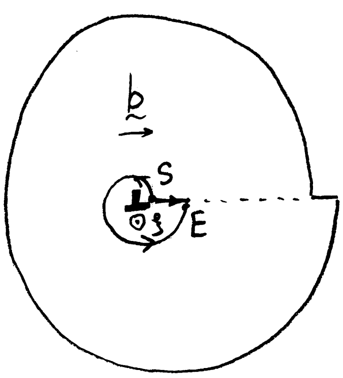
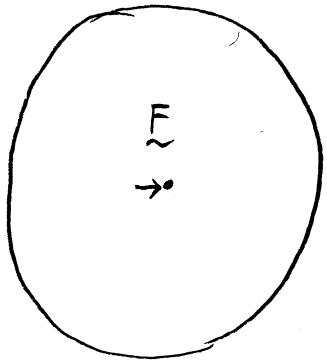
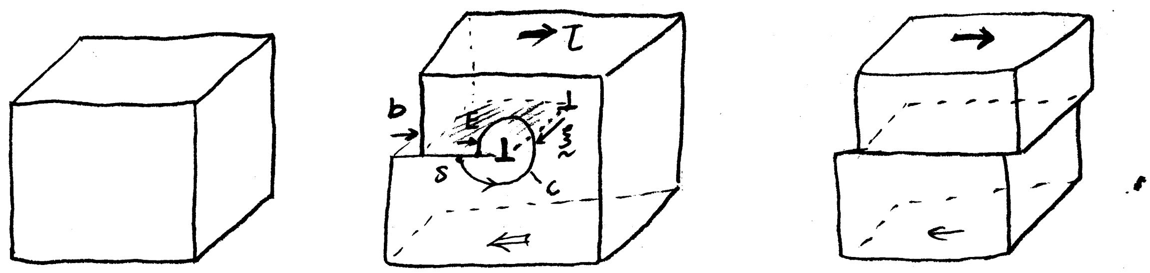
In this section, we discuss two important solutions, both having $\sigma \sim \frac{1}{r}$ singularity. They are the stress field of a dislocation line and that of a line force.




a line force is a body force concentrated at a point
i.e. $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{F}\,\delta(\underset{\sim}{x})$

## §1. Dislocation and Burgers Vector




starting with a stress-free elastic medium

a dislocation is introduced by making a cut on an internal surface and introducing a displacement jump across that surface.
The dislocation $\perp$ is the boundary line of the surface.

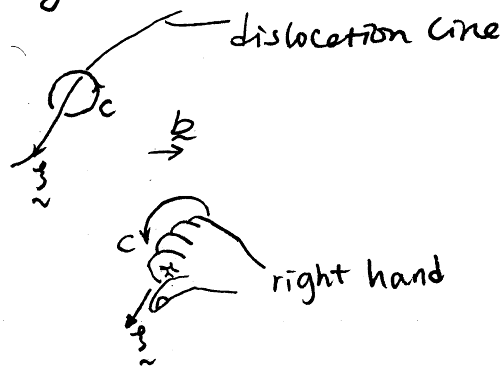
In this case, the dislocation experience a (Peach-Koehler) force to the right from the applied stress $\tau$.
By the time the dislocation escapes to the right, the entire upper half is displaced by $\underset{\sim}{b}$ wrt the lower half.

* Dislocations are important defects of crystals.
They are the fundamental carriers of plastic deformation in crystals. (see "Computer Simulations of Dislocations", Sect. 1.2–1.3)

Therefore, a medium containing a dislocation must have a displacement jump somewhere.

This can be written mathematically as

$$\oint_C \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \, d\boldsymbol{x} = \boldsymbol{b}$$




where C is some closed loop (called the <u>Burgers circuit</u>) around the dislocation line

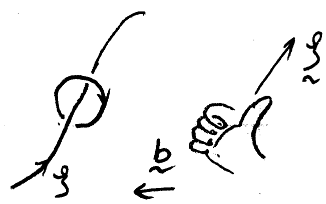
$\boldsymbol{b}$ is called the <u>Burgers vector</u>

Notice that the direction of $\boldsymbol{b}$ depend on the direction of c.

