Given a continuum medium subjected to external loading, we want to find:

1. the displacement field \( u_i (x) \) \( \quad \text{vector} \)
2. the strain field \( \varepsilon_{ij} (x) \) \( \quad \text{tensor} \) \( \text{rank 2} \)
3. the stress field \( \sigma_{ij} (x) \) \( \quad \text{tensor} \) \( \text{rank 2} \)

31. What is a vector?

We can represent a vector by:
- an arrow (in a figure)
- a symbol \( \sim \)
- its components (coordinates)

\[ u = u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3 = \sum_{i=1}^{3} u_i \hat{e}_i \]

index notation: repeated index is summed over from 1 to 3 (dummy index)

\[ u = u_i \hat{e}_i \]

Notice here we specify a vector \( u \) by linear combination of three (unit) vectors \( \hat{e}_i \):
\( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) form a coordinate system.

If we choose a different set of (unit) vectors \( \hat{e}_1', \hat{e}_2', \hat{e}_3' \) as our coordinate system, then the same vector \( u \) will have different coordinates \( (u_1', u_2', u_3') \):

\[ u = u_i \hat{e}_i = u_i \hat{e}_i' \]
define: \( Q_{ij} = (e_i' \cdot e_j') \) (dot product)

* \( Q_{ij} \) forms an orthogonal matrix

\[ Q_{ij} Q_{kj} = \delta_{ik} \]

notice \( j \) is a dummy variable. \( \delta_{ik} = 1 \) if \( i = k \), \( 0 \) if \( i \neq k \)

In Matrix notation

\[ Q \cdot Q^T = I \]

\( \uparrow \) transpose \( \downarrow \) identity matrix

notice

\[ (e_i' \cdot e_j') = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \]

\[ (e_i' \cdot e_j') = \delta_{ij} \]

\[ U_i e_i' = U_i e_i \] multiply both sides by \( e_j' \) (dot product)

\[ U'_j = U_i (e_j' \cdot e_i) \]

\[ U'_j = Q_{ji} U_i \]

\( \uparrow \)

\[ U_1 = Q_{11} U_1 + Q_{12} U_2 + Q_{13} U_3 \]
\[ U_2 = Q_{21} U_1 + Q_{22} U_2 + Q_{23} U_3 \]
\[ U_3 = Q_{31} U_1 + Q_{32} U_2 + Q_{33} U_3 \]

\( \uparrow \)

\[ U'_i = Q_{ij} U_j \] (\( j \) is a free index, \( i \) is a dummy index)

The same equations can be expressed using different choices of indices.
§2. displacement (vector) field \( \mathbf{u}(x) \)

A material point in the undeformed state is specified by a vector
\[ \mathbf{x} = x_i \mathbf{e}_i \]

The same material point in the deformed state is specified by another vector
\[ \mathbf{x}' = x'_i \mathbf{e}_i \]

**Displacement vector** \( \mathbf{u} \equiv \mathbf{x}' - \mathbf{x} \)

\( \mathbf{u}(\mathbf{x}) \) is a vector defined for every material point \( \mathbf{x} \), and is called a **vector field**, i.e. a vector as a function of another vector.

**Infinitesimal elasticity**, we assume \(|\mathbf{u}| \ll 1\), so that we do not distinguish \( \mathbf{u}(\mathbf{x}) \) v.s. \( \mathbf{u}(\mathbf{x}') \)

This is a great simplification. No longer valid for large deformation, which is treated in continuum mechanics.
§3. **Strain field** $\varepsilon_{ij}(x)$

A displacement field $u_i(x)$ does not necessarily lead to "deformation."

![Diagram showing deformation](image)

**Rigid-body translation**

$u_i(x) = u_i^0$, constant vector

**Rigid-body rotation**

$u_y = w \cdot x$

$u_x = -w \cdot y$

(to first order of $x, y$)

Taylor expand displacement field $u_i(x)$ up to 1st order.

$u_i(x) = u_i^0 + \frac{\partial u_i}{\partial x_j} \, dx_j$

$= u_i^0 + u_{i,j} \, dx_j$

$= u_i^0 + \frac{1}{2} (u_{i,j} + u_{j,i}) \, dx_j + \frac{1}{2} (u_{i,j} - u_{j,i}) \, dx_j$

$u_i = u_i^0 + \varepsilon_{ij} \, dx_j + \omega_{ij} \, dx_j$

$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$ Strain, $\varepsilon_{ij} = \varepsilon_{ji}$ symmetric

$\omega_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i})$ Rotation, $\omega_{ij} = -\omega_{ji}$ anti-symmetric

![Diagrams showing strain and rotation](image)
\[ E_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

This is the strain component when we express vectors \( \mathbf{u} \) and \( \mathbf{x} \) in the coordinate system of \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \).

Q: What happens if we choose a different coordinate system?

