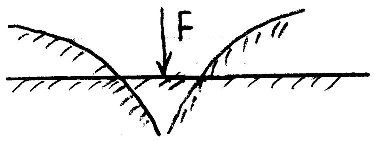
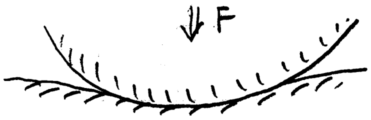
## §1. Compare 2D and 3D problems

elastic half-space subjected to plane-wave loading

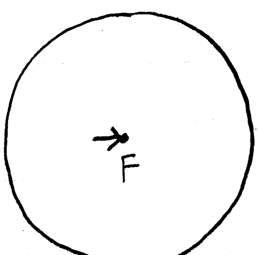
⇓ Fourier transform



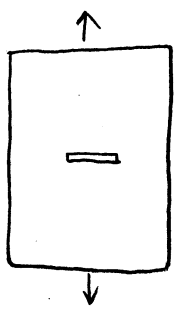

elastic half-space subjected to point force loading



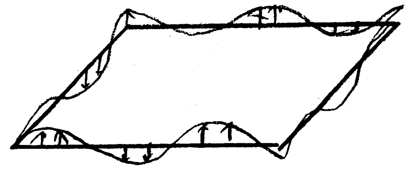

cylindrical contact

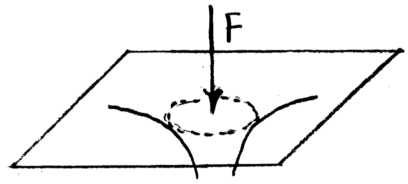


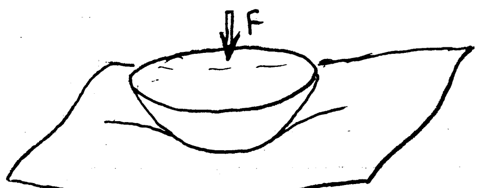

concentrated line force



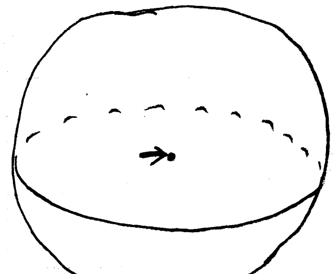
slit-like crack



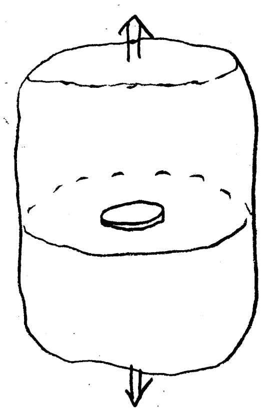



spherical contact



concentrated point force



penny-shaped crack

§2. Elasticity equations in 3D

equilibrium: $\sigma_{ij,i} + F_j = 0$

compatibility: (avoided by write strain in terms of displacements)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Generalized Hooke's Law: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$

$$= \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} \quad \text{(isotropic)}$$

write everything in terms of $u_i$

$$\begin{aligned}
0 &= \sigma_{ij,i} + F_j \\
&= (\lambda\,\varepsilon_{kk}\,\delta_{ij})_{,i} + 2\mu\,\varepsilon_{ij,i} + F_j \\
&= \lambda\,\varepsilon_{kk,j} + 2\mu\,\varepsilon_{ij,i} + F_j \\
&= \lambda\,u_{k,kj} + \mu(u_{i,j} + u_{j,i})_{,i} + F_j \\
&= \lambda\,u_{k,kj} + \mu\,u_{i,ji} + u_{j,ii} + F_j
\end{aligned}$$

$$\boxed{(\lambda+\mu)\,u_{i,ij} + \mu\,u_{j,ii} + F_j = 0}$$

recall $\lambda = \frac{2\mu\nu}{1-2\nu}$ $\qquad \lambda + \mu = \frac{\mu}{1-2\nu}$

$$\boxed{\frac{\mu}{1-2\nu}\,u_{i,ij} + \mu\,u_{j,ii} + F_j = 0}$$

In tensor notation

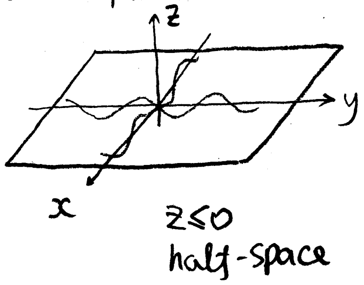
$$\boxed{\frac{\mu}{1-2\nu}\,\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) + \mu\,\nabla^2\underset{\sim}{u} + \underset{\sim}{F} = 0}$$

In the absense of body force ($\underset{\sim}{F} = 0$)

$$\boxed{\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) + (1-2\nu)\,\nabla^2\underset{\sim}{u} = 0}$$

independent of $\mu$ !