As a convention, let us define a line sense $\boldsymbol{\xi}$ along the dislocation. and let the direction of c follow $\boldsymbol{\xi}$ from the right-hand rule.

Thus, the orientation of $\boldsymbol{b}$ depends on the choice of line sense $\boldsymbol{\xi}$

If we reverse the choice of $\boldsymbol{\xi}$ for the same dislocation, the orientation of $\boldsymbol{b}$ will reverse as well.




A visual way to identify the Burgers vector $\boldsymbol{b}$ is to draw the Burgers circuit around the dislocation starting from the cut-plane. (see figure on page 1).

The vector connecting the starting point S and the end point E is the Burgers vector.

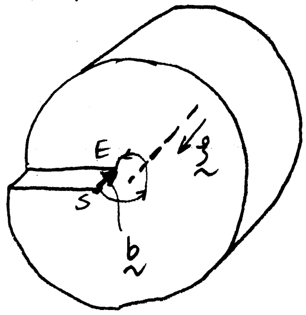
* Show that for the same dislocation, if we reverse $\boldsymbol{\xi}$ the Burgers vector reverse as well

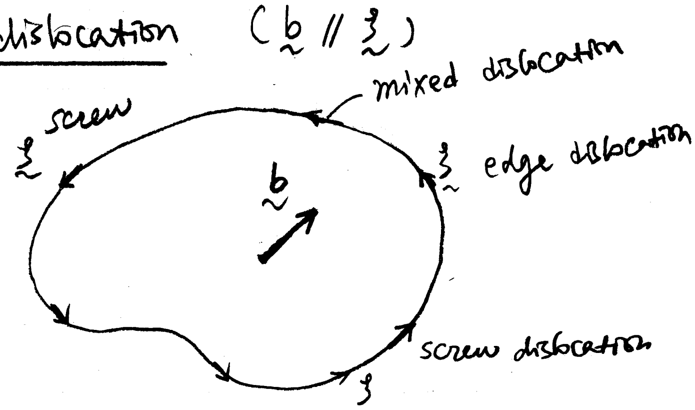
In the Figure on page 1, the Burgers vector $\underset{\sim}{b}$ is perpendicular to the line direction $\underset{\sim}{\xi}$

This is called an <u>edge dislocation</u> ($\underset{\sim}{b} \perp \underset{\sim}{\xi}$)

When $\underset{\sim}{b}$ is parallel to the line direction $\underset{\sim}{\xi}$

it is called a <u>screw dislocation</u> ($\underset{\sim}{b} \parallel \underset{\sim}{\xi}$)




In general, the dislocation line can be curved but the Burgers vector stays constant along the line.

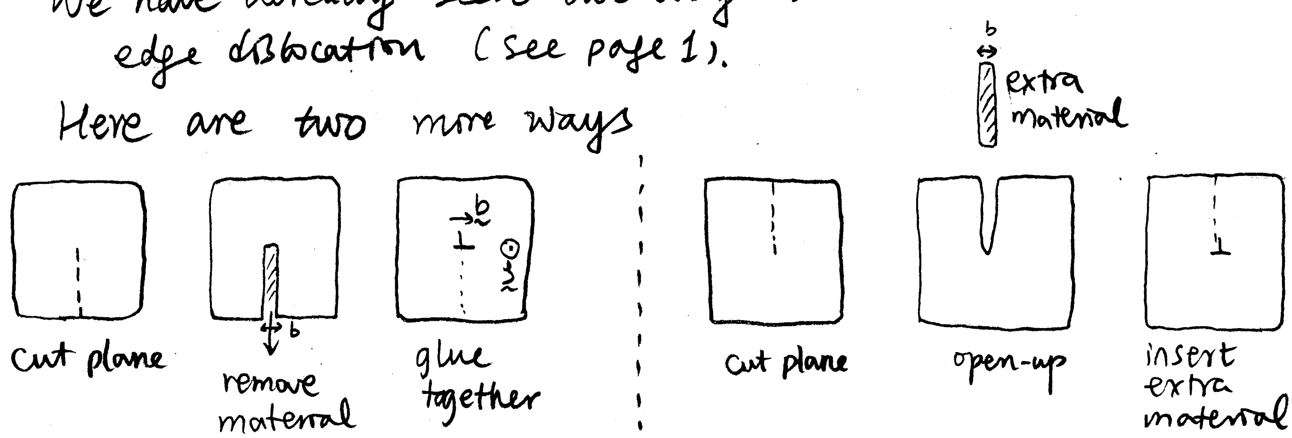
When $\underset{\sim}{b}$ is neither perpendicular nor parallel to $\underset{\sim}{\xi}$,

it is called a <u>mixed dislocation</u>.

