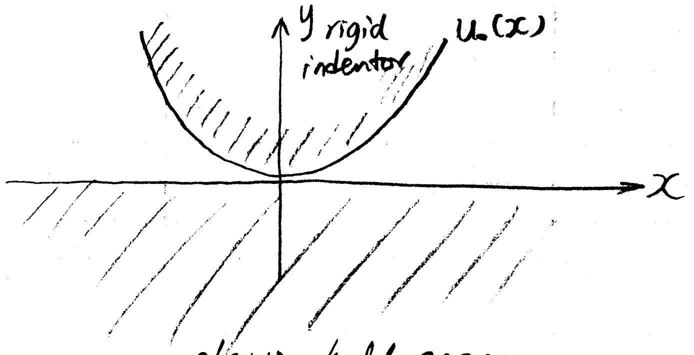
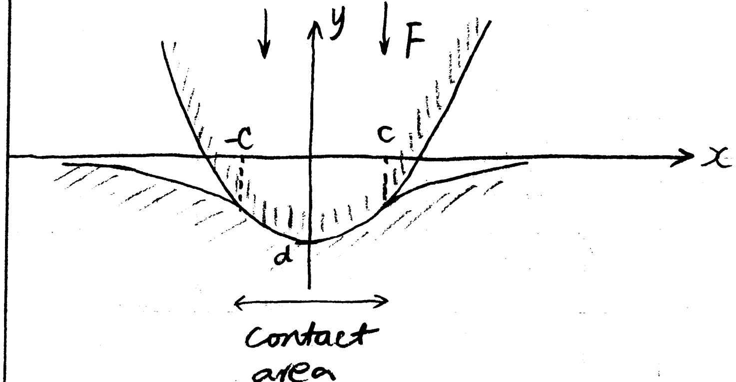
For simplicity, let us consider the problem of a rigid indentor in contact with an elastic half-space.



elastic half-space

$u_0(x)$: shape (profile) of the indentor



intuitively, the contact area $2c$ would increase with increasing indenting force F.

§1. Boundary condition of contact Problem

"contact area"
$-c \le x \le c$, $y=0$

$$\begin{cases} u_y = u_0(x) + d \\ \sigma_{yy} \le 0 \\ \sigma_{xy} = 0 \end{cases}$$

— the shape of elastic medium conforms to that of indentor,
* d is still unknown

frictionless compression

"gap" area
$|x| > c$, $y=0$

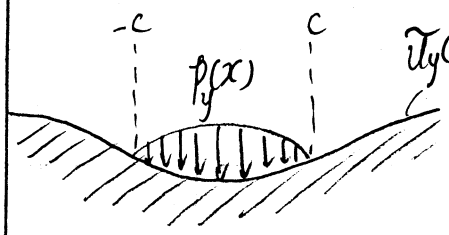
$$\begin{cases} u_y < u_0(x) + d \\ \sigma_{yy} = 0 \\ \sigma_{xy} = 0 \end{cases}$$

— existence of a gap

— free surface, zero tractions

- The contact area $2c$ and indentation depth d are <u>not</u> specified a priori
- The goal is to solve for c and d given the indentor shape $u_0(x)$ and indenting force F.

* usually very difficult.



Let $p_y(x)$ be the pressure exerted by the indentor onto the half-space.

$$\begin{cases} p_y(x) \geq 0 & -c \leq x \leq c \quad \text{in contact area} \\ p_y(x) = 0 & |x| > c \quad \text{outside} \end{cases}$$

Let $\tilde{u}_y(x)$ be the surface displacement in y direction

$$\begin{cases} \tilde{u}_y(x) = u_0(x) + d & -c \leq x \leq c \\ \tilde{u}_y(x) < u_0(x) + d & |x| > c. \end{cases}$$

From Lecture Notes on "Half-space", we have

$$\tilde{u}_y(x) = \frac{\kappa+1}{4\pi\mu} \int_{-\infty}^{+\infty} p_y(x') \cdot \log|x - x'| \cdot dx'$$

$$\tilde{u}_y(x) = \frac{\kappa+1}{4\pi\mu} \int_{-c}^{c} p_y(x') \cdot \log|x - x'| \cdot dx'$$

