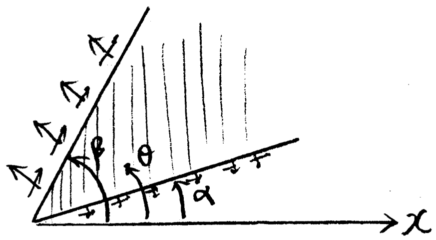
We can extend our study of the half-space problem by polar coordinates to wedges of arbitrary angle

§1. Example 1



wedge under uniform traction

σ_{rr}, $\sigma_{r\theta}$ independent of r (at least on the boundary)

trial solution

$$\phi = r^2 (A_1 \cos 2\theta + A_2 + A_3 \sin 2\theta + A_4 \theta)$$

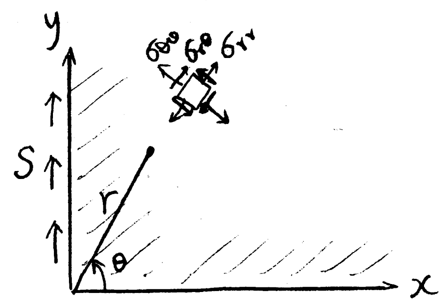
$$\sigma_{rr} = -2A_1 \cos 2\theta + 2A_2 - 2A_3 \sin 2\theta + 2A_4 \theta$$

$$\sigma_{r\theta} = 2A_1 \sin 2\theta + 0 - 2A_3 \cos 2\theta - A_4$$

$$\sigma_{\theta\theta} = 2A_1 \cos 2\theta + 2A_2 + 2A_3 \sin 2\theta + 2A_4 \theta$$

* We know at some point, the solutions
$r^2(\cos 2\theta \ln r - \theta \sin 2\theta)$, $r^2(\sin 2\theta \ln r + \theta \cos 2\theta)$
are needed. But they don't appear here. Why?
When are these solutions needed?
(See Lecture notes "Polar Coordinates II", p12.13.)

§2. Example 2



Uniform shear on a right-angle wedge

$\alpha = 0, \quad \beta = \pi/2$

B.C. $\quad \sigma_{r\theta} = \sigma_{\theta\theta} = 0 \qquad \theta = 0$

$\quad\quad \sigma_{r\theta} = S. \quad \sigma_{\theta\theta} = 0, \qquad \theta = \pi/2$

$\theta = 0$: $\quad \sigma_{r\theta} = -2A_3 - A_4 = 0$

$\quad\quad \sigma_{\theta\theta} = 2A_1 + 2A_2 = 0$

$\theta = \frac{\pi}{2}$: $\quad \sigma_{r\theta} = 2A_3 - A_4 = S$

$\quad\quad \sigma_{\theta\theta} = -2A_1 + 2A_2 + \pi A_4 = 0$

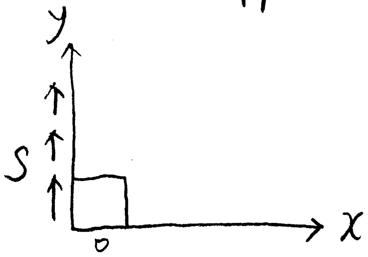
$\Rightarrow$

$$A_1 = -\frac{\pi S}{8}, \quad A_2 = \frac{\pi S}{8}$$

$$A_3 = \frac{S}{4}, \quad A_4 = -\frac{S}{2}$$

$$\phi = S\left(-\frac{\pi r^2 \cos 2\theta}{8} + \frac{\pi r^2}{8} + \frac{r^2 \sin 2\theta}{4} - \frac{r^2 \theta}{2}\right)$$

What happens at corner?

