In this section, we discuss two important solutions, both having $\sigma \sim \frac{1}{r}$ singularity. They are the stress field of a dislocation line and that of a line force.

A line force is a body force concentrated at a point, i.e., $f(x) = F \delta(x)$

§1. Dislocation and Burgers Vector

Starting with a stress-free elastic medium, a dislocation is introduced by making a cut on an internal surface and introducing a displacement jump across that surface. The dislocation is the boundary line of the surface.

In this case, the dislocation experiences a (Peach-Koehler) force to the right from the applied stress. By the time the dislocation escapes to the right, the entire upper half is displaced by $b$, while the lower half

* Dislocations are important defects of crystals. They are the fundamental carriers of plastic deformation in crystals. (See "Computer Simulations of Dislocations", Sect.12-13)
Therefore, a medium containing a dislocation must have a displacement jump somewhere. This can be written mathematically as

$$\oint_c \frac{\partial \ell}{\partial x} \, dx = \vec{b}$$

where $c$ is some closed loop (called the Burgers circuit) around the dislocation line.

$\vec{b}$ is called the Burgers vector.

Notice that the direction of $\vec{b}$ depends on the direction of $c$.

As a convention, let us define a line sense $\hat{s}$ along the dislocation, and let the direction of $c$ follow $\hat{s}$ from the right-hand rule.

Thus, the orientation of $\vec{b}$ depends on the choice of line sense $\hat{s}$.

If we reverse the choice of $\hat{s}$ for the same dislocation, the orientation of $\vec{b}$ will reverse as well.

A visual way to identify the Burgers vector $\vec{b}$ is to draw the Burgers circuit around the dislocation starting from the cut-plane. (See figure on page 1.)

The vector connecting the starting point $S$ and the end point $E$ is the Burgers vector.

* Show that for the same dislocation, if we reverse $\hat{s}$, the Burgers vector reverses as well.
In the Figure on page 1, the Burgers vector \( \mathbf{b} \) is perpendicular to the line direction \( \mathbf{z} \). This is called an edge dislocation \( (\mathbf{b} \perp \mathbf{z}) \).

When \( \mathbf{b} \) is parallel to the line direction \( \mathbf{z} \), it is called a screw dislocation \( (\mathbf{b} \parallel \mathbf{z}) \).

In general, the dislocation line can be curved but the Burgers vector stays constant along the line.

When \( \mathbf{b} \) is neither perpendicular nor parallel to \( \mathbf{z} \), it is called a mixed dislocation.

The same dislocation can be created by many different ways (with different choice of cut-planes). We have already seen two ways to create the same edge dislocation (see page 1).

Here are two more ways:

- **Cut plane**
- **Remove material**
- **Glue together**
- **Open-up**
- **Insert extra material**

*Show that in both cases, we get the same \( \mathbf{b} \) as before.*
§2. Dislocation motion and plastic strain

Dislocation nucleates from left surface → dislocation travels to the right → dislocation exits from right surface

Net result: upper half of the material (crystal) slips by bx with respect to lower half

Plastic deformation:

\[ \varepsilon_{xy}^{pl} = \frac{1}{2} \cdot \frac{bx}{Ly} = \frac{bx}{2} \cdot \frac{LxLz}{LyLz} = \frac{1}{2} \cdot \frac{b_x A^{tot}}{V} \]

\[ A^{tot} : \text{total area swept by dislocation} \]
\[ V : \text{material volume} \]

It can be shown that, in general, when a dislocation line swept an area \( A \) on a plane with normal vector \( n \), it produces a plastic strain of

\[ \varepsilon^{\text{pl}}_{ij} = \frac{1}{2} \cdot \frac{b_i n_j + b_j n_i}{V} \cdot A \]
33. Force on dislocation line from stress field.

Assuming a uniform traction force $T_x$ is applied to the top surface.

This leads to an applied (external) stress field $\sigma_{xy}^{ext} = T_x$.

The total stress field in the medium is the superposition of $\sigma_{ij}^{ext}$ and the internal stress field $\sigma_{ij}^{int}$ of the dislocation.

$$\sigma_{ij}^{tot} = \sigma_{ij}^{ext} + \sigma_{ij}^{int}$$

The total work done by the applied traction force as the dislocation moves from left end to right end is

$$\Delta W = \frac{T_x (L_x \cdot L_z) \cdot b_x}{\text{total force distance}} > 0 \quad \text{(when } T_x > 0)$$

This means it is energetically favorable for the dislocation to move from left to right when $T_x > 0$.

We can interprete $\Delta W$ as the work done by a generalized force $f_x$ (per unit length) exerted on the dislocation line.

$$\Delta W = \frac{f_x \cdot L_z \cdot L_x}{\text{total force distance}}$$

In general, the Peach-Koehler force is

$$f_x = (b \cdot \mathbf{e}) \times \frac{\mathbf{e}}{b}$$

The Peach-Koehler formula

$$f_x = \frac{\Delta W}{L_z L_x} = b_x T_x = b_x \sigma_{xy}$$
§4. Stress field of dislocation line and that of a line force in an infinite medium.

Look for \( \phi \) whose \( \delta r \) contains \( \theta \) (for discontinuity).

\[
\phi = A r \theta \sin \theta + B r \cos \theta \\
+ c r \ln r \cos \theta + d r \ln r \sin \theta
\]

A, C terms — even with \( \theta \)
B, D terms — odd with \( \theta \)

There should not be any net force integrated over the entire surface.

Solution strategy:

Find coefficients to give

- displacement jump but no net force
- net force but no displacement jump.

We expect \( \sigma(r, \theta) = f(r) g(\theta) \).

