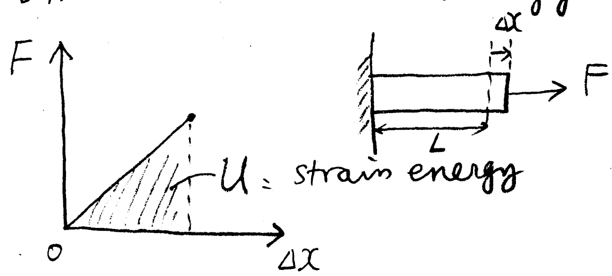
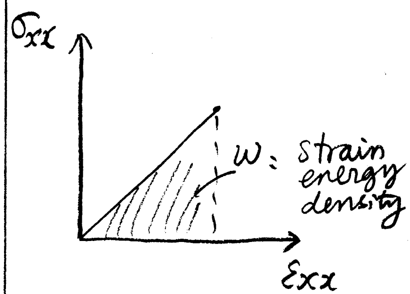
In this section, we will use the strain energy to show why the $\sigma \sim \frac{1}{r}$ singularity is not allowed for a crack tip while the $\sigma \sim \frac{1}{\sqrt{r}}$ singularity is allowed.

§1. Elastic Strain Energy



Consider a uni-axial tensile test of an elastic bar.

Let F be the applied force and Δx be the elongation



The elastic energy stored in the bar equals to the work done by the applied force (shade area on the F-Δx plot)

$$U = \frac{1}{2} F \cdot \Delta x$$

Recall that
$$F = \sigma_{xx} \cdot A$$
$$\Delta x = \varepsilon_{xx} \cdot L$$

$$U = \frac{1}{2} \sigma_{xx} \cdot A \cdot \varepsilon_{xx} \cdot L = \frac{1}{2} \sigma_{xx} \varepsilon_{xx} \cdot V$$

where V is the total volume of the sample.

Define strain energy density

$$w \equiv \frac{U}{V} = \frac{1}{2} \sigma_{xx} \varepsilon_{xx}$$

In this example, the strain energy density is uniform inside the sample. In general, w is a (scalar) field quantity

$$w(\underset{\sim}{x}) = \frac{1}{2} \sigma_{ij}(\underset{\sim}{x})\, \varepsilon_{ij}(\underset{\sim}{x})$$

The strain energy density w at point $\underset{\sim}{x}$ is a function of the local strain ε_{ij} at $\underset{\sim}{x}$.

$$w(\varepsilon_{ij}) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

Hence:

$$\sigma_{ij} = \frac{\partial w(\varepsilon_{ij})}{\partial \varepsilon_{ij}}$$

$$C_{ijkl} = \frac{\partial^2 w}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = \frac{\partial^2 w}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}} = C_{klij}.$$

— this is the cause of the major symmetry of C_{ijkl} tensor.

* We can also write w as a function of local stress σ_{ij}

$$w(\sigma_{ij}) = \frac{1}{2}S_{ijkl}\sigma_{ij}\sigma_{kl}$$

, where S_{ijkl} is the elastic compliance tensor.

$$\varepsilon_{ij} = \frac{\partial w(\sigma_{ij})}{\partial \sigma_{ij}}$$

$$S_{ijkl} = \frac{\partial^2 w}{\partial \sigma_{ij}\partial \sigma_{kl}} = S_{klij}$$

* Notice that $w \geq 0$ always. ($w=0$ when $\varepsilon_{ij}=0$)
i.e. any elastic deformation must increase elastic strain energy.

This put some constraints on elastic constants.

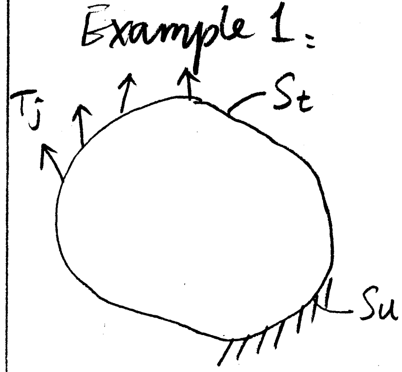
For isotropic material, $C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
we must have

