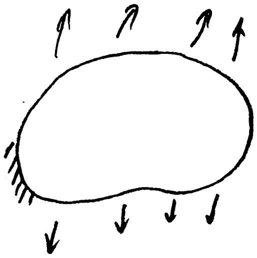

"continuum potato"

Given a continuum medium subjected to external loading, we want to find:

1. the displacement field $u_i(\underset{\sim}{x})$ — <u>vector</u>
2. the strain field $\varepsilon_{ij}(\underset{\sim}{x})$
3. the stress field $\sigma_{ij}(\underset{\sim}{x})$

$\Big\}$ <u>tensor</u> (rank 2)

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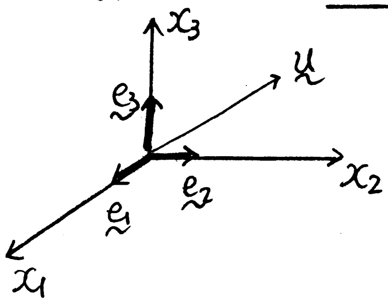
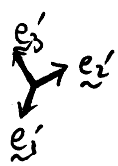
§1. What is a <u>vector</u>?



We can represent a vector by
- an arrow (in a figure)
- a symbol $\sim$
- its components (coordinates)

$$\underset{\sim}{u} = u_1 \underset{\sim}{e}_1 + u_2 \underset{\sim}{e}_2 + u_3 \underset{\sim}{e}_3 = \sum_{i=1}^{3} u_i \underset{\sim}{e}_i$$



<u>index notation</u>: repeated index is summed over from 1 to 3 (dummy index)

$$\underset{\sim}{u} = u_i \underset{\sim}{e}_i$$

Notice here we specify a vector $\underset{\sim}{u}$ by linear combination of three (unit) vectors $\underset{\sim}{e}_i$

$\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ form a <u>coordinate system</u>.

If we choose a different set of (unit) vectors $(\underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3')$ as our coordinate system, then the same vector $\underset{\sim}{u}$ will have different coordinates (u_1', u_2', u_3')

$$\underset{\sim}{u} = u_i \underset{\sim}{e}_i = u_i' \underset{\sim}{e}_i'$$

define $Q_{ij} = (\underset{\sim}{e}_i' \cdot e_j)$ (dot product)

* Q_{ij} forms an orthogonal matrix

$Q_{ij} Q_{Kj} = \delta_{iK}$

notice j is a dummy variable, $\delta_{iK} = \begin{cases} 0 & i \neq K \\ 1 & i = K \end{cases}$ is called <u>Kronecker delta</u>.

In Matrix notation

$Q \cdot Q^T = I$

transpose Identity matrix

notice $(\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$(\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j') = \delta_{ij}$

$u_i' \underset{\sim}{e}_i' = u_i \underset{\sim}{e}_i$ multiply both sides by $\underset{\sim}{e}_j'$ (dot product)

$u_j' = u_i (\underset{\sim}{e}_j' \cdot e_i)$

$\boxed{u_j' = Q_{ji} u_i}$ (i is dummy variable, summed over 1, 2, 3; j is a free index, can be 1, 2, 3)

$\Updownarrow$

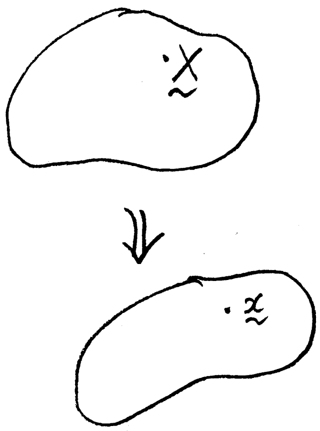
$$\boxed{\begin{aligned} u_1' &= Q_{11} u_1 + Q_{12} u_2 + Q_{13} u_3 \\ u_2' &= Q_{21} u_1 + Q_{22} u_2 + Q_{23} u_3 \\ u_3' &= Q_{31} u_1 + Q_{32} u_2 + Q_{33} u_3 \end{aligned}}$$

$\Updownarrow$

$\boxed{u_i' = Q_{ij} u_j}$ (i is a free index, j is a dummy index)

The same equation(s) can be expressed using different choices of indices.