The same dislocation can be created by many different ways (with different choice of cut-planes).

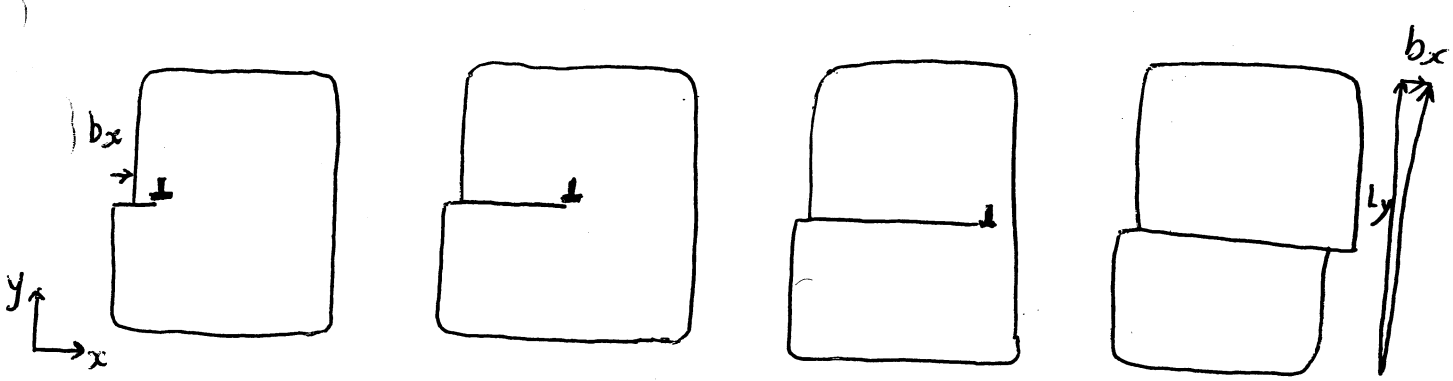
We have already seen two ways to create the same edge dislocation (see page 1).

Here are two more ways




* show that in both cases, we get the same $\underset{\sim}{b}$ as before.

## §2. Dislocation motion and plastic strain

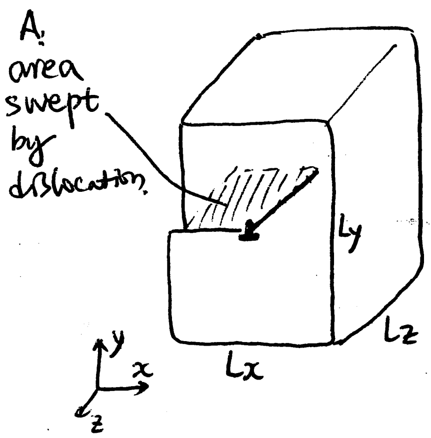



dislocation nucleates from left surface → dislocation travels to the right → dislocation exits from right surface

net result: upper half of the material (crystal) slips by $b_x$ with respect to lower half

plastic deformation:

$$\varepsilon_{xy}^{pl} = \frac{1}{2}\cdot\frac{b_x}{L_y} = \frac{b_x}{2}\,\frac{L_x L_z}{L_x L_y L_z} = \frac{1}{2}\,\frac{b_x \cdot A^{tot}}{V}$$



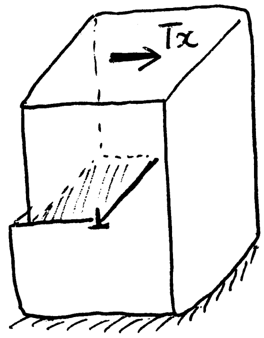

$A^{tot}$: total area swept by dislocation

$V$: material volume.