The total traction force in any circle with radius \( R \) must balance \( F \):

\[
\int_0^{2\pi} \sigma_{ij}(r, \theta) \cdot n_j(\theta) \, R \, d\theta = F
\]

\[ f(r) \cdot R = \text{const} \]
\[ f(r) \sim \frac{1}{R} \]

Look for \( \phi \) whose \( \sigma \sim \frac{1}{r} \).

\[
\phi = A r \theta \sin \theta + B r \cos \theta \\
+ c r \ln r \cos \theta + d r \ln r \sin \theta
\]

Same as trial solution for dislocations!

There should not be any displacement jump on any surface.
\[ b_x = U_r \bigg|_{\theta=2\pi} - U_r \bigg|_{\theta=0} = \frac{\pi}{2\mu} \left[ B (\kappa-1) - D (\kappa+1) \right] \]
\[ b_y = U_\theta \bigg|_{\theta=2\pi} - U_\theta \bigg|_{\theta=0} = \frac{\pi}{2\mu} \left[ A (\kappa-1) + C (\kappa+1) \right] \]

\[ F_x + \int_0^{2\pi} (\sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta) \, r \, d\theta = 0 \Rightarrow F_x = -2\pi A \]
\[ F_y + \int_0^{2\pi} (\sigma_{r\theta} \sin \theta + \sigma_{\theta\theta} \cos \theta) \, r \, d\theta = 0 \Rightarrow F_y = 2\pi B \]

Dislocation: \( b_x, b_y = 0, \quad F_x = 0, \quad F_y = 0. \]

\[ D = -\frac{2\mu}{\pi (\kappa+1)} \quad b_x = -\frac{\mu}{2\pi (1-\nu)} b_x \]

\( (A = B = C = 0) \)

\[ \Rightarrow \{ U_r, U_\theta \} \quad \sigma_{rr} = \frac{\mu b_x}{2\pi (1-\nu)} \quad \sigma_{r\theta} = \frac{\mu b_y}{2\pi (1-\nu)} \]

\[ b_x = 0, \quad b_y = 0, \quad F_x = 0, \quad F_y = 0. \]

\[ \kappa = 3-4\nu \quad \text{plane strain} \]

\[ \kappa + 1 = 4(1-\nu) \]
Line force. (Kelvin Solution)

\[ F_x, F_y = 0, \quad B_x = 0, \quad B_y = 0. \]

\[ A = -\frac{F_x}{2\pi} \]

\[ C = -\frac{k-1}{K+1} \cdot A = \frac{1-2\nu}{4\pi(1-\nu)} F_x \]

\[ B - D = 0. \]

\[ F_x = 0, \quad F_y, \quad B_x = 0, \quad B_y = 0. \]

\[ B = \frac{F_y}{2\pi} \]

\[ D = \frac{k-1}{K+1} \cdot B = \frac{1-2\nu}{4\pi(1-\nu)} F_y \]

Plane strain

\[ K = 3-4\nu \]

\[ \frac{k-1}{K+1} = \frac{2-4\nu}{4-4\nu} = \frac{1-2\nu}{2(1-\nu)} \]

\[ \begin{align*}
\sigma_{rr} &= \frac{F_x}{r} (\cdots) \\
\sigma_{\theta\theta} &= \sigma_{\phi\phi} \\
\end{align*} \]

\[ \begin{align*}
\sigma_{rr} &= \frac{F_x}{r} (\cdots) \\
\sigma_{\theta\theta} &= \sigma_{\phi\phi} \\
\end{align*} \]
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Stress field of dislocation  \( b_y = 0 \)

\[ \phi = -\frac{M b_x}{2\pi (1-\nu)} y \ln r \sin \theta \]

\[ \phi(x,y) = -\frac{M b_x}{4\pi (1-\nu)} y \ln (x^2+y^2) \]

\[ \sigma_{xx} = \frac{\partial \phi}{\partial y^2} = -\frac{M b_x}{2\pi (1-\nu)} \frac{y(3x^2+y^2)}{(x^2+y^2)^2} \]

\[ \sigma_{yy} = \frac{\partial \phi}{\partial x} = \frac{M b_x}{2\pi (1-\nu)} \frac{y(x^2-y^2)}{(x^2+y^2)^2} \]

\[ \sigma_{xy} = -\frac{\partial \phi}{\partial xy} = \frac{M b_x}{2\pi (1-\nu)} \frac{x(x^2-y^2)}{(x^2+y^2)^2} \]

\[ \delta_{xx} < 0, \delta_{yy} < 0 \]

\[ \delta_{xx} > 0, \delta_{yy} > 0 \]

Stress field of dislocation  \( b_x = 0 \), \( b_y \)

\[ \phi = \frac{M b_y}{2\pi (1-\nu)} y \ln r \cos \theta \]

\[ \phi(x,y) = \frac{M b_y}{4\pi (1-\nu)} x \ln (x^2+y^2) \]

\[ \sigma_{xx} = \frac{\partial \phi}{\partial y^2} = \frac{M b_y}{2\pi (1-\nu)} \frac{x(x^2-y^2)}{(x^2+y^2)^2} \]

\[ \sigma_{yy} = \frac{\partial \phi}{\partial x} = \frac{M b_y}{2\pi (1-\nu)} \frac{x(x^2+3y^2)}{(x^2+y^2)^2} \]

\[ \sigma_{xy} = -\frac{\partial \phi}{\partial xy} = \frac{M b_y}{2\pi (1-\nu)} \frac{y(x^2-y^2)}{(x^2+y^2)^2} \]

\[ \delta_{xx} < 0, \delta_{yy} > 0 \]

\[ \delta_{xx} > 0, \delta_{yy} > 0 \]

Notice that  \( \sigma_{ij} (\lambda x, \lambda y) = \frac{1}{\lambda} \sigma_{ij} (x,y) \)

\[ \sigma_{ij} \sim \frac{1}{r} \]