* so we constrain $|x'| \leq c$

If we also constrain $|x| \leq c$, then

$$u_0(x) + d = \frac{\kappa+1}{4\pi\mu} \int_{-c}^{c} p_y(x') \cdot \log|x - x'| \cdot dx'$$

To get rid of the unknown constant d, $\frac{d}{dx}$ on both sides.

$$\frac{d u_0(x)}{dx} = \frac{\kappa+1}{4\pi\mu} \int_{-c}^{c} \frac{p_y(x')}{x - x'} dx' \quad (\text{for } |x| \leq c)$$

$$* \ \frac{d}{dx} \log|x - x'| = \frac{1}{x - x'}$$

Our task is to invert this equation to find $p_y(x')$, after that we will find $\tilde{u}_y(x)$ everywhere (outside the contact area)

§2.

To invert an integral equation is difficult, and we will need some special mathematical tools.

Now that both x and x' are limited to the domain $[c, c]$, It is natural to introduce angular variables ϕ, θ such that $x = c\cos\phi$ and $x' = c\cos\theta$ and $\phi, \theta \in [0, \pi]$

$$dx = -c\sin\phi\, d\phi \quad , \quad dx' = -c\sin\theta\, d\theta$$

$u_0(x)$ Can be rewritten as a function of ϕ, $u_0(\phi)$

$p_y(x')$ can be rewritten as a function of θ, $p_y(\theta)$

$$-\frac{1}{c\sin\phi}\cdot\frac{du_0(\phi)}{d\phi} = \frac{\kappa+1}{4\pi\mu}\int_\pi^0 \frac{p_y(\theta)}{c\,(\cos\phi-\cos\theta)}(-c\sin\theta)\cdot d\theta$$

$$-\frac{1}{c\sin\phi}\frac{du_0(\phi)}{d\phi} = \frac{\kappa+1}{4\pi\mu}\int_0^\pi \frac{p_y(\theta)\sin\theta\, d\theta}{\cos\phi-\cos\theta}$$

Now we need a special mathematical formula.

$$\boxed{-\frac{\sin n\phi}{\sin\phi} = \frac{1}{\pi}\int_0^\pi \frac{\cos n\theta\, d\theta}{\cos\phi-\cos\theta}} \qquad n = 0, 1, 2, \cdots$$

* This formula will be proved at the end of this lecture note.

In order to take advantage of this formula, we need to expand

$$\frac{du_0(\phi)}{d\phi} = \sum_{n=1}^{\infty} w_n \sin n\phi$$ (the n=0 term makes zero contribution)

$$p_y(\theta)\sin\theta = \sum_{n=0}^{\infty} p_n \cos n\theta$$

i.e. $$p_y(\theta) = \sum_{n=0}^{\infty}\frac{p_n\cos n\theta}{\sin\theta}$$

$$-\frac{1}{c}\sum_{n=1}^{\infty}\frac{w_n \sin n\phi}{\sin\phi} = \frac{\kappa+1}{4\mu}\cdot\frac{1}{\pi}\sum_{n=0}^{\infty}\int_0^{\pi}\frac{p_n \cos(n\theta)\,d\theta}{\cos\phi-\cos\theta}$$

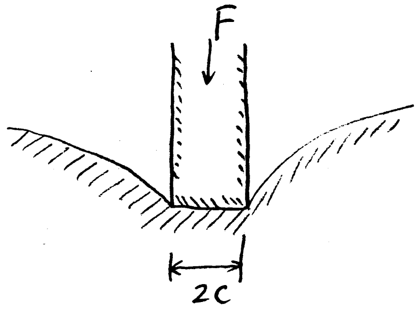
$$= -\frac{\kappa+1}{4\mu}\sum_{n=1}^{\infty}\frac{p_n \sin n\phi}{\sin\phi}$$

$\therefore \quad w_n = \frac{\kappa+1}{4\mu}\cdot p_n \cdot c \quad$ for $n=1,2,3,\cdots$

$$\boxed{p_n = \frac{4\mu}{(\kappa+1)\cdot c}\cdot w_n} \qquad n=1,2,3,\cdots$$

p_0 is arbitrary

§3. Flat Punch



The flat punch problem is easier than a general contact problem because 2C is already known.

