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Properties of the Eshelby Tensor and Existence of the Equivalent Ellipsoidal Inclusion Solution

D. M. Barnett and Wei Cai*
Department of Materials Science and Engineering,
Department of Mechanical Engineering, Stanford University,
Stanford, CA 94305

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Abstract

We show that the Eshelby tensor, $\mathcal{S}^E$, when written in the 6×6 matrix (Voigt) form, is weakly positive definite, i.e. it can be written as a product of two positive definite matrices. All eigenvalues of $\mathcal{S}^E$ are real and lie between 0 and 1, for an arbitrary anisotropic elastic medium with a positive definite elastic stiffness tensor $\mathbf{C}$. The weakly positive definiteness property leads to a direct proof of the existence of Eshelby's equivalent inclusion solution for a "transformed" ellipsoidal inhomogeneity in an infinite elastic medium.

1 Introduction

The stress field generated by inclusions and inhomogenities in an elastic matrix is a fundamental problem in micromechanics and has important consequences in the strength of multi-component alloys [1]. Both inclusions and inhomgeneities are associated with a uniform transformation strain (i.e. eigenstrain), which is the stress-free strain if they were removed from the surrounding elastic matrix. The inclusions have the same elastic stiffness tensor C_{ijkl} as the elastic matrix, while the inhomogenities have a different elastic stiffness tensor C'_{ijkl}. Eshelby showed that the stress and strain fields inside a transformed ellipsoidal inclusion are uniform [2]. The Eshelby

*Email: caiwei@stanford.edu

tensor $\mathcal{S}^E$ relates the eigenstrain e^* to the "constrained" strain e^c, which is the actual strain of the inclusion when embedded in the matrix,

$$e^c_{ij} = \mathcal{S}^E_{ijkl} e^*_{kl} \ . \tag{1}$$

Given the Eshelby tensor, the stress field inside the inclusion can be easily obtained,

$$\sigma^{\mathrm{I}}_{ij} = C_{ijkl}(e^c_{kl} - e^*_{kl}) = C_{ijkl}(\mathcal{S}^E_{klmn} e^*_{mn} - e^*_{kl}) \ . \tag{2}$$

The expression for the Eshelby tensor has been obtained for an arbitrary ellipsoidal shape of the inclusion (specified by semiaxes a, b, c). In an arbitrarily anisotropic elastic medium the expression involves a double integral over a unit sphere [3, 4, 5]. In an isotropic elastic medium, explicit expressions for the Eshelby tensor have been obtained. Based on the Eshelby tensor, it is also possible to construct the solution for) a transformed inhomogeneity by finding an "equivalent inclusion" (see below).

Given the importance of the Eshelby tensor $\mathcal{S}^E$, it is somewhat surprising that its mathematical properties have not been well explored. Part of the difficulties originates from the fact that $\mathcal{S}^E$ lacks major symmetry, i.e. $\mathcal{S}^E_{ijkl} \neq \mathcal{S}^E_{klij}$ (unlike the elastic stiffness tensor, $\mathbf{C}$, for which $C_{ijkl} = C_{klij}$). It is also known that $\mathcal{S}^E$ need not be positive definite, i.e. there may exist some (real symmetric) strain e_{ij} such that $e_{ij}\mathcal{S}^E_{ijkl}e_{kl} < 0$. In this paper, we show that despite the lack of the major symmetry and positive definiteness, the Eshelby tensor still possesses a number of interesting properties, especially when it is written in the "reduced" Voigt form as a 6×6 matrix (see Appendix A for the Voigt notation). The properties of the Eshelby tensor to be shown in this paper are as follows.

P1. $\mathcal{S}^E$ is weakly positive definite (WPD), which for the purpose of the present work may be defined as a matrix obtained from a (real symmetric) positive definite matrix by a similarity transformation [6]. It has been shown that a matrix is weakly positive definite if and only if it can be written as the product of two positive definite matrices. Consequently, the determinant of a weakly positive definite matrix is positive, i.e. $\det(\mathcal{S}^E) > 0$.

P2. All 6 eigenvalues of $\mathcal{S}^E$ are real and lie between 0 and 1. This complements the known property) [7] that the sum of all 6 eigenvalues of $\mathcal{S}^E$ is 3.)

The stress field of a "transformed" ellipsoidal inhomogeneity in an infinite matrix [8, 9, 10, 11] can be obtained using the "equivalent inclusion" concept of Eshelby. The inhomogeneity with eigenstrain $e^{*'}_{ij}$ and elastic constants (C'_{ijkl}) is replaced by an inclusion with an (unknown) eigenstrain e^*_{ij}, such that the stress and total strain in the inhomogeneity are equal to those in the equivalent inclusion. The eigenstrain e^*_{ij} of the equivalent inclusion is to be determined by solving the following equation,

$$[(C'_{ijkl} - C_{ijkl})\mathcal{S}^E_{klmn} + C_{ijmn}]e^*_{mn} = C'_{ijkl}e^{*'}_{kl}\,. \tag{3}$$