§2. <u>displacement (vector) field</u> $u_i(\underset{\sim}{x})$



a material point in the undeformed state is specified by a vector

$$\underset{\sim}{X} = X_i \, \underset{\sim}{e_i}$$

The same material point in the deformed state is specified by another vector

$$\underset{\sim}{x} = x_i \, \underset{\sim}{e_i}$$

<u>displacement vector</u> $\underset{\sim}{u} \equiv \underset{\sim}{x} - \underset{\sim}{X}$

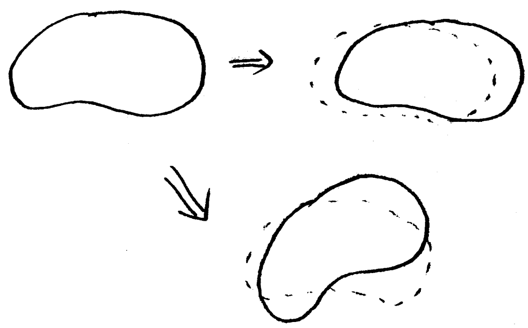
$\underset{\sim}{u}(\underset{\sim}{x})$ is a vector defined for every material point $\underset{\sim}{x}$.
and is called a <u>vector field</u>.
i.e. a vector as a function of another vector.

<u>Infinitesimal elasticity</u>: we assume $|\underset{\sim}{u}| \ll 1$, so that
we do not distinguish $\underset{\sim}{u}(\underset{\sim}{x})$ v.s. $\underset{\sim}{u}(\underset{\sim}{X})$

This is a great simplification.
No longer valid for large deformation, which is treated in continuum mechanics.

§3. <u>strain field</u> $\varepsilon_{ij}(\underset{\sim}{x})$

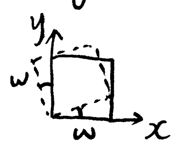
a displacement field $u_i(\underset{\sim}{x})$ does not necessarily lead to "deformation."



rigid-body translation

$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{u}_0$ constant vector

rigid-body rotation



$u_y = +w \cdot x$

$u_x = -w \cdot y$

(to first order of x, y)

Taylor expand displacement field $u_i(\underset{\sim}{x})$ up to 1st order.

index notation

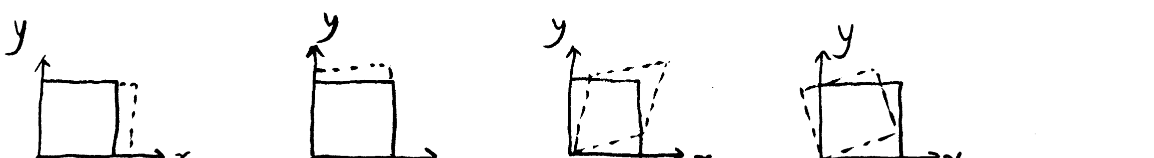
$$u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}$$

$$u_i(\underset{\sim}{x}) = u_i^0 + \frac{\partial u_i}{\partial x_j} dx_j$$

$$= u_i^0 + u_{i,j}\, dx_j$$

$$= u_i^0 + \tfrac{1}{2}(u_{i,j} + u_{j,i})\, dx_j + \tfrac{1}{2}(u_{i,j} - u_{j,i})\, dx_j$$

$$\boxed{\begin{aligned} u_i &= u_i^0 + \varepsilon_{ij}\, dx_j + \omega_{ij}\, dx_j \\ \varepsilon_{ij} &\equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}) && \text{strain: } \varepsilon_{ij} = \varepsilon_{ji} \text{ symmetric} \\ \omega_{ij} &\equiv \tfrac{1}{2}(u_{i,j} - u_{j,i}) && \text{rotation: } \omega_{ij} = -\omega_{ji} \text{ anti-symmetric} \end{aligned}}$$



ε_{11} strain | ε_{22} strain | ε_{12} strain | ω_{12} rotation

$$\varepsilon_{ij} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

This is the strain component when we express vectors $\underset{\sim}{u}$ and $\underset{\sim}{x}$ in the coordinate system of $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$

Q: What happens if we choose a different coordinate system?

$$\underset{\sim}{u} = u_i' \underset{\sim}{e}_i'$$
$$\underset{\sim}{x} = x_i' \underset{\sim}{e}_i'$$
$$\varepsilon_{ij}' \equiv \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'}\right)$$

Q: how does ε_{ij} transform with a change of coordinate system?