$E > 0$ Youngs modulus

$-1 < \nu < \frac{1}{2}$ Poisson's ratio

$\mu > 0$ Shear modulus

Example 1:



consider an elastic medium subjected to arbitrary surface traction $T_j(\underset{\sim}{x})$

The elastic energy stored in the medium must equal the work done at the surface:

$$U = \frac{1}{2}\oint_S T_j(\underset{\sim}{x})\, u_j(\underset{\sim}{x})\, dS(\underset{\sim}{x})$$

$$= \frac{1}{2}\oint_S \sigma_{ij}\, u_j\, n_i\, dS$$

apply Gauss's theorem $\longrightarrow$

$$= \frac{1}{2}\int_V (\sigma_{ij} u_j)_{,i}\, dV$$

$$= \frac{1}{2}\int_V (\cancel{\sigma_{ij,i} u_j} + \sigma_{ij} u_{j,i})\, dV$$

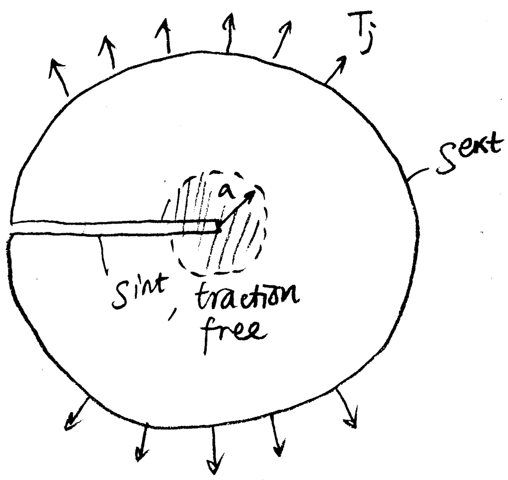
equilibrium condition $\sigma_{ij,i} = 0$ $\longrightarrow$

$$= \frac{1}{2}\int_V \sigma_{ij}\, \varepsilon_{ij}\, dV$$

$$= \int_V w(\underset{\sim}{x})\, dV(\underset{\sim}{x})$$

This confirms that the volume integral of $w(\underset{\sim}{x})$ is the total elastic strain energy U.

§2. Strain Energy of a Crack Tip. (or notch)



consider an elastic medium with a crack (or notch) that is loaded by T_j on external surfaces S^{ext} (away from the crack tip)

We expect the surface displacement u_j to be finite. hence the total stored elastic energy to be finite

$$U = \frac{1}{2}\int_{S^{ext}} T_j\, u_j\, dS \longrightarrow \text{finite}$$

Also notice that

$$U = \int_V w(\underset{\sim}{x})\, dV \quad \text{and} \quad w(\underset{\sim}{x}) \geq 0 \text{ everywhere}$$

i.e. there are no "cancellation"s of elastic energy.

This means the elastic energy in any sub volume of the medium also has to be finite.

$$U_{sub} = \int_{V_{sub}} w(\underset{\sim}{x})\, dV \longrightarrow \text{finite.}$$

Now imagine that the stress field at the crack tip is

$$\sigma_{ij} \sim \frac{1}{r^p}$$

$$\varepsilon_{ij} \sim \frac{1}{r^p}$$

then $$w \sim \frac{1}{r^{2p}}$$

The strain energy stored inside the circle of radius a around the crack tip

$$U_a = \int_0^a \int_0^{2\pi} \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, r \, dr \, d\theta$$

$$\sim \int_0^a r^{1-2p} \, dr$$

when $p<1$, $1-2p>-1$, $U_a \to$ finite

when $p>1$, $1-2p<-1$, $U_a \to$ infinite

when $p=1$, $U_a \sim \int_0^a r^{-1} dr = \ln a - \ln 0 \to$ infinite

Hence $U_a \to \infty$ for $p \geqslant 1$.

But from previous analysis, the strain energy stored in any sub volume has to be finite.

Therefore, for a crack (or notch) loaded from far away, the stress singularity $\sigma_{ij} \sim \frac{1}{r^p}$ must satisfy $p<1$.