## §3. Plane-wave loading - Trial solution




B.C.

$T_x(x,y) = \sigma_{xz}|_{z=0} = 0$

$T_y(x,y) = \sigma_{yz}|_{z=0} = 0$

$T_z(x,y) = \sigma_{zz}|_{z=0} = S\cdot e^{i(k_x x + k_y y)}$

Trial solution:

$u_x = (A_1 + B_1 z)\, e^{i(k_x x + k_y y)}\, e^{k_z z}$

$u_y = (A_2 + B_2 z)\, e^{i(k_x x + k_y y)}\, e^{k_z z}$

$u_z = (A_3 + B_3 z)\, e^{i(k_x x + k_y y)}\, e^{k_z z}$

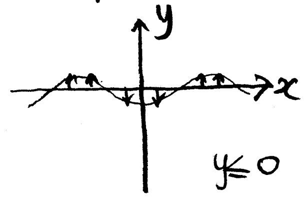
$$k_z = \sqrt{k_x^2 + k_y^2}$$

six unknowns : $A_1\ B_1\ A_2\ B_2\ A_3\ B_3$

3 equations from equilibrium cond.

3 equations from B.C.

compare with 2D problem




B.C. $T_x(x) = \sigma_{xy}|_{y=0} = 0$

$T_y(x) = \sigma_{yy}|_{y=0} = -A\cos kx$

Trial solution:

$\phi = \cos kx\,(A + By)\, e^{ky}$

Solution: $B = -Ak$

$\sigma_{xx} = -Ak^2 \cos kx\,(1+ky)\, e^{ky}$

$\sigma_{yy} = -Ak^2 \cos kx\,(1-ky)\, e^{ky}$

$\sigma_{xy} = -Ak^3 \sin kx\; y\, e^{ky}$

$u_x = -\frac{Ak}{E}\sin kx\left[(1-\nu-2\nu^2) + (1+\nu)ky\right]e^{ky}$

$u_y = -\frac{Ak}{E}\cos kx\left[(2-2\nu^2) - (1+\nu)ky\right]e^{ky}$

Equilibrium equation: $\nabla(\nabla\cdot \underset{\sim}{u}) + (1-2\nu)\nabla^2 \underset{\sim}{u} = 0$

$\nabla^2 u_x = 2B_1 k_z\, e^{i(k_x x + k_y y)}\, e^{k_z z}$

$\nabla^2 u_y = 2B_2 k_z\, e^{i(k_x x + k_y y)}\, e^{k_z z}$

$\nabla^2 u_z = 2B_3 k_z\, e^{i(k_x x + k_y y)}\, e^{k_z z}$

$\nabla\cdot \underset{\sim}{u} = \big[(A_1 + B_1 z)\, i k_x + (A_2 + B_2 z)\, i k_y$
$\quad + (A_3 + B_3 z)\, i k_z + B_3\big]\cdot$
$\quad e^{i(k_x x + k_y y)}\, e^{k_z z}$

$$\nabla(\nabla\cdot\underset{\sim}{u}) + (1-2\nu)\nabla^2 \underset{\sim}{u} = 0$$

$$\rightarrow \begin{cases} -B_1 K_x - B_2 K_y + i B_3 K_z = 0 \\ K_x(-K_x A_1 - K_y A_2 + i B_3 + i K_z A_3) + 2(1-2\nu) B_1 K_z = 0 \\ K_y(-K_x A_1 - K_y A_2 + i B_3 + i K_z A_3) + 2(1-2\nu) B_2 K_z = 0 \end{cases}$$

(these are the only three independent equations)

$$\rightarrow \begin{cases} -B_1 K_x - B_2 K_y + i B_3 K_z = 0 \\ B_1 K_y - B_2 K_x = 0 \\ K_x A_1 + K_y A_2 - i K_z A_3 = (3-4\nu) i B_3 \end{cases}$$