Using the "reduced" Voigt notation Eq. (3) may be written as

$$[(C'_{IK} - C_{IK})\mathcal{S}^E_{KM} + C_{IM}]e^*_{M} = C'_{IM}e^{*'}_{M}\,, \tag{4}$$

where all upper case indices range from 1 to 6. Solutions of Eq. (4) require that the 6×6 matrix $\mathbf{A} \equiv [(\mathbf{C}' - \mathbf{C})\mathcal{S}^E + \mathbf{C}]$ be non-singular. Eshelby did not discuss whether this matrix would always be nonsingular, or whether there exists some set of conditions that preclude obtaining such a solution. In a previous paper [12] we gave a proof that the matrix $\mathbf{A}$ is always non-singular, and hence invertible. That proof considers a family of matrices $\mathbf{A}(\epsilon)$ and makes use of the fact that the energy of an infinite elastic medium containing a finite-sized inclusion is inherently non-negative and bounded. More recently, another proof was constructed by Freidin and Kucher using a Fourier analysis approach [13] that does not involve the indirect "blow-up" argument in [12]. The Freidin-Kucher work required only that the elastic stiffness matrices $\mathbf{C}$ and $\mathbf{C}'$) satisfy the strong ellipticity condition. Our present work requires that $\mathbf{C}$ and $\mathbf{C}'$) be positive definite (strong convexity) so that uniqueness of elastic solutions is guaranteed, which we think is preferable.

Here we show that using Property P1 of the Eshelby tensor, a more direct and simpler proof of the invertibility of matrix $\mathbf{A}$ can be constructed. In particular, we show that $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{A}$ is a symmetric positive definite matrix, so that $\det[(\mathcal{S}^E)^{\mathrm{T}}\mathbf{A}] > 0$, where the superscript $^{\mathrm{T}}$ indicates transposition. Given that (P1) $\det(\mathcal{S}^E) > 0$, we have $\det(\mathbf{A}) > 0$, and hence $\mathbf{A}$ is always invertible.

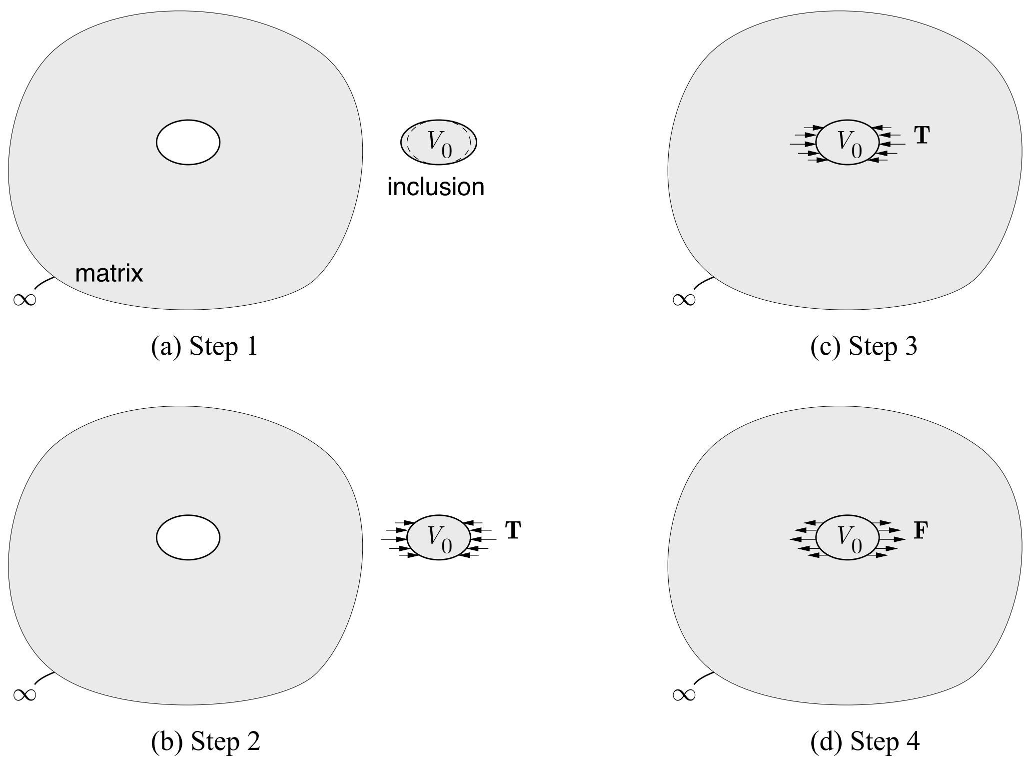
The rest of paper is organized as follows. First, we briefly review Eshelby's method of the equivalent inclusion and give the analytic (integral) expressions of the Eshelby tensor in Section 2. In Section 3, we prove that (P1) the Voigt matrix representation of the Eshelby tensor $\mathcal{S}^E$ is weakly positive definite and (P2) all of its eigenvalues lie between 0 and 1. In Section 4, we provide numerical case studies that confirm these two properties. In Section 5 we provide a direct proof of the invertibility of matrix $\mathbf{A}$. Some conclusive remarks are given in Section 6.)

2 Eshelby's Method

2.1 Eshelby's inclusion problem

Eshelby obtained the stress, strain and elastic energy associated with a transformed ellipsoidal inclusion with a uniform eigenstrain e^*_{ij} in an infinite matrix using a thought experiment that consists of four steps as shown in Figure 1.

Figure 1: Steps to solve Eshelby's inclusion problem.