We can show that

$$\boxed{\varepsilon_{ij}' = Q_{ip} Q_{jq} \varepsilon_{pq}}$$

p, q are dummy indices
i, j are free indices.

* recall $u_i' = Q_{ij} u_j$

The "rule" is that: the first index of Q is the free index
the second index of Q is the dummy.

* The relation in the box can be shown through the "chain rule"

notice that $x_j' = Q_{jq} x_q$ — because $\underset{\sim}{x}$ is a vector

(be careful!) $\rightarrow$ $x_q = Q_{jq} x_j'$ — because $Q \cdot Q^T = I$
Q's inverse is its transpose

we have $\dfrac{\partial x_q}{\partial x_j'} = Q_{jq}$

$$\frac{\partial}{\partial x_j'} f = \left(\frac{\partial}{\partial x_q} f\right) \cdot \left(\frac{\partial x_q}{\partial x_j'}\right) = Q_{jq} \frac{\partial}{\partial x_q} f$$

$f_{,j'} = Q_{jq} f_{,q}$ — spatial derivatives transform as vector

ε_{ij} is a matrix of 9 numbers that transform according to $\varepsilon'_{ij} = Q_{ip} Q_{jq} \varepsilon_{pq}$ — which is the definition of a rank-2 tensor.

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* An arbitrary ranked tensor $A_{ijklm\cdots n}$ is a set of numbers that transform as

$$A'_{ijklm\cdots n} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} Q_{mt} \cdots Q_{nu} A_{pqrst\cdots u}$$

Always: the first index of Q is free
the second index of Q is dummy
$Q_{ip} \equiv (\underset{\sim}{e}'_i \cdot \underset{\sim}{e}_p)$

---

* The differential operator is a vector

$$\underset{\sim}{\nabla} = \frac{\partial}{\partial x_i}(\cdot)\ \underset{\sim}{e}_i \qquad \left(\text{e.g. } \underset{\sim}{\nabla} f = \left(\frac{\partial f}{\partial x_i}\right)\underset{\sim}{e}_i\right)$$

* A tensor can also be expressed in vector notation

$$\underset{\approx}{\varepsilon} = \varepsilon_{ij}\,(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) = \frac{1}{2}\left[(\underset{\sim}{\nabla}\,\underset{\sim}{u}) + (\underset{\sim}{\nabla}\,\underset{\sim}{u})^T\right]$$

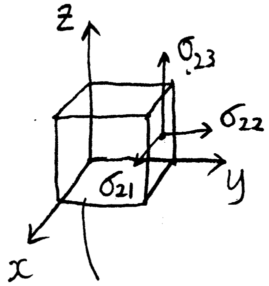
$\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ is a tensor product
— also called dyadic notation

Reading assignment:

Sadd Section 1.1 – 1.8

* see section 1.9 for complications that can arise in curvilinear (e.g. cylindrical or polar) coordinate systems.

§4. stress field.

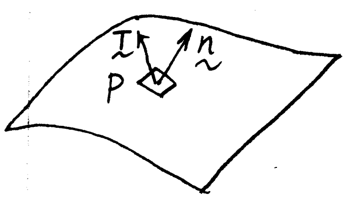


"the stress cube"

σ_{ij} : force per unit area on i-th face in j-th direction (* note correction from previous version!)

* This is the Cauchy stress in continuum mechanics.