Flat Punch $\Leftrightarrow$ $u_0 = \text{const} \quad -c \le x \le c.$

$$\frac{d u_0(\phi)}{d\phi} = 0$$

i.e. when we expand $\frac{d u_0(\phi)}{d\phi} = \sum_{n=1}^{\infty} w_n \sin n\phi$

then $\quad w_n = 0 \quad$ for all $n=1,2,3,\cdots$

Therefore $\quad p_n = 0 \quad$ for all $n=1,2,3,\cdots$

The only non-zero component is p_0

$$p_y(\theta) = \sum_{n=0}^{\infty}\frac{p_n \cos n\theta}{\sin\theta} = \frac{p_0}{\sin\theta}$$

Recall that $x' = c\cos\theta$, hence $\sin\theta = \sqrt{1-\left(\frac{x'}{c}\right)^2}$

$$p_y(x') = \frac{p_0}{\sqrt{1-\left(\frac{x'}{c}\right)^2}}$$

How to determine the unknown coefficient p_0 and $2c$?

Intuitively, we expect the contact pressure $p_y(x)$ to be non-singular everywhere (unlike the flat punch).

This is because a singular solution has very high energy. The system should automatically remove the singularity by adjusting the contact area $2c$.

Notice that $\frac{1}{\sin\theta} = \frac{1}{\sqrt{1-(\frac{x'}{c})^2}}$ is singular at $x' = \pm c$

Hence the numerator $p_0 + p_2 \cos 2\theta$ must contain a multiplication factor $\sin\theta$ to cancel the denominator.

This is the case when $p_0 = -p_2$

$$p_0 + p_2 \cos 2\theta = p_0 (1 - \cos 2\theta) = 2 p_0 \sin^2\theta$$

$$p(\theta) = \frac{p_0 + p_2 \cos 2\theta}{\sin\theta} = 2 p_0 \sin\theta$$

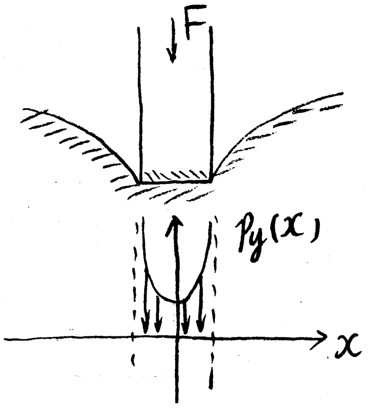
$$p_y(x') = 2 p_0 \sqrt{1-(\tfrac{x'}{c})^2}$$ — the inverse of flat punch problem

The total indenting force

$$F = \int_{-c}^{c} p_y(x')\, dx' = p_0 \pi c$$

$$p_0 = \frac{F}{\pi c}$$

$$p_y(x') = \frac{2F}{\pi c} \sqrt{1-\left(\frac{x'}{c}\right)^2} = \frac{2F}{\pi c^2} \sqrt{c^2 - x'^2}$$



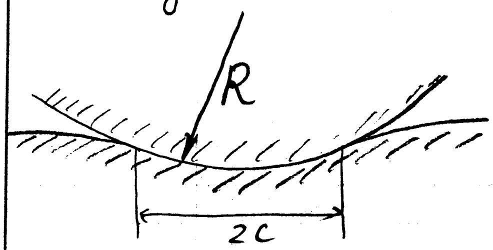
Total indenting force

$$F = \int_{-c}^{c} p_y(x')\,dx' = p_0 \pi c$$

$$\therefore \quad p_y(x') = -\frac{F}{\pi c \sqrt{1-\left(\frac{x'}{c}\right)^2}}$$

$$\boxed{p_y(x') = -\frac{F}{\pi\sqrt{c^2 - x'^2}}}$$

§4. Cylindrical Punch (Hertz contact problem)



Let R be the radius of curvature of the rigid indentor.