In Step 1 (Figure 1a), both matrix and the ellipsoidal inclusion are unstressed. The inclusion is taken out of the matrix and is given the uniform stress-free transformation strain e^*_{ij}. In Step 2 (Figure 1b), tractions $-\sigma^*_{ij} n_i$ are applied to the inclusion boundary to "elastically" back-strain it to its original shape and size, while the matrix remains undeformed. Here $\sigma^*_{ij} = C_{ijkl} e^*_{kl}$ and n_i is the outward facing unit normal of the inclusion). In Step 3 (Figure 1c), the inclusion is inserted back into the ellipsoidal "hole" in the matrix, and the discontinuity in traction across the matrix-inhomogeneity interface is recognized as being a layer of body force acting on the interface between the matrix and inclusion

$$T_j = \sigma^*_{ij} n_i \mathrm{d}S \tag{5}$$

distributed over S_0, the interface, where $\mathrm{d}S$ is a differential surface element of the interface. In Step 4 (Figure 1d), this body force is relaxed to zero, which is equivalent to adding on a solution corresponding to applying an opposite body force $F_j = -T_j$

everywhere on S_0. The strain produced by Step 4 is the "constrained" strain, e^c_{ij}, which is the total (final) strain of the inclusion. The displacement field u^c_i produced by Step 4 can be written in terms of the Green's function $G_{ij}(\mathbf{x})$ of the elastic medium,

$$u^c_i(\mathbf{x}) = \int_{S_0} \sigma^*_{lk}\, n_k(\mathbf{x}')\, G_{il}(\mathbf{x}-\mathbf{x}')\, \mathrm{d}S(\mathbf{x}')\,. \tag{6}$$

Therefore, the constrained strain field $e^c_{ij} = \mathcal{S}^E_{ijkl} e^*_{kl}$ can be written as

$$e^c_{ij}(\mathbf{x}) = \frac{1}{2}\int_{S_0} \sigma^*_{lk}\, n_k(\mathbf{x}')\ \left[G_{il,j}(\mathbf{x}-\mathbf{x}') + G_{jl,i}(\mathbf{x}-\mathbf{x}')\right]\, \mathrm{d}S(\mathbf{x}')\,. \tag{7}$$

It can be shown that for field points inside the inclusion ($\mathbf{x} \in V_0$), the constrained field e^c_{ij} and hence the Eshelby tensor $\mathcal{S}^E$ are uniform. In particular, the Eshelby tensor can be written in the following analytic form [3, 4].

$$\mathcal{S}^E_{ijmn} = \mathcal{D}_{ijkl}\, C_{klmn}\,, \tag{8}$$

$$\mathcal{D}_{ijkl} = \frac{abc}{8\pi} \oiint_{|\mathbf{z}|=1} \frac{z_i z_l\,(zz)^{-1}_{jk} + z_j z_l\,(zz)^{-1}_{ik}}{\left(a^2 z_1^2 + b^2 z_2^2 + c^2 z_3^2\right)^{3/2}}\, \mathrm{d}S\,, \tag{9}$$

where a, b, c are semi-axes of the ellipsoid, $\mathbf{z} = (z_1, z_2, z_3)$ is a unit vector along a radial direction for the unit sphere and the integral is over the unit sphere. Matrix $(zz)^{-1}$ is the inverse of the acoustic matrix (zz), where $(zz)_{ij} = z_m C_{imjn} z_n = (zz)_{ji}$. Because $C_{klmn} = C_{lkmn}$, $\mathcal{S}^E_{ijmn}$ can also be written as

$$\mathcal{S}^E_{ijmn} = P_{ijkl}\, C_{klmn} \tag{10}$$

where

$$\begin{aligned} P_{ijkl} &= \frac{1}{2}\left(\mathcal{D}_{ijkl} + \mathcal{D}_{ijlk}\right) \\ &= \frac{abc}{16\pi} \oiint_{|\mathbf{z}|=1} \frac{z_i z_l\,(zz)^{-1}_{jk} + z_j z_l\,(zz)^{-1}_{ik} + z_i z_k\,(zz)^{-1}_{jl} + z_j z_k\,(zz)^{-1}_{il}}{\left(a^2 z_1^2 + b^2 z_2^2 + c^2 z_3^2\right)^{3/2}}\, \mathrm{d}S \end{aligned} \tag{11}$$

is called the Hill polarization tensor. It has been shown that the Hill tensor $\mathbf{P}$ satisfies both the minor and major symmetries of the elastic stiffness tensor [14], i.e., $P_{ijkl} = P_{jikl} = P_{ijkl} = P_{klij}$. For example, the major symmetry can be shown as follows.

$$\begin{aligned} P_{klij} &= \frac{abc}{16\pi} \oiint_{|\mathbf{z}|=1} \frac{z_k z_j\,(zz)^{-1}_{li} + z_l z_j\,(zz)^{-1}_{ki} + z_k z_i\,(zz)^{-1}_{lj} + z_l z_i\,(zz)^{-1}_{kj}}{\left(a^2 z_1^2 + b^2 z_2^2 + c^2 z_3^2\right)^{3/2}}\, \mathrm{d}S \\ &= P_{ijkl} \end{aligned} \tag{12}$$

In Voigt notation (see Appendix A), $\mathcal{S}^E$, $\mathbf{P}$ and $\mathbf{C}$ can be written as 6×6 matrices, with

$$\mathcal{S}^E_{IM} = P_{IK}\, C_{KM} \tag{13}$$

where both P_{IK} and C_{KM} are real symmetric matrices.