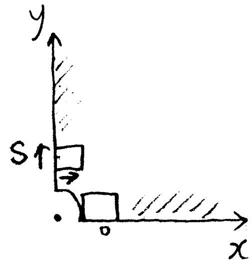


$\sigma_{xy} \neq \sigma_{yx}$?

$$\phi = S\left(-\frac{\pi}{8}(x^2-y^2) + \frac{\pi}{8}(x^2+y^2) + \frac{xy}{2} + \frac{x^2+y^2}{2}\arctan\frac{y}{x}\right)$$

answer:

$$\sigma_{xy} = \sigma_{yx} = -\frac{\partial^2\phi}{\partial x \partial y} = -\frac{S y^2}{x^2+y^2}$$

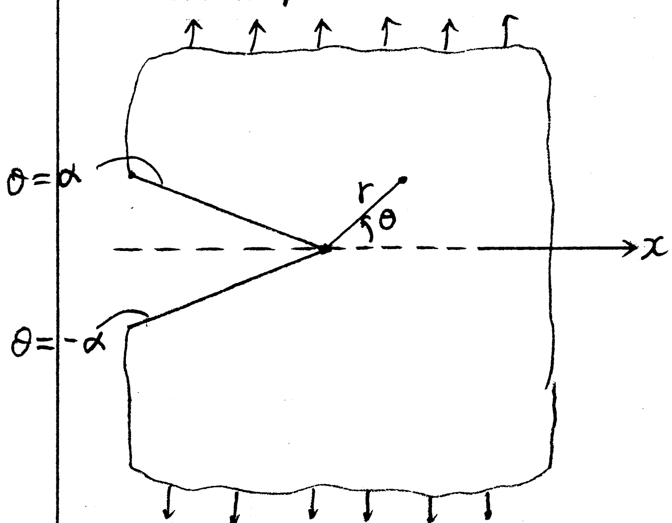


if $y=0$, $x>0$, $\sigma_{xy}=0$

if $x=0$, $y>0$, $\sigma_{xy}=-S$

if $x=0$, $y=0$, σ_{xy} undefined.

§3. Example 3   Notch & Crack = re-entrant corner



intuitively we expect stress field to be singular at the corner. ($r \to 0$)

At the same time, the notch surface ($\theta = \pm\alpha$) is traction-free.

i.e. $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$, $\theta = \pm\alpha$.

We would like to know "how singular" is the stress field at the corner.

→ look for stress functions that produce singular stress fields. William's solutions.

$$\phi = r^{n+2}\{A_1\cos(n+2)\theta + A_2\cos n\theta + A_3\sin(n+2)\theta + A_4\sin n\theta\}$$

Let $n = \lambda - 1$.   (* λ does NOT have to be an integer!)

$$\phi = r^{\lambda+1}\{A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta\}$$

B.C. $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$, $\theta = \pm\alpha$

$$\underbrace{\begin{bmatrix} (\lambda+1)\sin(\lambda+1)\alpha & (\lambda-1)\sin(\lambda-1)\alpha \\ (\lambda+1)\cos(\lambda+1)\alpha & (\lambda+1)\cos(\lambda-1)\alpha \end{bmatrix}}_{M_1} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

$$\underbrace{\begin{bmatrix} (\lambda+1)\cos(\lambda+1)\alpha & (\lambda-1)\cos(\lambda-1)\alpha \\ (\lambda+1)\sin(\lambda+1)\alpha & (\lambda+1)\sin(\lambda-1)\alpha \end{bmatrix}}_{M_2} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = 0$$

to have a non-trivial solution. (trivial solution is: $A_1 = A_2 = A_3 = A_4 = 0$)

we need:

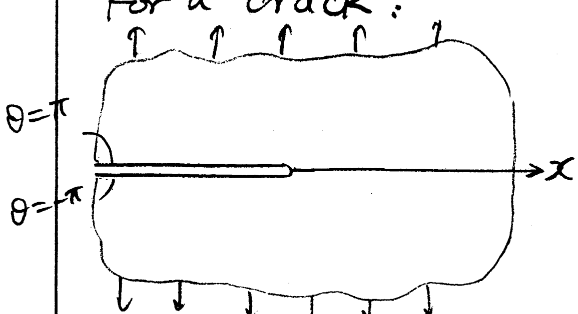
$\det(M_1) = 0 \longrightarrow \lambda \sin 2\alpha + \sin 2\lambda\alpha = 0$

or

$\det(M_2) = 0 \longrightarrow \lambda \sin 2\alpha - \sin 2\lambda\alpha = 0$

$\Big\}\ \lambda \sin 2\alpha \pm \sin 2\lambda\alpha = 0$

For a crack:



$\alpha = \pi. \rightarrow \sin 2\alpha = 0.$

$\sin 2\pi\lambda = 0$

$\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

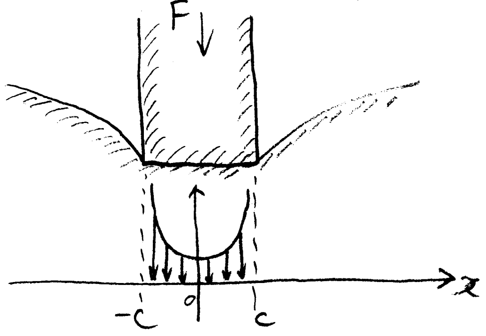
* Notice that $\lambda = 0$ is always a solution to $\lambda \sin 2\alpha \pm \sin 2\lambda\alpha = 0$. It corresponds to $\sigma \sim \frac{1}{r}$ and we will show later on that this singularity is too strong — unphysical.

* $\lambda = 1, \frac{3}{2}, \dots$ corresponds to non-singular stress fields.

Hence the leading <u>singularity at the crack ~~tip~~</u> is

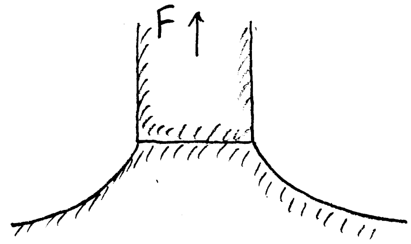
$\phi \sim \sqrt{r}$, $\begin{matrix}\sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta}\end{matrix} \sim \frac{1}{\sqrt{r}}$ square root singularity.

* §4. Crack tip singularity from flat punch solution



① From lecture note "Contact" P.5 the pressure arising from the flat punch:

$$p_y(x) = \frac{F}{\pi\sqrt{c^2-x^2}} = -\sigma_{yy}(x, y=0)$$

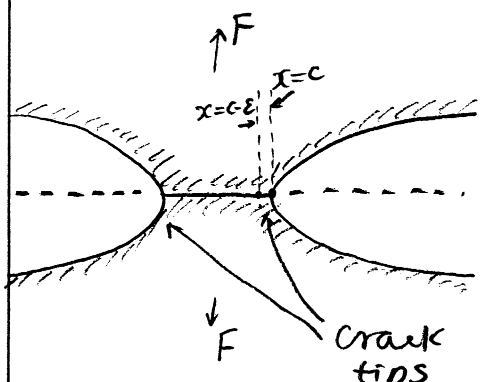


② How is this solution related to a crack?

Imagine we "glue" the flat punch to the half space and pull it.

The stress field will simply change its sign. i.e.

$$\sigma_{yy}(x, y=0) = \frac{F}{\pi\sqrt{c^2-x^2}}$$



③ Next imagine we glue together two half-spaces over the region $-c \le x \le c$, and then pull them apart with force F.

By symmetry, the y-displacement in the region $-c \le x \le c$ must be 0.

So the solution is identical to that in step ②.

Consider the point $x = c-\varepsilon$, $y=0$. $\varepsilon \ll c$.

$$\sigma_{yy}(x=c-\varepsilon, y=0) = \frac{F}{\pi\sqrt{c^2-(c^2-2c\varepsilon+\varepsilon^2)}} = \frac{F}{\pi\sqrt{2c\varepsilon-\varepsilon^2}}$$

$$\approx \frac{F}{\pi\sqrt{2c\varepsilon}} \propto \frac{1}{\sqrt{\varepsilon}}$$ square root singularity.