We can approximately write

$$u_0(x) = \frac{x^2}{2R}$$

Since $x = c\cos\phi$, $u_0(\phi) = \frac{c^2}{2R}\cos^2\phi$

We shall rewrite $\frac{du_0(\phi)}{d\phi}$ into the form of $\sum_{n=1}^{\infty} w_n \sin n\phi$.

$$u_0(\phi) = \frac{c^2}{4R}(\cos 2\phi - 1)$$

$$\frac{du_0(\phi)}{d\phi} = -\frac{c^2}{2R}\sin 2\phi = \sum_{n=1}^{\infty} w_n \sin n\phi$$

$$\therefore \quad w_2 = -\frac{c^2}{2R} \qquad w_1 = w_3 = w_4 = \cdots = 0$$

$$p_2 = -\frac{4\mu}{(\kappa+1)\cdot c}\cdot\frac{c^2}{2R} = -\frac{2\mu c}{(\kappa+1)\cdot R} \qquad p_1 = p_3 = p_4 = \cdots = 0$$

p_0 is still unknown

$$p(\theta) = \frac{p_0}{\sin\theta} + \frac{p_2\cos 2\theta}{\sin\theta} = \frac{p_0 + p_2\cos 2\theta}{\sin\theta}$$

Recall $p_2 = -\dfrac{2\mu c}{(\kappa+1)\cdot R}$, $p_0 = -p_2$, $p_0 = \dfrac{F}{\pi c}$

$$\therefore \quad -\frac{2\mu c}{(\kappa+1)R} = -\frac{F}{\pi c}$$

$$F = \frac{2\pi\mu}{\kappa+1}\cdot\frac{c^2}{R}, \qquad \boxed{c = \sqrt{\frac{F(\kappa+1)R}{2\pi\mu}}}$$

* contact area $\sim \sqrt{F}$

Q: What is the indentation depth d ?

Q: What if the indentor profile is $u_0(x) = x^4$?

*§5. Hilbert transform. (David Hilbert 1862–1943 German mathematician)

The Hilbert transform of a function $s(t)$ is defined as

$$r(t) = \mathcal{H}\{s(t)\} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{s(t')}{t-t'}\,dt'$$

Hilbert transform is its own inverse (with a minus sign)

$$s(t) = -\mathcal{H}\{r(t)\} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{r(t')}{t-t'}\,dt'$$

Some Hilbert transform pairs

Signal $s(t)$	Hilbert transform $\mathcal{H}\{s(t)\}$
$\delta(t)$	$\frac{1}{\pi t}$
$\sin t$	$-\cos t$
$\cos t$	$\sin t$
$e^{i\omega t}$	$-i\,\mathrm{sgn}(\omega)\,e^{i\omega t}$

Recall that $\dfrac{d\tilde{u}_y(x)}{dx} = \dfrac{\kappa+1}{4\pi\mu}\displaystyle\int_{-\infty}^{+\infty} \frac{p_y(x')}{x-x'}\,dx'$

Hence $\dfrac{d\tilde{u}_y(x)}{dx}$ is the Hilbert transform of $\dfrac{\kappa+1}{4\mu}\,p_y(x)$

Applying the reverse Hilbert transform

$$\frac{\kappa+1}{4\mu}\,p_y(x) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\frac{d\tilde{u}_y(x')}{dx'}}{x-x'}\,dx'$$

Unfortunately, this does not help us finding $p_y(x)$ — which is zero if $|x|>c$,

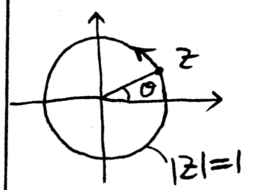
$\dfrac{d\tilde{u}_y(x)}{dx}$ is non-zero and unknown if $|x|>c$.