2.2 Eshelby's transformed inhomogeneity problem

Eshelby solved the "transformed" inhomogeneity problem by noting that an ellipsoidal inhomogeneity (elastic stiffnesses $\mathbf{C}'$ and eigenstrain $e^{*\prime}$) can be replaced by an "equivalent" inclusion with elastic stiffness $\mathbf{C}$ and eigenstrain e^*. We can apply Eshelby's procedure in §2.1 to find the total strain and elastic strain inside the inhomogeneity and the equivalent inclusion:

$$e^{\text{tot}}_K = e^c_K \qquad \text{(total strain)} \tag{14}$$

$$e^{\text{I,el}}_K = e^c_K - e^*_K \qquad \text{(elastic strain for the inclusion)} \tag{15}$$

$$e^{\text{I,el}}_K = e^c_K - e^{*\prime}_K \qquad \text{(elastic strain for the inhomogeneity).} \tag{16}$$

If the total strains are the same, the inclusion may be replaced by the inhomogeneity with displacement continuity maintained at the inhomogeneity/matrix interface. Traction continuity is maintained with this switch if the stress in both inclusion and inhomogeneity are equal, i.e. if

$$C'_{IK}(e^c_K - e^{*\prime}_K) = C_{IK}(e^c_K - e^*_K). \tag{17}$$

This is the equation from which Eshelby's equivalent inclusion method stems and is equivalent to Eq. (3) and Eq. (4). Eshelby assumed that an equivalent inclusion (e^*) can always be found for each transformed inhomogeneity ($e^{*\prime}$, C'_{IK}). This requires $\mathbf{A} \equiv (\mathbf{C}' - \mathbf{C})\mathcal{S}^E + \mathbf{C}$ to be invertible, which is the property we aim to prove in this paper. For convenience, we rewrite matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{C}'\mathcal{S}^E + \mathbf{C}(\mathbf{I} - \mathcal{S}^E) \tag{18}$$

where $\mathbf{I}$ is the 6×6 identity matrix.

3 Proving the Range of Eigenvalues of $\mathcal{S}^E$

In this section, we prove two properties of the Eshelby tensor: (P1) $\mathcal{S}^E$ is weakly positive definite, and (P2) all of its eigenvalues lie between 0 and 1. We start with a brief review of the definition of weakly positive definite matrices.

3.1 Weakly positive definite matrices

All matrices in this paper are **real** matrices. Therefore, if such a matrix is symmetric, then it is also Hermitian, and all of its eigenvalues are real. The Voigt matrix expressions of the elastic stiffness tensor $\mathbf{C}$ and the Hill tensor $\mathbf{P}$ are real symmetric matrices, and so their eigenvalues are real. A real symmetric matrix $\mathbf{G}$ is positive definite (PD) if for any non-zero real vector $\mathbf{x}$,

$$\mathbf{x}^{\mathrm{T}}\,\mathbf{G}\,\mathbf{x} > 0 \tag{19}$$

All eigenvalues of a real symmetric PD matrix are positive. For example, the elastic stiffness tensor $\mathbf{C}$ is PD. We will show that the Hill tensor $\mathbf{P}$ is also PD.

Wigner [6] defined a matrix $\mathbf{W}$ to be weakly positive if it can be written as $\mathbf{W} = \mathbf{X}\,\mathbf{G}\,\mathbf{X}^{-1}$ where $\mathbf{G}$ is a Hermitian positive definite matrix and $\mathbf{X}$ is a non-singular matrix. In other words, $\mathbf{W}$ is related to a Hermitian positive definite matrix $\mathbf{G}$ through a similarity transformation. As a result, $\mathbf{W}$ has the same eigenvalues as $\mathbf{G}$, and hence all eigenvalues of $\mathbf{W}$ are real and positive. In this paper, we shall call such a matrix $\mathbf{W}$ a weakly positive definite (WPD) matrix. Wigner [6] has shown that a matrix is WPD if and only if it can be written as the product of two (Hermitian) PD matrices,

$$\mathbf{W} = \mathbf{G}_1\,\mathbf{G}_2\,. \tag{20}$$

Given that $\mathcal{S}^E = \mathbf{P}\,\mathbf{C}$, and that $\mathbf{C}$ is PD, then $\mathcal{S}^E$ is WPD if we can show $\mathbf{P}$ is also PD. Note that a PD matrix is also WPD, but the converse does not hold; the former is easily proved by noting that a PD matrix $\mathbf{G}$ may be written as $\mathbf{G}\,\mathbf{I}$.

Nilssen [15] generalized the definition of WPD matrices to those matrices that are related to a non-symmetric real PD matrix through a similarity transformation. If matrix $\mathbf{G}$ is PD but not symmetric, then the eigenvalues of $\mathbf{G}$ (and $\mathbf{W}$) are not necessarily real, and we can only say that the real part of their eigenvalues are positive [15, 16]. However, in this paper, we do not need to use Nilssen's generalized definition of WPD matrices. Given that both $\mathbf{P}$ and $\mathbf{C}$ are real symmetric matrices, we will show that $\mathcal{S}^E$ is WPD according to the stricter definition of Wigner.

