Both stress and strain are symmetric, rank-2 tensors. that are defined through

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \varepsilon_{ij} = \varepsilon_{ji}$$

$$T_j = \sigma_{ij}\, n_i \qquad \sigma_{ij} = \sigma_{ji}$$

and they transform as follows

$$\varepsilon'_{ij} = Q_{ip}\, Q_{jq}\, \varepsilon_{pq}$$

$$\sigma'_{ij} = Q_{ip}\, Q_{jq}\, \sigma_{pq}$$

<u>spherical strain</u>

$$\tilde{\varepsilon}_{ij} \equiv \frac{1}{3}\varepsilon_{kk}\,\delta_{ij}$$

<u>deviatoric strain</u>

$$\hat{\varepsilon}_{ij} \equiv \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\,\delta_{ij}$$

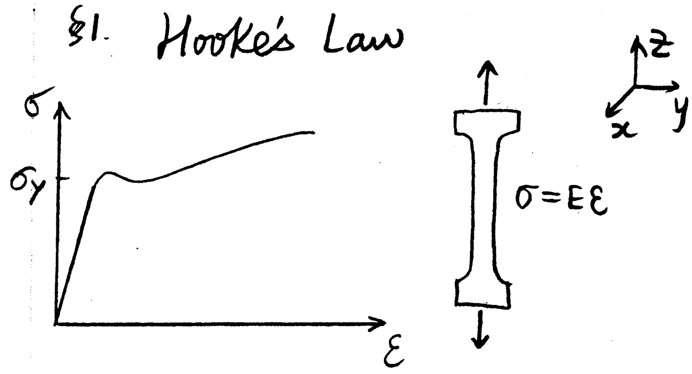
<u>spherical stress</u>

$$\tilde{\sigma}_{ij} \equiv \frac{1}{3}\sigma_{kk}\,\delta_{ij}$$

<u>deviatoric strain</u>

$$\hat{\sigma}_{ij} \equiv \sigma_{ij} - \frac{1}{3}\sigma_{kk}\,\delta_{ij}$$

§1. Hooke's Law



— Assuming <u>isotropic material</u>

<u>tensile test</u>

$$\sigma_{zz} = E\,\varepsilon_{zz} \qquad E = \text{Young's modulus}$$

$$\sigma_{xx} = \sigma_{yy} = 0$$

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$$

i.e. σ_{zz} is the only non-zero stress component ($\sigma_{zz} > 0 \Leftrightarrow$ tension)

Q: Is ε_{zz} the only non-zero strain component?

$$\varepsilon_{xx} = \varepsilon_{yy} = -\nu\,\varepsilon_{zz} \qquad (\varepsilon_{xx} < 0 \Leftrightarrow \text{contraction})$$

Poisson effect. ν — Poisson's ratio ~0.2–0.3 for most mater.

$$\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

If we put an isotropic material under pure shear,

i.e. $\varepsilon_{xy} \neq 0$, $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0$.

then $\sigma_{xy} = 2\mu\, \varepsilon_{xy}$ $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = \sigma_{31} = 0$

(we will see why the factor of "2" later)

μ: shear modulus.

But an isotropic material only has <u>2 independent</u> elastic constants. (we will see why later)

$$\boxed{E = 2\mu\,(1+\nu)}$$

— We need to remember this one.

§2. Generalized Hooke's Law.

$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$ — k, l dummy indices

C_{ijkl} — elastic stiffness tensor. (rank-4) usually called elastic constants

As a tensor, C_{ijkl} transform as follows

$$\boxed{C'_{ijkl} = \qquad\qquad\qquad C_{pqrs}}$$

← fill in the blanks

Because $\sigma_{ij} = \sigma_{ji}$, $\varepsilon_{ij} = \varepsilon_{ji}$, C_{ijkl} has symmetries as well

$C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$ ← minor symmetries

We also have

$C_{ijkl} = C_{klij}$ ← major symmetry

(We will see later that it is a consequence of)

σ_{ij} or ε_{ij} 9 components

$(\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}, \cdots)$

$\downarrow$ due to symmetry $\sigma_{ij} = \sigma_{ji}$

6 <u>independent components</u>

$(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12})$

C_{ijkl} 81 components

$(C_{1111}, C_{1112}, C_{1113}, C_{1121}, \cdots)$

$\downarrow$ due to major and minor symmetries

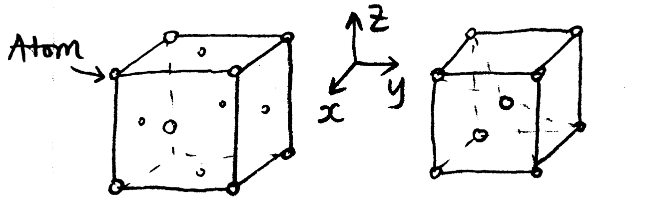
21 <u>independent components</u>

(can you enumerate them?)

for a general anisotropic material.

But real materials usually has more symmetries, which further reduces the number of independent elastic constants.

Most engineering materials are made of crystals having a cubic symmetry.



face-centered cubic crystal structure (FCC) e.g. Al, Cu, Au

body-centered cubic crystal structure (BCC) e.g. Fe, W, Nb

The cubic symmetry reduces the number of independent elastic constants to <u>3</u>

$$\boxed{\begin{aligned} C_{1111} &= C_{2222} = C_{3333} \equiv C_{11} \\ C_{1122} &= C_{2233} = C_{3311} \equiv C_{12} \\ C_{1212} &= C_{1313} = C_{2323} \equiv C_{44} \end{aligned}}$$

other components are obtained either by symmetry or are zero.