*§6. Proof of the formula

$$-\frac{\sin n\phi}{\sin\phi} = \frac{1}{\pi}\int_0^{\pi}\frac{\cos n\theta}{\cos\phi-\cos\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos n\theta}{\cos\phi-\cos\theta}\,d\theta$$

by complex analysis.

Define complex variable $z \equiv e^{i\theta}$, $\quad \frac{1}{z} = e^{-i\theta}$



$$z+\frac{1}{z} = 2\cos\theta, \qquad dz = e^{i\theta}\cdot i\cdot d\theta = iz\,d\theta$$

$$z^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

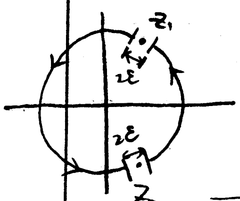
$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos n\theta + i\sin n\theta}{\cos\phi-\cos\theta}\,d\theta = \frac{1}{2\pi}\oint_{|z|=1}\frac{z^n\cdot\frac{dz}{iz}}{\cos\phi-\frac{1}{2}(z+\frac{1}{z})}$$

Define complex variable

$$z_1 = \cos\phi + i\sin\phi \qquad = e^{i\phi}$$

$$z_2 = z_1^* = \cos\phi - i\sin\phi = e^{-i\phi}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos n\theta+i\sin n\theta}{\cos\phi-\cos\theta}\,d\theta = \frac{1}{\pi}\oint_{|z|=1}\frac{i z^n\,dz}{z^2-2\cos\phi\cdot z+1}$$



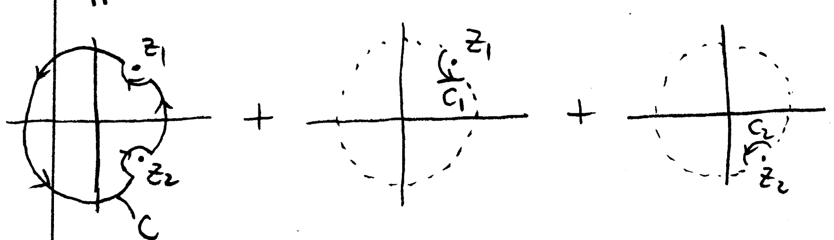
original integration contour

— Cauchy Principal value

$$= \frac{1}{\pi}\oint_{|z|=1}\frac{iz^n\,dz}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{\pi}\left\{\oint_C + \int_{C_1} + \int_{C_2}\right\}\frac{iz^n\,dz}{(z-z_1)(z-z_2)}$$

||



The contour integral $\oint_C = 0$ because the integrand does not contain any singularity inside C.

On the other hand, for the integral $\int_{C_1}$, we can introduce a new angular variable α, such that

$$z = z_1 + \varepsilon e^{i\alpha} \quad , \quad \varepsilon \to 0.$$

$$\int_{C_1} \frac{i z^n}{(z-z_1)(z-z_2)} dz = \int_{C_1} \frac{i z_1^n \varepsilon e^{i\alpha} i d\alpha}{\varepsilon e^{i\alpha}(z_1 - z_2)}$$

$$= -\frac{z_1^n}{z_1 - z_2} \int_{C_1} d\alpha = -\frac{\pi z_1^n}{2i\sin\phi}$$

Similarly,

$$\int_{C_2} \frac{i z^n}{(z-z_1)(z-z_2)} dz = \frac{\pi z_2^n}{2i\sin\phi}$$

$$\therefore \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos n\theta + i\sin n\theta}{\cos\phi - \cos\theta} d\theta = \frac{1}{\pi}\left\{-\frac{\pi z_1^n}{2i\sin\phi} + \frac{\pi z_2^n}{2i\sin\phi}\right\}$$

$$= -\frac{\sin n\phi}{\sin\phi} \qquad \left(z_1^n - z_2^n = 2i\sin n\phi\right)$$

$$\therefore \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos n\theta}{\cos\phi - \cos\theta} d\theta = -\frac{\sin n\phi}{\sin\phi} \qquad n = 0, 1, 2, \cdots$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\sin n\theta}{\cos\phi - \cos\theta} d\theta = 0$$