It can be shown that, in general, when a dislocation line swept an area $A$ on a plane with normal vector $\underset{\sim}{n}$, it produces a plastic strain of

$$\varepsilon_{ij}^{pl} = \frac{1}{2}\cdot\frac{b_i n_j + b_j n_i}{V}\cdot A$$

§3. Force on dislocation line from stress field.




Assuming a uniform traction force $T_x$ is applied to the top surface.

This leads to an applied (external) stress field

$$\sigma_{xy}^{ext} = T_x$$

The total stress field in the medium is the superposition of $\sigma_{ij}^{ext}$ and the internal stress field $\sigma_{ij}^{int}$ of the dislocation.

$$\sigma_{ij}^{tot} = \sigma_{ij}^{ext} + \sigma_{ij}^{int}$$

The total work done by the applied traction force as the dislocation moves from left end to right end is

$$\Delta W = \underbrace{T_x (L_x \cdot L_z)}_{\text{total force}} \cdot \underbrace{b_x}_{\text{distance}} \quad > 0 \quad (\text{when } T_x > 0)$$

This means it is energetically favorable for the dislocation to move from left to right when $T_x > 0$.

We can interprete $\Delta W$ as the work done by a generalized force $f_x$ (per unit length) exerted on the dislocation line.

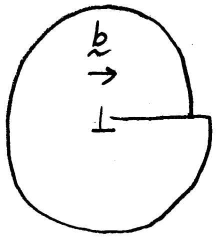
$$\Delta W = \underbrace{f_x \cdot L_z}_{\text{total force}} \cdot \underbrace{L_x}_{\text{distance}}$$

> In general, the peach-Koehler force is
>
> $$\underset{\sim}{f} = (\underset{\sim}{b} \cdot \underset{\approx}{\sigma}) \times \underset{\sim}{\xi}$$

L The Peach-Koehler formula

$$f_x = \frac{\Delta W}{L_z L_x} = b_x T_x = b_x \cdot \sigma_{xy}$$

## §4. Stress field of a dislocation line and that of a line force in an infinite medium

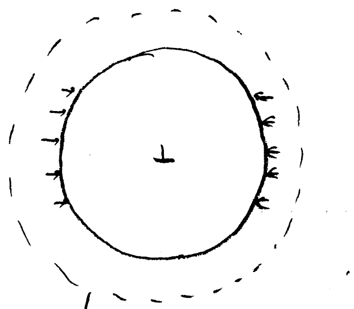



Look for $\phi$ whose $u_r$ contains $\theta$ (for discontinuity)

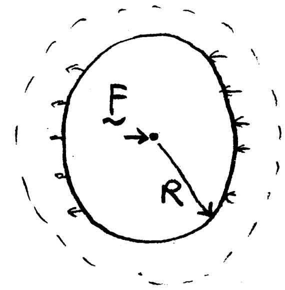
$$\phi = A r\theta \sin\theta + B r\theta\cos\theta + C r\ln r\cos\theta + D r\ln r \sin\theta$$

A, C terms — even with $\theta$

B, D terms — odd with $\theta$



there should not be any net force integrated over the entire surface




We expect $\sigma(r,\theta) = f(r)\,g(\theta)$

The total traction force on any circle with radius R must balance F.

$$\int_0^{2\pi} \sigma_{ij}(R,\theta)\cdot n_j(\theta)\; R\, d\theta = F$$

$$f(R)\cdot R = \text{const}$$

$$f(R) \sim \frac{1}{R}.$$

or $f(r) \sim \frac{1}{r}$.

Look for $\phi$ whose $\sigma \sim \frac{1}{r}$

$$\phi = A r\theta\sin\theta + B r\theta\cos\theta + C r\ln r\cos\theta + D r\ln r\sin\theta$$

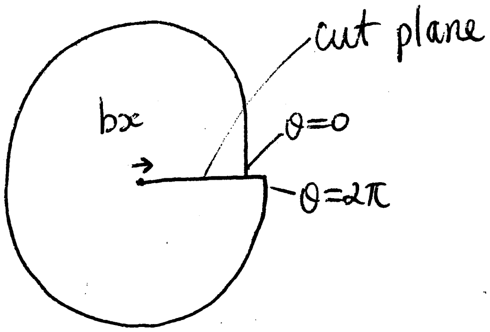
Same as trial solution for dislocations!

