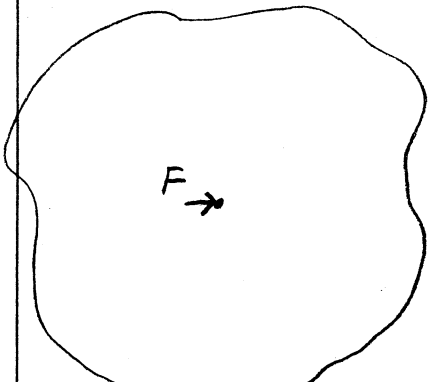
§1. Kelvin's Problem & Green function

(* Story of poor Green and Lord Kelvin on coursework. "The Green of Green Functions")



Point force F acting on the interior of an infinite elastic medium.

Suppose the force F has unit magnitude, acts on the origin, and points to the j-direction, then the body force distribution can be expressed as

$$F_i(\underset{\sim}{x}) = \delta(\underset{\sim}{x})\,\delta_{ij}$$

Recall equilibrium condition: $\sigma_{ki,k} + F_i = 0$

$\therefore\ \sigma_{ki,k} = -\delta(x)\,\delta_{ij}$

Let $u_i(\underset{\sim}{x})$ be the resulting displacement in response the point force (along j-direction)

Obviously, $u_i(\underset{\sim}{x})$ depends on j

Define $G_{ij}(\underset{\sim}{x}) \equiv u_i(x)$ when point force is along j-dir.

displacement direction (i) — point force direction (j).

> $G_{ij}(\underset{\sim}{x})$ is the displacement at point $\underset{\sim}{x}$ in the i-direction in response to a unit point force at origin in the j-direction.
> $G_{ij}(\underset{\sim}{x})$ is called the <u>Green function</u> of the infinite medium

Given $G_{ij}(\underset{\sim}{x})$, the displacement field $u_i(\underset{\sim}{x})$ in response to an arbitrary body force distribution $F_j(\underset{\sim}{x})$ is

$$u_i(\underset{\sim}{x}) = \iiint F_j(\underset{\sim}{x}')\, G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}')\, d^3\underset{\sim}{x}'$$

§2. Solving Kelvin's Problem by Galerkin vector

Galerkin Vector Representation

define vector field $\underset{\sim}{V}$ that is related to displacement field $\underset{\sim}{u}$ through:

$$2\mu \underset{\sim}{u} = 2(1-\nu)\nabla^2 \underset{\sim}{V} - \nabla(\nabla\cdot\underset{\sim}{V})$$

$$2\mu\, \nabla\cdot\underset{\sim}{u} = 2(1-\nu)\nabla^2(\nabla\cdot\underset{\sim}{V}) - \nabla^2(\nabla\cdot\underset{\sim}{V}) = (1-2\nu)\nabla^2(\nabla\cdot\underset{\sim}{V})$$

$$\frac{\mu}{1-2\nu}\nabla(\nabla\cdot\underset{\sim}{u}) + \mu\nabla^2\underset{\sim}{u} = (1-\nu)\nabla^4\underset{\sim}{V}$$

Equilibrium condition

$$(1-\nu)\nabla^4\underset{\sim}{V} + \underset{\sim}{F} = 0$$

$$\boxed{\nabla^4\underset{\sim}{V} = -\frac{1}{1-\nu}\underset{\sim}{F}}$$

To be specific, let

$$F_x = 0$$
$$F_y = 0$$
$$F_z = \delta(\underset{\sim}{x})$$

(point force applied in the $j=3$ direction)

Solution:

$$V_x = 0$$
$$V_y = 0$$
$$\nabla^4 V_z = -\frac{1}{1-\nu}\delta(\underset{\sim}{x})$$

Here we need to invoke another "magic" formula from electrostatics

$$\nabla^2 \frac{1}{R} = -4\pi\,\delta(\underset{\sim}{x}) \quad , \quad \text{where } R=\sqrt{x^2+y^2+z^2}$$

* Poisson's equation in electrostatics

$\Phi(\underset{\sim}{x})$ is the potential field produced by charge density field $\rho(\underset{\sim}{x})$

$$\nabla^2 \Phi(\underset{\sim}{x}) = -\frac{\rho(\underset{\sim}{x})}{\epsilon_0}$$

When $\rho(\underset{\sim}{x}) = Q\,\delta(\underset{\sim}{x})$, i.e. a point charge Q at origin.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{R} \qquad R=\sqrt{x^2+y^2+z^2}$$

Therefore,

$$\nabla^2 \frac{1}{R} = -4\pi\,\delta(\underset{\sim}{x})$$

It can also be verified that

$$\nabla^2 R = \frac{2}{R}$$

$$R=\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial R}{\partial x} = \frac{x}{R}$$

$$\frac{\partial^2 R}{\partial x^2} = \frac{1}{R} - \frac{x}{R^2}\cdot\frac{\partial R}{\partial x} = \frac{1}{R} - \frac{x^2}{R^3}$$

$$\nabla^2 R = \frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} = \frac{1}{R} - \frac{x^2}{R^3} + \frac{1}{R} - \frac{y^2}{R^3} + \frac{1}{R} - \frac{z^2}{R^3}$$

$$= \frac{3}{R} - \frac{x^2+y^2+z^2}{R^3} = \frac{2}{R}$$

Hence

$$\nabla^4 R = -8\pi\, \delta(\underset{\sim}{x})$$

$$\therefore\ V_z(\underset{\sim}{x}) = \frac{1}{8\pi(1-\nu)} \cdot R \qquad R = \sqrt{x^2+y^2+z^2}$$

displacement field

$$\underset{\sim}{u} = \frac{1-\nu}{\mu} \nabla^2 \underset{\sim}{V} - \frac{1}{2\mu} \nabla(\nabla \cdot \underset{\sim}{V})$$

$$\nabla \cdot \underset{\sim}{V} = \frac{\partial}{\partial z} V_z = \frac{1}{8\pi(1-\nu)} \frac{\partial R}{\partial z}$$

$$u_x = \frac{1-\nu}{\mu} \nabla^2 V_x - \frac{1}{2\mu} \frac{\partial}{\partial x} \frac{\partial}{\partial z} V_z = -\frac{1}{2\mu} \frac{\partial^2}{\partial x \partial z} V_z$$

$$u_y = -\frac{1}{2\mu} \frac{\partial^2}{\partial y \partial z} V_z$$

$$u_z = \frac{1-\nu}{\mu} \nabla^2 V_z - \frac{1}{2\mu} \frac{\partial^2}{\partial z^2} V_z$$

$$u_i = \left[\frac{1-\nu}{\mu} \nabla^2 R\, \delta_{ij} - \frac{1}{2\mu} \frac{\partial^2}{\partial x_i \partial x_j} R \right] \cdot \frac{1}{8\pi(1-\nu)}$$

in this case, $j=3$. but this result can be generalized to arbitrary j.

$$\boxed{G_{ij}(\underset{\sim}{x}) = \frac{1}{8\pi\mu} \left[\delta_{ij} R_{,kk} - \frac{1}{2(1-\nu)} R_{,ij} \right]}$$

$$G_{ij}(\underset{\sim}{x}) = \frac{1}{16\pi\mu(1-\nu)} \cdot \frac{1}{R} \left[(3-4\nu)\delta_{ij} + \frac{x_i x_j}{R^2} \right]$$