibrium: $\sigma_{ij,i} + F_j = 0$

elastic constitutive law:
– Generalized Hooke's Law for isotropic medium

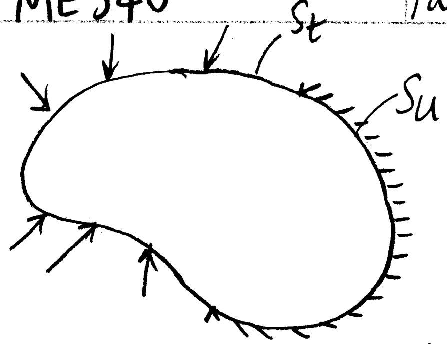
$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$

or $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$

All elasticity problems in this class satisfy the same equations as above.

Different applications correspond to different boundary conditions, which will lead to different solutions.

Boundary Value Problem (B.V.P.): Find the u_i, ε_{ij}, σ_{ij} fields that satisfy compatibility, equilibrium, and constitutive equations and the specified boundary conditions.



In general, two types of boundary conditions can be applied on the surface.

$\left\{ \begin{array}{l} \text{traction boundary condition: } T_j(\underset{\sim}{x}) = g_j(\underset{\sim}{x}) \text{ on } S_t \\ \quad \text{i.e. } \sigma_{ij}(\underset{\sim}{x}) \cdot n_i(\underset{\sim}{x}) = g_j(x) \text{ on } S_t \\ \text{or} \\ \text{displacement boundary condition } u_i(\underset{\sim}{x}) = h_i(x) \text{ on } S_u. \end{array} \right.$

(or some combination of the two)

In the remaining of this class, we will

1. practice how to <u>formulate</u> the B.V.P.
2. learn the tools to <u>solve</u> the B.V.P.

General strategies to solve B.V.P.

1. Avoid compatibility eq. by working with u_i directly
(<u>displacement formulation</u>)

$\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$

$\downarrow$ plug into equilibrium cond.

$\mu u_{i,kk} + (\lambda+\mu) u_{k,ki} + F_i = 0$

$\mu \nabla^2 \underset{\sim}{u} + (\lambda+\mu) \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{F} = 0$

Boundary Condition:

S_u: is easy

S_t: is a little more complicated.

mostly for <u>3D</u> problems

2. <u>Stress formulation</u>

need to work with compatibility condition.

rewrite compatibility cond. in terms of stress

\+ equilibrium cond. $\sigma_{ij,i} + F_j = 0$

$\Rightarrow$ Beltrami-Michel compatibility eq.

$$\sigma_{ij,kk} + \frac{1}{1+\nu} \sigma_{kk,ij} = -\frac{\nu}{1-\nu} \delta_{ij} F_{k,k} - F_{i,j} - F_{j,i}$$

mostly for <u>2D</u> problems