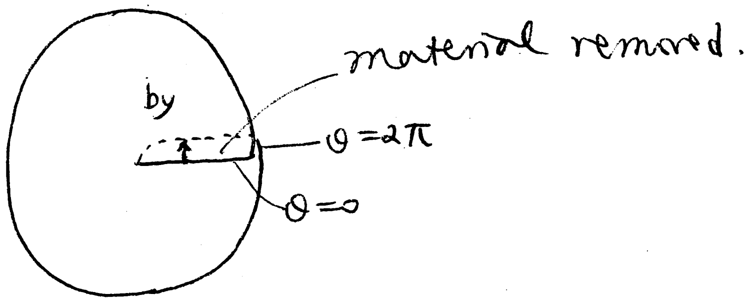
there should not be any displacement jump on any surface

Solution strategy:

find coefficients to give

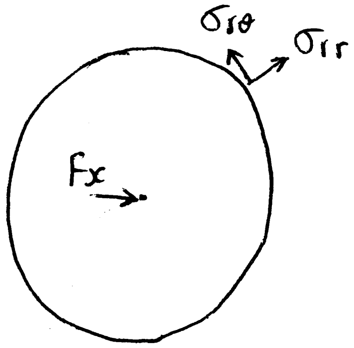
| | |
|---|---|
| displacement jump but no net force | net force but no displacement jump. |

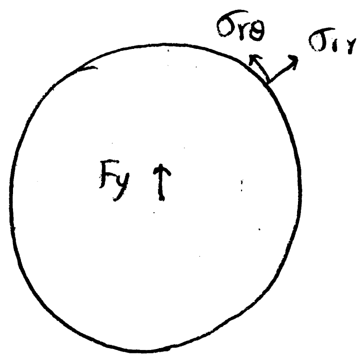



$$b_x = u_r\big|_{\theta=2\pi} - u_r\big|_{\theta=0} = \frac{\pi}{2\mu}\left[B(\kappa-1) - D(\kappa+1)\right]$$

$$b_y = u_\theta\big|_{\theta=2\pi} - u_\theta\big|_{\theta=0} = \frac{\pi}{2\mu}\left[A(\kappa-1) + C(\kappa+1)\right]$$




$$F_x + \int_0^{2\pi} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\, r\, d\theta = 0 \Rightarrow F_x = -2\pi A$$

$$F_y + \int_0^{2\pi} (\sigma_{rr}\sin\theta + \sigma_{r\theta}\cos\theta)\, r\, d\theta = 0 \Rightarrow F_y = 2\pi B$$

---

dislocation: $b_x$, $b_y = 0$, $F_x = 0$, $F_y = 0$. $\kappa = 3-4\nu$ plane strain

$\kappa + 1 = 4(1-\nu)$

$$D = -\frac{2\mu}{\pi(\kappa+1)} b_x = -\frac{\mu}{2\pi(1-\nu)} b_x$$

$(A = B = C = 0)$ $\Rightarrow \begin{cases} u_r \\ u_\theta \end{cases}$, $\begin{matrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{matrix} = \frac{\mu b_x}{2\pi(1-\nu)r}(\ldots)$

↑ angular dependence

---

$b_x = 0$, $b_y$, $F_x = 0$, $F_y = 0$.

$$C = \frac{2\mu}{\pi(\kappa+1)} b_y = \frac{\mu}{2\pi(1-\nu)} b_y \quad (A = B = D = 0)$$

Line force. (Kelvin Solution)

$F_x$, $F_y = 0$, $b_x = 0$, $b_y = 0$.

plane strain

$\kappa = 3 - 4\nu$

$$A = -\frac{F_x}{2\pi}$$

$$\frac{\kappa - 1}{\kappa + 1} = \frac{2 - 4\nu}{4 - 4\nu} = \frac{1 - 2\nu}{2(1-\nu)}$$

$$C = -\frac{\kappa - 1}{\kappa + 1} \cdot A = \frac{1 - 2\nu}{4\pi(1-\nu)} F_x$$

$$B = D = 0.$$

$$\Rightarrow \begin{cases} \sigma_{rr} = \frac{F_x}{r}(\cdots) \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{cases}$$

angular dependence.