B.C.

$$\sigma_{xz} = 2\mu \varepsilon_{xz} = \mu(u_{x,z} + u_{z,x})$$

$$\sigma_{yz} = 2\mu \varepsilon_{yz} = \mu(u_{y,z} + u_{z,y})$$

$$\sigma_{zz} = (\lambda+2\mu)\varepsilon_{zz} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{xx}$$

$$= (\lambda+2\mu) u_{z,z} + \lambda u_{y,y} + \lambda u_{x,x}$$

at $z=0$:

$$u_{x,z} + u_{z,x} = 0$$

$$u_{y,z} + u_{z,y} = 0$$

$$(\lambda+2\mu)u_{z,z} + \lambda u_{y,y} + \lambda u_{x,x} = S \cdot e^{i(K_x X + K_y Y)} \cdot e^{K_z z}$$

$$\rightarrow \begin{cases} \text{three more algebraic equations} \\ \text{for } A_1, A_2, A_3, B_1, B_2, B_3 \end{cases}$$

[half-space_3d.m] on coursework

solution

$$\begin{cases} A_1 = -\dfrac{i K_x (1-2\nu)}{2\mu K_z^2} \cdot S \\ A_2 = -\dfrac{i K_y (1-2\nu)}{2\mu K_z^2} \cdot S \\ A_3 = \dfrac{1-\nu}{\mu K_z} \cdot S \end{cases} \qquad \begin{cases} B_1 = -\dfrac{i \ K_x}{2\mu K_z} \cdot S \\ B_2 = -\dfrac{i \ K_y}{2\mu K_z} \cdot S \\ B_3 = -\dfrac{1}{2\mu} \cdot S \end{cases}$$

define $\varphi = e^{i(K_x x + K_y y)} \, e^{K_z z}$

$$u_x = -\frac{i K_x}{2\mu K_z^2} \cdot \left[(1-2\nu) + K_z z\right] \cdot \varphi \cdot S$$

$$u_y = -\frac{i K_y}{2\mu K_z^2} \left[(1-2\nu) + K_z z\right] \cdot \varphi \cdot S$$

$$u_z = \frac{1}{2\mu K_z} \left[2(1-\nu) - K_z z\right] \cdot \varphi \cdot S$$

normal displacement on the surface

$$\tilde{u}_z = u_z(z=0) = \frac{1-\nu}{\mu} \cdot \frac{e^{i(K_x x + K_y y)}}{K_z} \cdot S$$

$$K_z = \sqrt{K_x^2 + K_y^2}$$

Summary:

Elastic half space subjected to traction force:

$$T_x = T_y = 0, \qquad T_z = S \, e^{i(K_x x + K_y y)}$$

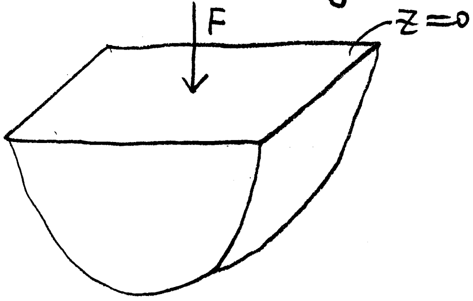
The displacement on the surface is

$$\tilde{u}_z = S \cdot \frac{1-\nu}{\mu} \cdot \frac{e^{i(K_x x + K_y y)}}{\sqrt{K_x^2 + K_y^2}}$$

$$\tilde{u}_z = \frac{1-\nu}{\mu} \frac{T_z}{\sqrt{K_x^2 + K_y^2}}$$

* Let $K_y \to 0$ reduce to 2D solution

§4. Point Loading on half-space




B.C. on the surface $z=0$

$T_x(x,y)=0,$

$T_y(x,y)=0.$

$T_z(x,y)=-F\,\delta(x,y)$

Solution can be obtained by inverse Fourier transform.