3.2 $\mathcal{S}^E$ is weakly positive definite

We have previously proved [12] that $\mathbf{K} \equiv \mathbf{C}\,\mathcal{S}^E$ is a positive definite (PD) matrix, meaning that

$$e^*_K C_{KI} \mathcal{S}_{IM} e^*_M > 0 \tag{21}$$

for any non-zero e_K^*. Because $\mathcal{S}^E = \mathbf{P}\,\mathbf{C}$, and both $\mathbf{C}$ and $\mathbf{P}$ are symmetric, then

$$\mathbf{K} = \mathbf{C}\,\mathbf{P}\,\mathbf{C} \tag{22}$$

is a symmetric matrix as well. Let $\mathbf{S} = \mathbf{C}^{-1}$ be the matrix corresponding to the elastic compliance tensor (note that $\mathbf{S}$ is symmetric and PD). Then the matrix form of the Hill tensor can be written as

$$\mathbf{P} = \mathbf{S}\,\mathbf{K}\,\mathbf{S} \tag{23}$$

Because matrix $\mathbf{K}$ is PD, it follows that the Hill tensor $\mathbf{P}$ is also PD. This can be shown by substitution into the definition of positive definiteness, yielding

$$\mathbf{x}^{\mathrm{T}}\mathbf{P}\,\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{S}\,\mathbf{K}\,\mathbf{S}\,\mathbf{x} = (\mathbf{S}\mathbf{x})^{\mathrm{T}}\,\mathbf{K}\,(\mathbf{S}\mathbf{x}) > 0 \tag{24}$$

Now that both $\mathbf{P}$ and $\mathbf{C}$ are real, symmetric and PD, we have proved that $\mathcal{S}^E = \mathbf{P}\,\mathbf{C}$ is WPD following the definition of Wigner [6].

3.3 All eigenvalues of $\mathcal{S}^E$ are between 0 and 1

Given that $\mathcal{S}^E$ is WPD, all of its eigenvalues are real and positive. In addition, the determinant of $\mathcal{S}^E$ is also positive, because

$$\det(\mathcal{S}^E) = \det(\mathbf{P}) \cdot \det(\mathbf{C}) > 0 \tag{25}$$

Similarly, we have also proved [12] that $\mathbf{C}(\mathbf{I} - \mathcal{S}^E) = \mathbf{C} - \mathbf{K}$ is a PD matrix (note it is also symmetric). Therefore, the matrix $\mathbf{I} - \mathcal{S}^E = \mathbf{S}(\mathbf{C} - \mathbf{K})$ is also WPD. It then follows that all eigenvalues of $\mathbf{I} - \mathcal{S}^E$ are positive. Therefore, the eigenvalues λ of $\mathcal{S}^E$ satisfy the condition $0 < \lambda < 1$, since the eigenvalues of $\mathbf{I} - \mathcal{S}^E$ are $1 - \lambda$.

4 Numerical Case Studies

4.1 Isotropic elastic media

The Eshelby tensor $\mathcal{S}^E$ in an isotropic elastic medium can be expressed analytically in terms of elliptic integrals [1]. $\mathcal{S}^E$ is function of the Poisson's ratio ν as well as the semi-axes, a, b, c, of the elliposidal inclusion. (It is independent of the shear modulus μ of the elastic medium.) This expression is implemented in Matlab to compute the $\mathcal{S}^E$ matrix and its 6 eigenvalues for any input values of ν, a, b, c.

We considered 1000 random cases in which ν is randomly generated from a uniform distribution from -1 to 0.5, and a, b, c are randomly generated from a uniform distribution from 0 to 1. In every case, we find that eigenvalues of $\mathbf{S}^E$ are all real valued (confirming property P1), so that we can sort them as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_6$. Figure 2 plots the smallest eigenvalue λ_1 and the largest eigenvalue λ_6 as functions of ν, from which we can observe that all eigenvalues lie between 0 and 1 (confirming property P2). Furthermore, we notice that $\lambda_1 \leq 0.5$ and $\lambda_6 \geq 0.5$. This is consistent with the property) [7] that the sum of all 6 eigenvalues is 3). Indeed, we have verified that $\sum_{I=1}^{6} \lambda_I = 3$ for every Eshelby tensor that we considered. For completeness, a short proof of this result is given in Appendix) C. Obviously, if the 6 eigenvalues must sum up to 3, then average value of the eigenvalues is always 0.5; the smallest eigenvalue cannot exceed the average while the largest eigenvalue cannot fall below the average.

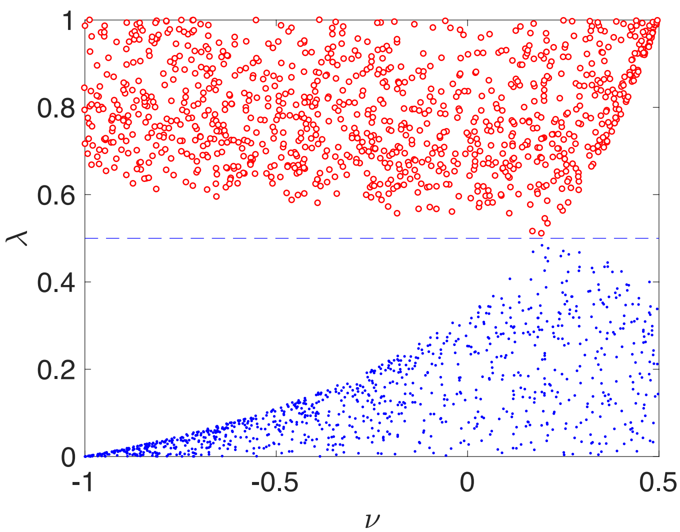


Figure 2: The smallest and largest eigenvalues of the Eshelby tensor in an isotropic elastic medium with Poisson's ratio ν. Dots: smallest eigenvalue λ_1. Circles: largest eigenvalue λ_6.