---

$F_x = 0$. $F_y$. $b_x = 0$. $b_y = 0$.

$$B = \frac{F_y}{2\pi}$$

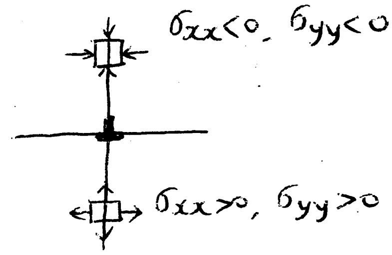
$$D = \frac{\kappa - 1}{\kappa + 1} B = \frac{1 - 2\nu}{4\pi(1-\nu)} F_y$$

$$\Rightarrow \begin{cases} \sigma_{rr} = \frac{F_y}{r}(\cdots) \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{cases}$$

Stress field of dislocation. $b_x$, $b_y = 0$.

$$\phi = -\frac{\mu b_x}{2\pi(1-\nu)}\, r \ln r \sin\theta \qquad r\sin\theta = y, \quad r = \sqrt{x^2+y^2}$$

$$\phi(x,y) = -\frac{\mu b_x}{4\pi(1-\nu)}\, y \ln(x^2+y^2)$$

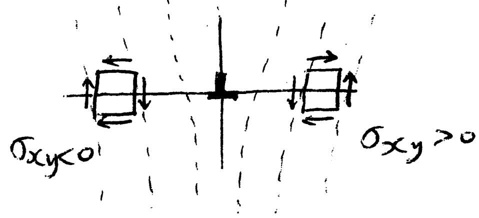



$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = -\frac{\mu b_x}{2\pi(1-\nu)}\frac{y(3x^2+y^2)}{(x^2+y^2)^2}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{\mu b_x}{2\pi(1-\nu)}\frac{y(x^2-y^2)}{(x^2+y^2)^2}$$

see disloc_stress.m

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \frac{\mu b_x}{2\pi(1-\nu)}\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

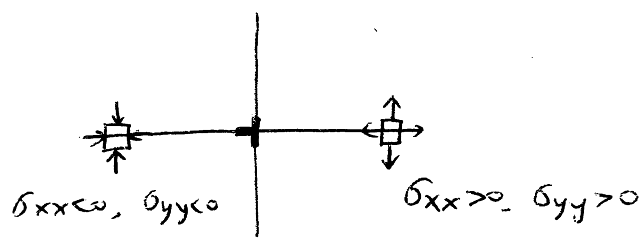



---

Stress field of dislocation $b_x = 0$, $b_y$

$$\phi = \frac{\mu b_y}{2\pi(1-\nu)}\, r\ln r\cos\theta$$

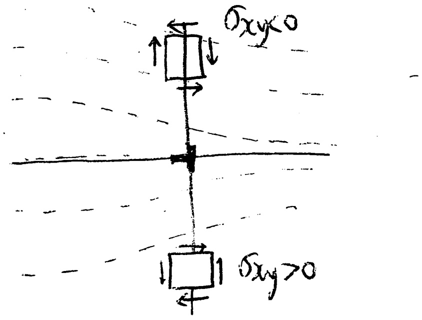
$$\phi(x,y) = \frac{\mu b_y}{4\pi(1-\nu)}\, x\ln(x^2+y^2)$$




$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \frac{\mu b_y}{2\pi(1-\nu)}\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

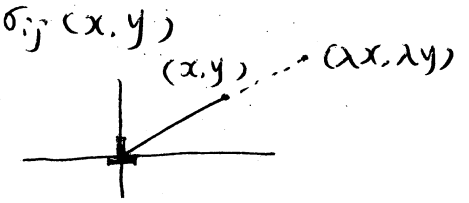
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{\mu b_y}{2\pi(1-\nu)}\frac{x(x^2+3y^2)}{(x^2+y^2)^2}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \frac{\mu b_y}{2\pi(1-\nu)}\frac{y(x^2-y^2)}{(x^2+y^2)^2}$$




---

Notice that $\sigma_{ij}(\lambda x, \lambda y) = \frac{1}{\lambda}\sigma_{ij}(x,y)$




$$\sigma_{ij} \sim \frac{1}{r}$$