Define 2D Fourier transform

$$\hat{f}(k_x,k_y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,e^{-i(k_x x+k_y y)}\,dx\,dy$$

inverse transform

$$f(x,y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{f}(k_x,k_y)\,e^{i(k_x x+k_y y)}\,dk_x\,dk_y$$

Fourier transform pairs

| real space | Fourier space |
|---|---|
| $\delta(x,y)$ | $1$ |
| $T_z(x,y)=-F\,\delta(x,y)$ $\rightarrow$ | $\hat{T}_z(k_x,k_y)=-F$ |
| | $\downarrow$ |
| $\tilde{u}_z(x,y)=-\frac{F(1-\nu)}{2\pi\mu}\frac{1}{\sqrt{x^2+y^2}}$ $\leftarrow$ | $\hat{u}_z(k_x,k_y)=-F\cdot\frac{1-\nu}{\mu}\cdot\frac{1}{\sqrt{k_x^2+k_y^2}}$ |

$$=-\frac{F(1-\nu)}{2\pi\mu}\cdot\frac{1}{r}$$

$$r=\sqrt{x^2+y^2}$$

notice that $\frac{1}{r}\to 0$ when $r\to\infty$.

$\frac{1}{r}$ only has singularity at $r\to 0$.

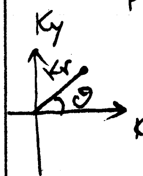
* compare with the 2D solution $\ln r$, which has singularity at both $r\to\infty$ and $r\to 0$.

* Proof of inverse transform

$$\tilde{u}_z(x,y) \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(-F)\frac{1-\nu}{\mu}\frac{e^{i(k_x x + k_y y)}}{\sqrt{k_x^2+k_y^2}}\,dk_x\,dk_y = -\frac{F(1-\nu)}{\mu}\cdot\frac{1}{2\pi r}$$

by symmetry we expect $\hat{u}_z(x,y)$ to be a function of $r \equiv \sqrt{x^2+y^2}$

Hence, without loss of generality, we can consider the point $x = r$, $y = 0$.




$$k_x x + k_y y = k_r \cdot r \cdot \cos\theta \qquad k_r = \sqrt{k_x^2+k_y^2}$$

$$\hat{u}_z(r) = \frac{1}{(2\pi)^2}\frac{(-F)(1-\nu)}{\mu}\int_0^{\infty}\int_0^{2\pi}\frac{e^{ik_r\cdot r\cdot\cos\theta}}{k_r}\cdot k_r\,dk_r\,d\theta$$

$$= -\frac{F(1-\nu)}{2(2\pi)^2\mu}\int_{-\infty}^{+\infty}\int_0^{2\pi} e^{ik_r r\cos\theta}\,dk_r\,d\theta$$

$$= -\frac{F(1-\nu)}{4\pi\mu}\int_0^{2\pi}\delta(r\cos\theta)\,d\theta$$

because $\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk = \delta(x)$

$$= -\frac{F(1-\nu)}{4\pi\mu}\cdot\frac{1}{r}\int_0^{2\pi}\delta(\cos\theta)\,d\theta$$

because $\delta(ax) = \frac{1}{a}\delta(x)$

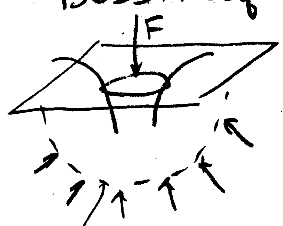
$$= -\frac{F(1-\nu)}{2\pi\mu}\cdot\frac{1}{r}$$

$$= -\frac{F(1-\nu^2)}{\pi E}\frac{1}{r}$$

because

$$\int_0^{2\pi}\delta(\cos\theta)\,d\theta = 2\int_0^{\pi}\delta(\cos\theta)\,d\theta = 2\int_{-1}^{1}\frac{\delta(x)}{\sqrt{1-x^2}}\,dx = 2$$

Bossinesq solution:



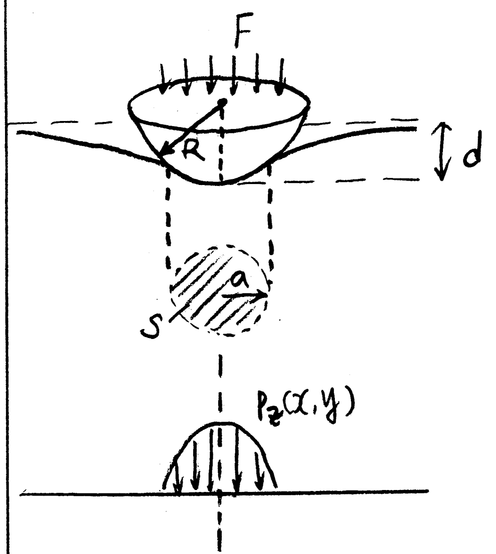

$$\boxed{\begin{aligned} u_z\big|_{z=0} &= -\frac{F(1-\nu)}{2\pi\mu}\frac{1}{r} \\ u_r\big|_{z=0} &= -\frac{F(1-2\nu)}{4\pi\mu}\frac{1}{r}\end{aligned}}$$