4.2 Cubic elastic media

In order to compute the Eshelby tensor in an anisotropic elastic medium, we implemented Eq. (8) and Eq. (9), where the integrals are performed numerically using the `quadv` function in Matlab.

We considered 500 random cases in which the elastic stiffness tensors satisfy the cubic symmetry, i.e. they can be specified in terms of C_{11}, C_{12}, and C_{44}. Positive definiteness of the elastic stiffness tensor requires $C_{11} - C_{12} > 0$, $C_{11} + 2C_{12} > 0$ and $C_{44} > 0$. To satisfy these conditions, we generate values for $C_{11} - C_{12}$, $C_{11} + 2C_{12}$ and C_{44} from uniform distributions from 0 to 1. The semi-axes a, b, c of the ellipsoidal inclusion are also randomly generated from a uniform distribution from 0 to 1.

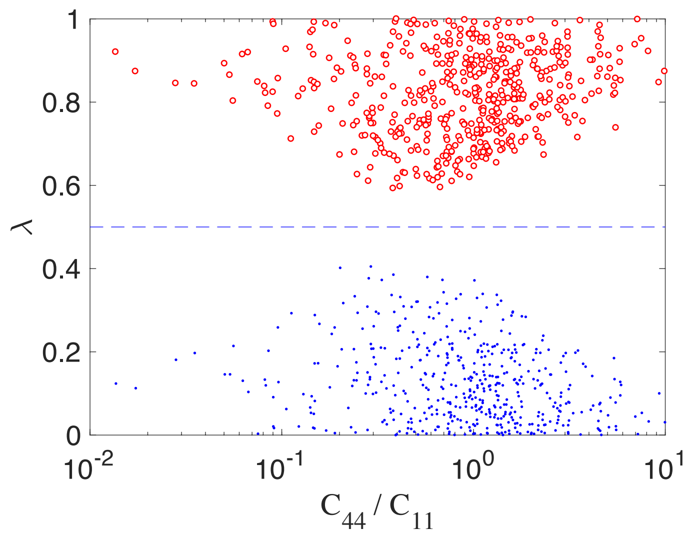


Figure 3: The smallest and largest eigenvalues of the Eshelby tensor in an elastic medium with cubic symmetry. Dots: smallest eigenvalue λ_1. Circles: largest eigenvalue λ_6.

Again eigenvalues of $\mathbf{S}^E$ are all real and lie between 0 and 1. Figure 2 plots the smallest eigenvalue λ_1 and the largest eigenvalue λ_6 as functions of C_{44}/C_{11}, from which we can observe that $\lambda_1 \leq 0.5$ and $\lambda_6 \geq 0.5$. We also confirmed) that in every case $\sum_{I=1}^{6} \lambda_I = 3$.

4.3 Generally anisotropic elastic media

To create generally anisotropic elastic tensors that are positive definite, we start with a 6×6 real symmetric matrix $\mathbf{R}$ whose elements are random numbers uniformly distributed from 0 to 1. We then diagonize matrix $\mathbf{R}$, i.e. finding matrices $\mathbf{U}$ and $\mathbf{D}_1$ such that $\mathbf{R} = \mathbf{U}\,\mathbf{D}_1\,\mathbf{U}^{\mathrm{T}}$, where $\mathbf{U}$ is an orthogonal matrix and $\mathbf{D}_1$ is a diagonal matrix. We then construct matrix $\mathbf{D}_2$ whose elements are the absolute values of those in $\mathbf{D}_1$. Finally, we construct $\mathbf{C} = \mathbf{U}\,\mathbf{D}_2\,\mathbf{U}^{\mathrm{T}}$, whose eigenvalues are all positive.

We considered 500 random cases in which $\mathbf{C}$ is generated using the approach described above, and a, b, c of the ellipsoidal inclusion are randomly generated from

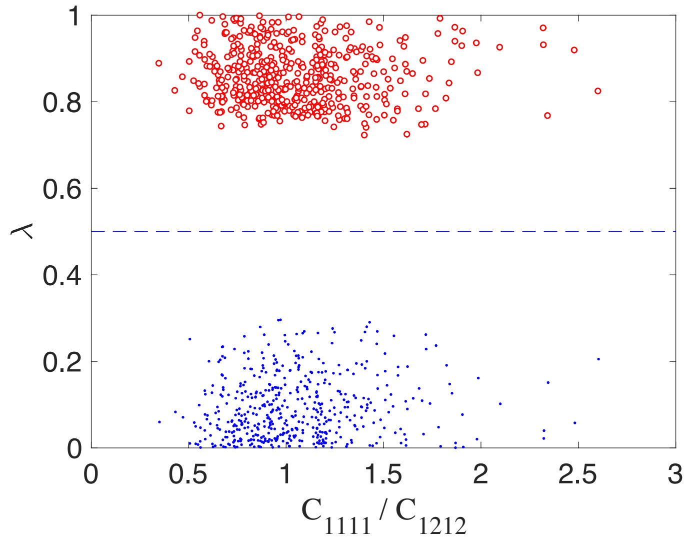


Figure 4: The smallest and largest eigenvalues of the Eshelby tensor in a generally anisotropic elastic medium. Dots: smallest eigenvalue λ_1. Circles: largest eigenvalue λ_6.