Given the stress field, we can obtain the traction force $\underset{\sim}{T}$ per unit area on any surface element with normal vector $\underset{\sim}{n}$:



$$\boxed{T_j = \sigma_{ij} \, n_i}$$

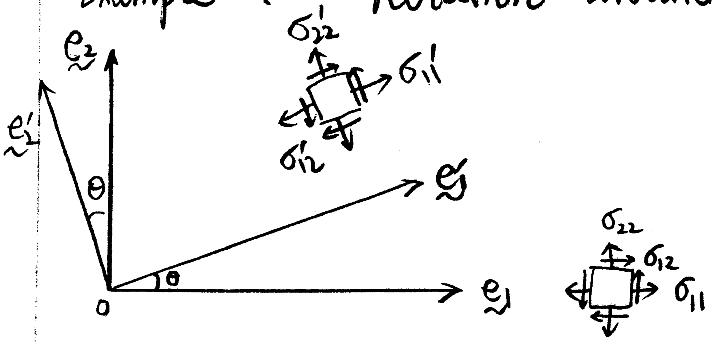
We can show (but not here) that the stress is also a rank-2 (symmetric) tensor, i.e. it transforms as

$$\boxed{\sigma'_{ij} = Q_{ip} \, Q_{jq} \, \sigma_{pq}} \qquad \sigma_{ij} = \sigma_{ji}, \quad \sigma'_{ij} = \sigma'_{ji}$$

* both $\underset{\sim}{T}$ and $\underset{\sim}{n}$ are vectors. i.e. $n'_i = Q_{ip} \, n_p$
$T'_j = Q_{jq} \, T_q$

In fact, we can use these two relations to prove the relation in the box.

Example 1: Rotation around z-axis



$Q_{11} = e_1' \cdot e_1 = \cos\theta \quad Q_{12} = e_1' \cdot e_2 = \sin\theta$

$Q_{21} = e_2' \cdot e_1 = -\sin\theta \quad Q_{22} = e_2' \cdot e_2 = \cos\theta$

$Q_{13} = Q_{23} = Q_{31} = Q_{32} = 0$

$Q_{33} = 1$

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

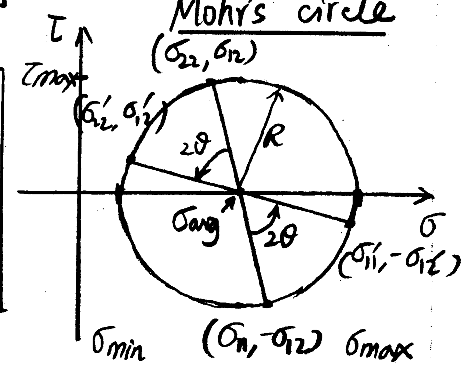
$$\boxed{\begin{aligned} \sigma_{11}' &= \cos^2\theta\, \sigma_{11} + \sin^2\theta\, \sigma_{22} + 2\sin\theta\cos\theta\, \sigma_{12} \\ \sigma_{22}' &= \sin^2\theta\, \sigma_{11} + \cos^2\theta\, \sigma_{22} - 2\sin\theta\cos\theta\, \sigma_{12} \\ \sigma_{12}' &= -\sin\theta\cos\theta\, \sigma_{11} + \sin\theta\cos\theta\, \sigma_{22} + (\cos^2\theta - \sin^2\theta)\, \sigma_{12} \end{aligned}}$$

$\Leftarrow \sigma_{ij}' = Q_{ip} Q_{jq} \sigma_{pq}$

or equivalently

$$\boxed{\begin{aligned} \sigma_{11}' &= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta \\ \sigma_{22}' &= \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta - \sigma_{12}\sin 2\theta \\ \sigma_{12}' &= -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \end{aligned}}$$

Mohr's circle



principle stress

$$\sigma_{avg} = \frac{\sigma_{11}+\sigma_{22}}{2}$$

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

$$\sigma_{max} = \sigma_{avg} + R$$

$$\sigma_{min} = \sigma_{avg} - R$$

Stress points rotate twice as fast in Mohr's circle as in real space (2θ v.s. θ)

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* Strain tensor satisfies a similar transformation rule. Hence there exist a Mohr's circle for strain.

* These relations are useful in polar coordinates, which are convenient for these problems