Area $\sim 4\pi r^2$: from dimensional analysis, we expect

$$\sigma \sim \frac{1}{r^2}$$

$$\therefore\ u \sim \frac{1}{r}$$

§5. Hertz contact Problem




For simplicity, consider a rigid sphere of radius R indenting an elastic half-space.

(in general, the indentor can be a deformable ellipsoid)

By symmetry, we expect the contact area to be a circle of radius a.
Let F be the magnitude of indenting force and d be the indentation depths.

We want to know the F-d relationship and F-a relationship.


We also want to know the pressure distribution on the surface

$$p_z(x,y) = -\sigma_{zz}\big|_{z=0}$$

$$\begin{cases} p_z(x,y) \geq 0 & x^2+y^2 \leq a^2 \quad \text{(inside contact area } S\text{)} \\ p_z(x,y) = 0 & x^2+y^2 > a^2 \end{cases}$$

The shape of the indentor can be described by

$$u_0(x,y) = \frac{x^2}{2R} + \frac{y^2}{2R}$$

(see lecture notes "Contact", p.5)

Let $\tilde{u}_z(x,y)$ be the surface displacement of the half-space.

then
$$\begin{cases} \tilde{u}_z(x,y) = u_0(x,y) - d & x^2+y^2 \leq a^2 \\ \tilde{u}_z(x,y) < u_0(x,y) - d & x^2+y^2 > a^2 \end{cases}$$

using the Boussinesq solution, $\tilde{u}_z(x,y)$ and $p_z(x,y)$ are related to each other

$$-\frac{1-\nu^2}{\pi E} \iint_{S:\ x'^2+y'^2\le a^2} \frac{p_z(x',y')}{\sqrt{(x-x')^2+(y-y')^2}}\, dx'dy' = \tilde{u}_z(x,y)$$

For $(x,y)$ inside area $S$: $(x^2+y^2\le a^2)$

$$\iint_S \frac{p_z(x',y')}{\sqrt{(x-x')^2+(y-y')^2}}\, dx'dy' = -\frac{\pi E}{1-\nu^2}\left(-d+\frac{x^2}{2R}+\frac{y^2}{2R}\right)$$

What distribution of $p_z(x',y')$ solves the above equation?

Magic formula:

$$\iint_{x'^2+y'^2\le a^2} \frac{\sqrt{1-\left(\frac{x'}{a}\right)^2-\left(\frac{y'}{a}\right)^2}}{\sqrt{(x-x')^2+(y-y')^2}}\, dx'dy' = \frac{\pi}{4a}\left(2a^2-x^2-y^2\right)$$

* For proof and more discussions, see notes "Potential Field of a Uniformly Charged Ellipsoid".

* This is closely related to the fact that the potential field of a uniformly charged ellipsoid is a quadratic function inside the ellipsoid, something supposedly well known to physicists familiar with electrostatics — such as Lev Landau.

* This property is also closely related to the Eshelby's inclusion (that has ellipsoidal shape) that will be discussed in ME340B "Micromechanics".

By matching the above two expressions, we get

$$p_z(x', y') = p_0 \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2}$$

$p_0$ is related to the ~~total~~ force $F = \iint_S p_z(x', y')\,dx'dy' = p_0 \frac{2\pi a^2}{3}$

$$\boxed{p_0 = \frac{3F}{2\pi a^2}}$$

Also by matching the expression, we find

contact radius : $\boxed{a = \left(\frac{3(1-\nu^2)FR}{4E}\right)^{1/3}}$

indent depth : $\boxed{d = \left(\frac{3(1-\nu^2)}{4E}\right)^{2/3} \frac{F^{2/3}}{R^{1/3}}}$