a uniform distribution from 0 to 1. Figure 4 plots the smallest eigenvalue λ_1 and the largest eigenvalue λ_6 as functions of C_{1111}/C_{1212}, from which we can observe that all eigenvalues are between 0 and 1, and that $\lambda_1 \leq 0.5$ and $\lambda_6 \geq 0.5$. We confirmed) that in every case $\sum_{I=1}^{6} \lambda_I = 3$. We also performed calculations for a class of triclinic media described in [17], in which all the above mentioned) properties of the Eshelby tensor are confirmed as well.

We have also confirmed that the Hill tensor $\mathbf{P}$ is symmetric in all three test cases described above. The Matlab programs for these calculations can be downloaded from our web site [18].

5 Proving A is Invertible

We now present a direct proof of the invertibility of matrix $\mathbf{A}$, using the property (P1) that $\mathcal{S}^E$ is WPD. This proof does not use the property P2) concerning the eigenvalues of $\mathcal{S}^E$. We shall consider the matrix $\mathbf{M}$ given by

$$\mathbf{M} \equiv (\mathcal{S}^E)^{\mathrm{T}}\mathbf{A} = (\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}'\mathcal{S}^E + (\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}(\mathbf{I} - \mathcal{S}^E) \tag{26}$$

and will show that it is symmetric and PD. This will be accomplished by showing that both terms, $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}'\mathcal{S}^E$ and $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}(\mathbf{I} - \mathcal{S}^E)$, are symmetric and PD.

First, let us prove that $\mathbf{M}$ is symmetric. It is obvious that matrix $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}'\mathcal{S}^E$ is symmetric, given that $\mathbf{C}'$ is symmetric. Recall that $\mathcal{S}^E = \mathbf{P}\,\mathbf{C}$, where $\mathbf{P}$ is symmetric, hence

$$(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}(\mathbf{I} - \mathcal{S}^E) = \mathbf{C}\,\mathbf{P}\,\mathbf{C} - \mathbf{C}\,\mathbf{P}\,\mathbf{C}\,\mathbf{P}\,\mathbf{C} \tag{27}$$

which is clearly symmetric as well. Therefore, $\mathbf{M} \equiv (\mathcal{S}^E)^{\mathrm{T}}\mathbf{A}$ is a symmetric matrix.

Next, we show that $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}'\mathcal{S}^E$ is PD. To do so, we consider any real non-zero 6×1 column vector $\mathbf{X}$. Define $\mathbf{V} \equiv \mathcal{S}^E\mathbf{X}$ which is also non-zero. Since $\mathbf{C}'$ is PD,

$$\mathbf{X}^{\mathrm{T}}(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}'\mathcal{S}^E\mathbf{X} = \mathbf{V}^{\mathrm{T}}\mathbf{C}'\mathbf{V} > 0 \tag{28}$$

Therefore, $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}'\mathcal{S}^E$ is PD.

Lastly, we consider matrix $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}(\mathbf{I} - \mathcal{S}^E)$. It can be shown that the strain energy stored in the matrix surrounding the inclusion (with arbitrary eigenstrain $\mathbf{e}^*$) can be written as (see Appendix B)

$$E^{\mathrm{M}} = \frac{1}{2}(\mathbf{e}^*)^{\mathrm{T}}(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}(\mathbf{I} - \mathcal{S}^E)\mathbf{e}^*\,V_0 \tag{29}$$

Given our assumption of the strong convesity of the elastic stiffness tensor $\mathbf{C}$, the stain energy density function is positive everywhere for all non-zero strain states. As a result, the strain energy integrated over any region, such as the elastic energy in the matrix, E^{M}, must be positive for all possible non-zero eigenstrains $\mathbf{e}^*$.) Consequently $(\mathcal{S}^E)^{\mathrm{T}}\mathbf{C}(\mathbf{I} - \mathcal{S}^E)$ is PD!

Because the sum of two PD matrices is also PD, from Eq. (26) we conclude that matrix $\mathbf{M} \equiv (\mathbf{S}^E)^{\mathrm{T}}\mathbf{A}$ is PD. Therefore,

$$\det[(\mathcal{S}^E)^{\mathrm{T}}\mathbf{A}] = \det(\mathcal{S}^E)\,\det(\mathbf{A}) > 0 \tag{30}$$

Combined with Eq. (25), $\det(\mathcal{S}^E) > 0$, we have

$$\det(\mathbf{A}) > 0\ . \tag{31}$$

Therefore, matrix $\mathbf{A}$ is non-singular and always invertible. This proves that the equivalent inclusion can always be found for an arbitrary ellipsoidal shaped transformed inhomogeneity in an arbitrarily anisotropic elastic medium with a positive definite elastic stiffness tensor.

6 Conclusions

We proved several mathematical properties of the Eshelby tensor: (P1) $\mathcal{S}^E$ is weakly positive definite, and (P2) all of its eigenvalues satisfy $0 < \lambda_I < 1$ for $I = 1, 2, \cdots, 6$. Property P1 allows us to constrct a direct and concise proof of the non-singular nature of matrix $\mathbf{A}$, which guarantees that the equivalent inclusion solution can always be obtained for an ellipsoidal transformed inhomogeneity in an infinite elastic medium.