* This is the case only when the x, y, z axes of the coordinate system are aligned with the cubic axes of the crystal.
Otherwise we get C'_{ijkl} which follows the tensor's transformation rule and is in general non-zero. (Homework)

---

cubic material

Generalized Hooke's Law

$$\boxed{\begin{aligned}
\sigma_{11} &= C_{11}\,\varepsilon_{11} + C_{12}\,\varepsilon_{22} + C_{12}\,\varepsilon_{33}\\
\sigma_{22} &= C_{12}\,\varepsilon_{11} + C_{11}\,\varepsilon_{22} + C_{12}\,\varepsilon_{33}\\
\sigma_{33} &= C_{12}\,\varepsilon_{11} + C_{12}\,\varepsilon_{22} + C_{11}\,\varepsilon_{33}\\
\sigma_{12} &= 2\,C_{44}\,\varepsilon_{12}\\
\sigma_{23} &= 2\,C_{44}\,\varepsilon_{23}\\
\sigma_{31} &= 2\,C_{44}\,\varepsilon_{31}
\end{aligned}}$$

Q: what is the physical meaning of C_{11}, C_{12}, C_{44} ?

Isotropic material

can be regarded as a special case of a cubic material which satisfies $\boxed{C_{11} = C_{12} + 2\,C_{44}}$

* We can define anisotropic factor $A \equiv \dfrac{2\,C_{44}}{C_{11} - C_{12}}$

$$A = 1 \iff \text{isotropic material}$$

Many engineering materials, e.g. metals, are isotropic because they are polycrystals, i.e. made of many small randomly oriented grains, even though each crystal grain is elastically anisotropic

For isotropic materials, the Lamé constants (λ, μ) are usually used

$$\boxed{\begin{aligned} C_{12} &= \lambda \\ C_{44} &= \mu \\ C_{11} &= \lambda + 2\mu \end{aligned}}$$

In index notation

$$\boxed{C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

← can be derived by plugging the above expr. to $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$. a good exercise for index notation.

(* notice the factor of "2")

The inverse of expression $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ is $\varepsilon_{ij} = S_{ijkl} \sigma_{kl}$, S_{ijkl} is the elastic compliance tensor.

For isotropic material

$S_{ijkl} =$ ________________ (Homework)

$\varepsilon_{ij} =$ ________________

$$\boxed{\begin{aligned} \varepsilon_{11} &= \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} \\ \varepsilon_{22} &= -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} \\ \varepsilon_{33} &= -\frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{33} \\ \varepsilon_{12} &= \frac{1}{2\mu}\sigma_{12} \\ \varepsilon_{23} &= \frac{1}{2\mu}\sigma_{23} \\ \varepsilon_{31} &= \frac{1}{2\mu}\sigma_{31} \end{aligned}}$$

(isotropic material)

§3. Thermo-elasticity

The Generalized Hooke's Law describes the relationship between <u>stress</u> and <u>elastic strain</u>, more precisely,

$$\sigma_{ij} = C_{ijkl}\, \varepsilon^{el}_{kl}$$

(* don't ever say "elastic stress")

Temperature change can also cause shape change (thermal expansion)

Thermal strain is the strain under zero stress

$$\varepsilon^{T}_{ij} = \alpha_{ij}(T - T_0)$$

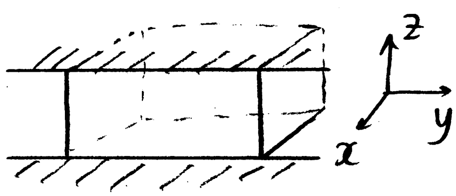
T_0 : reference temperature
α_{ij} : linear thermal expansion coefficient

isotropic material $\alpha_{ij} = \alpha\,\delta_{ij}$

$$\varepsilon^{T}_{ij} = \alpha(T-T_0)\cdot\delta_{ij}$$

$$\varepsilon^{tot}_{ij} = \varepsilon^{el}_{ij} + \varepsilon^{T}_{ij}$$

Example 1: Stress can be developed if material is not allowed to expand freely.



A material constrained between two rigid plates

rigid plates $\Longleftrightarrow$ $\varepsilon^{tot}_{zz} = 0$

free expansion in x, y $\Leftrightarrow$ $\sigma_{xx} = \sigma_{yy} = 0$

our task is to find the remaining stress. strain components. e.g. σ_{zz}, ε_{xx}, ε_{yy}

$$0 = \varepsilon^{tot}_{zz} = \varepsilon^{el}_{zz} + \varepsilon^{T}_{zz} \qquad \varepsilon^{el}_{zz} = -\varepsilon^{T}_{zz} = -\alpha(T-T_0)$$

$$\varepsilon^{el}_{zz} = \frac{1}{E}(-\nu\cancel{\sigma_{xx}} - \nu\cancel{\sigma_{yy}} + \sigma_{zz}) \qquad \sigma_{zz} = E\,\varepsilon^{el}_{zz} = -\alpha E(T-T_0)$$

$$\varepsilon^{el}_{xx} = \frac{1}{E}(\cancel{\sigma_{xx}} - \nu\cancel{\sigma_{yy}} - \nu\sigma_{zz}) = -\frac{\nu}{E}\sigma_{zz} = \alpha\nu(T-T_0) = \varepsilon^{el}_{yy}$$