\( u = u' e' \)
\( x = x' e' \)

\[ E'_{ij} = \frac{1}{2} \left( \frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right) \]

Q: How does \( E_{ij} \) transform with a change of coordinate system?

We can show that

\[ E'_{ij} = Q_{ip} Q_{jq} E_{pq} \]

\( p, q \) are dummy indices
\( i, j \) are free indices.

* Recall \( u_i = Q_{ij} u_j \). The "rule" is that: the first index of \( Q \) is the free index

* The relation in the box can be shown through the "chain rule"

Notice that \( x'_j = Q_{jk} x_k \) — because \( \mathbf{x} \) is a vector

\( (\text{be careful!}) \rightarrow x'_q = Q_{qr} x'_r \) — because \( Q \cdot Q^T = I \)

\( Q \)'s inverse is its transpose

We have \( \frac{\partial x_p}{\partial x_j} = Q_{pj} \)

\[ \frac{\partial}{\partial x_j} f = \left( \frac{\partial}{\partial x_q} f \right) \cdot \left( \frac{\partial x_q}{\partial x_j} \right) = Q_{jr} \frac{\partial}{\partial x_q} f \]

\[ f_{,ij} = Q_{ij} f_{,q} \quad \text{— derivatives transform as vector} \]
$E_{ij}$ is a matrix of 9 numbers that transform according to $E_{ij} = Q_{ip} Q_{jq} E_{pq}$ — which is the definition of a rank-2 tensor.

* An arbitrary ranked tensor $A_{ijkl\ldots n}$ is a set of numbers that transform as

$$A_{ijkl\ldots n} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} Q_{mt} \ldots Q_{nu} A_{pqrst\ldots u}$$

Always: the first index of $Q$ is free, the second index of $Q$ is dummy.

$Q_{ip} \equiv (e_i \cdot e_p)$

* The differential operator is a vector

$$\nabla = \frac{\partial}{\partial x_i}(\cdot) \, e_i \quad (e.g. \, \nabla f = \left(\frac{\partial f}{\partial x_i}\right) e_i)$$

* A tensor can also be expressed in vector notation

$$\hat{E} = E_{ij} (e_i \otimes e_j) \quad e_i \otimes e_j \text{ is a tensor product —also called dyadic notation}$$

$$= \frac{1}{2} [(\nabla u) + (\nabla u)^T]$$

Reading assignment:

Sadd Section 1.1 - 1.8

* See Section 1.9 for complications that can arise in curvilinear (e.g. cylindrical or polar) coordinate systems.
\( \sigma_{ij} \): force per unit area on \( i \)-th face in \( j \)-th direction (note correction from previous version)

* This is the Cauchy stress in continuum mechanics.

Given the stress field, we can obtain the traction force \( T_j \) per unit area on any surface element with normal vector \( \mathbf{n} \).

\[ T_j = \sigma_{ij} n_i \]

We can show (but not here) that the stress is also a rank-2 (symmetric) tensor, i.e. it transforms as

\[ \bar{\sigma}_{ij} = Q_{ip} Q_{jq} \sigma_{pq} \]

\( \sigma_{ij} = \sigma_{ji}, \quad \sigma_{ij} = \sigma_{ji} \)

* Both \( I \) and \( \mathbf{n} \) are vectors. i.e.

\[ \mathbf{n}' = Q_{ip} \mathbf{n}_p \]

\[ T_j' = Q_{ij} T_i \]

In fact, we can use these two relations to prove the relation in the box.
Example 1: Rotation around 2-axis

\[ \mathbf{Q} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \sigma_{1}' = \cos^2 \theta \sigma_{11} + \sin^2 \theta \sigma_{22} + 2 \sin \theta \cos \theta \sigma_{12} \]
\[ \sigma_{22}' = \sin^2 \theta \sigma_{11} + \cos^2 \theta \sigma_{22} - 2 \sin \theta \cos \theta \sigma_{12} \]
\[ \sigma_{12}' = -\sin \theta \cos \theta \sigma_{11} + \sin \theta \cos \theta \sigma_{22} + (\cos^2 \theta - \sin^2 \theta) \sigma_{12} \]

or equivalently

\[ \sigma_{11}' = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \sigma_{12} \sin 2\theta \]
\[ \sigma_{22}' = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta - \sigma_{12} \sin 2\theta \]
\[ \sigma_{12}' = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \sigma_{12} \cos 2\theta \]

\[ \sigma_{\text{avg}} = \frac{\sigma_{11} + \sigma_{22}}{2} \]
\[ \sigma_{\text{max}} = \sigma_{\text{avg}} + R \]
\[ \sigma_{\text{min}} = \sigma_{\text{avg}} - R \]

Stress points rotate twice as fast in Mohr's circle as in real space (2\(\theta\) v.s. \(\theta\))

* Strain tensor satisfies a similar transformation rule. Hence there exist a Mohr's circle for strain.

* These relations are useful in polar coordinates, which are convenient for these problems.