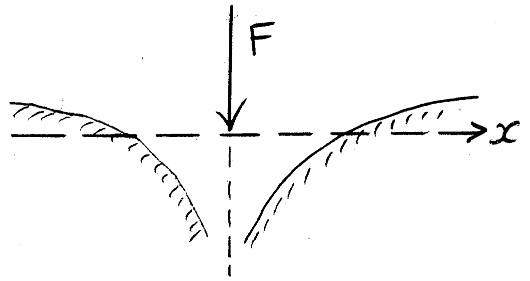
Thus the leading singularity of crack tip is

$\sigma_{ij} \sim \frac{1}{\sqrt{r}}$ (see notes "Wedge". p.3)

* However, $\sigma \sim \frac{1}{r}$ singularity will be allowed if this is required by the Boundary Condition, i.e. in order to satisfy B.C., we need to do work that goes to infinity. Part of this work goes to the singularity that has infinite energy.

(See notes "dislocations")

For example, recall our solution of point force on an elastic half-space



surface displacement

$\tilde{u}_y \sim \log|x|$

$\therefore \; \varepsilon_{xy} \sim \frac{1}{x}$ (for $y=0$)

$\varepsilon_{xy} \sim \frac{1}{r}$ in general

Notice that the displacement at origin (where F exerts) also diverges

$\tilde{u}_y(x=0) \to \infty$

$\therefore$ the work done by F is ∞.

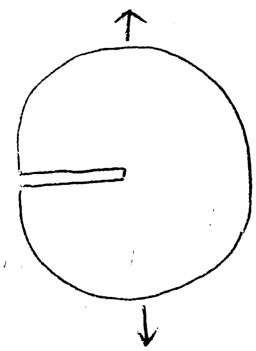
So the solution is self-consistent. (work, energy $\to \infty$)

In practice, the singularities (either $\frac{1}{\sqrt{r}}$, or $\frac{1}{r}$) do not exist. The crack cannot be infinitely sharp and the force cannot be infinitely concentrated.

* But this is not the reason why the crack tip singularity cannot be $\sigma \sim \frac{1}{r}$.

Knowing that $\sigma, \varepsilon \sim \frac{1}{\sqrt{r}}$ for a crack tip helps us to design more efficient numerical methods to handle cracks. (faster convergence).

$\lambda = \frac{1}{2}$.

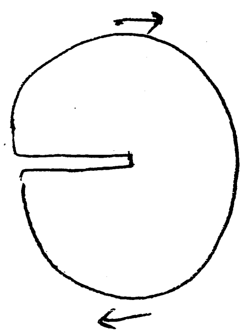


$$A_1 = A(\lambda-1)\sin(\lambda-1)\alpha$$
$$A_2 = -A(\lambda+1)\sin(\lambda+1)\alpha.$$

A is an arbitray constant.

Let $K_I = 3A\sqrt{\frac{\pi}{2}}$. — mode I stress intensity factor

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- even with } \theta$$
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \quad \text{--- even with } \theta$$
$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- odd with } \theta$$



K_{II} — mode II stress intensity factor.

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{-\frac{5}{4}\sin\left(\frac{\theta}{2}\right) + \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- odd with } \theta$$
$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{-\frac{3}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad \text{----- odd with } \theta$$
$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{\frac{1}{4}\cos\left(\frac{\theta}{2}\right) + \frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \quad \text{---- even with } \theta.$$

Stress intensity factor is linked to the driving force for crack tip extension.

$$J = \frac{1-\nu}{2\mu} K_I^2 \qquad \text{for mode I loading}$$

$$J = \frac{1-\nu}{2\mu} K_{II}^2 \qquad \text{for mode II loading}$$

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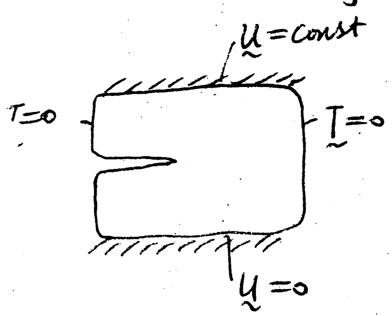
* calculation of J is a subject of micromechanics.



If loading condition is such that external force does no work when crack tip moves, then

$$J = -\frac{\Delta U^{tot}}{\Delta x}$$

ΔU^{tot} is the change of elastic energy as crack tip moves forward by Δx.