We note that our proof of the invertibility of the matrix $\mathbf{A}$ also justifies the validity of Eshelby's use of the variations of the equivalent inclusion method to not only the problem of the transformed ellipsoidal inhomogeneity in an infinite medium with no remote stress applied at infinity, but also to treat the following problems.)

(a) A transformed ellipsoidal inhomogeneity in an infinite medium with a constant remote applied stress state at infinity.)

(b) An ellipsoidal inhomogeneity (no transformation strain) in an infinite medium with a constant remote applied stress state at infinity.)

(c) An ellipsoidal hole with prescribed tractions on the hole boundary in an infinite medium with no remote applied stress at infinity. The prescribed tractions have the form of) $T_j^0 = \sigma_{ij}^0 n_i$, where) σ_{ij}^0 is a uniform stress and) n_i is the local hole boundary normal. An example is an ellipsoidal hole pressurized by an ideal fluid in a state of pure hydrostatic pressure.)

As mentioned by Eshelby) [21] (for isotropy) and, in more detail for anisotropy) [5], the invertibility of the) $\mathbf{A}$ matrix governs the validity of the Eshelby equivalent inclusion method applied to the above problems also.)

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A Voigt Notation

A symmetric fourth-rank tensor, such as the elastic stiffness tensor, may be reduced to a symmetric 6×6 matrix using the Voigt form. Correspondingly, a symmetric second-rank tensor, such as stress and strain, is reduced to a 1×6 column vector in the Voigt form. However, this reduction in order requires the use of weights to recover losses associated with the symmetries of the tensor. There are different approaches to the weights, and) in this paper the Nye notation will be used [20]. In the fourth-rank tensors presented below the 81 components reduce to 36 distinct components represented in a 6×6 matrix due to minor symmetries (i.e. $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$).

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} =$$

$$\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{pmatrix}$$

The corresponding matrix representations of stress and strain tensors are

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} \qquad \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{13} \\ 2e_{12} \end{pmatrix}$$

So that the generalized Hooke's law, $\sigma_{ij} = C_{ijkl}e_{kl}$, can be equivalently written in the matrix form as $\sigma_I = C_{IK}e_K$, where I and K are indices from 1 to 6.

The corresponding matrix representations of the Eshelby tensor $\mathcal{S}^E_{ijkl}$ and Hill tensor P_{ijkl} are as follows.

$$\begin{pmatrix} \mathcal{S}^E_{11} & \mathcal{S}^E_{12} & \mathcal{S}^E_{13} & \mathcal{S}^E_{14} & \mathcal{S}^E_{15} & \mathcal{S}^E_{16} \\ \mathcal{S}^E_{21} & \mathcal{S}^E_{22} & \mathcal{S}^E_{23} & \mathcal{S}^E_{24} & \mathcal{S}^E_{25} & \mathcal{S}^E_{26} \\ \mathcal{S}^E_{31} & \mathcal{S}^E_{32} & \mathcal{S}^E_{33} & \mathcal{S}^E_{34} & \mathcal{S}^E_{35} & \mathcal{S}^E_{36} \\ \mathcal{S}^E_{41} & \mathcal{S}^E_{42} & \mathcal{S}^E_{43} & \mathcal{S}^E_{44} & \mathcal{S}^E_{45} & \mathcal{S}^E_{46} \\ \mathcal{S}^E_{51} & \mathcal{S}^E_{52} & \mathcal{S}^E_{53} & \mathcal{S}^E_{54} & \mathcal{S}^E_{55} & \mathcal{S}^E_{56} \\ \mathcal{S}^E_{61} & \mathcal{S}^E_{62} & \mathcal{S}^E_{63} & \mathcal{S}^E_{64} & \mathcal{S}^E_{65} & \mathcal{S}^E_{66} \end{pmatrix} =$$

$$\begin{pmatrix} \mathcal{S}^E_{1111} & \mathcal{S}^E_{1122} & \mathcal{S}^E_{1133} & \mathcal{S}^E_{1123} & \mathcal{S}^E_{1113} & \mathcal{S}^E_{1112} \\ \mathcal{S}^E_{2211} & \mathcal{S}^E_{2222} & \mathcal{S}^E_{2233} & \mathcal{S}^E_{2223} & \mathcal{S}^E_{2213} & \mathcal{S}^E_{2212} \\ \mathcal{S}^E_{3311} & \mathcal{S}^E_{3322} & \mathcal{S}^E_{3333} & \mathcal{S}^E_{3323} & \mathcal{S}^E_{3313} & \mathcal{S}^E_{3312} \\ 2\mathcal{S}^E_{2311} & 2\mathcal{S}^E_{2322} & 2\mathcal{S}^E_{2333} & 2\mathcal{S}^E_{2323} & 2\mathcal{S}^E_{2313} & 2\mathcal{S}^E_{2312} \\ 2\mathcal{S}^E_{1311} & 2\mathcal{S}^E_{1322} & 2\mathcal{S}^E_{1333} & 2\mathcal{S}^E_{1323} & 2\mathcal{S}^E_{1313} & 2\mathcal{S}^E_{1312} \\ 2\mathcal{S}^E_{1211} & 2\mathcal{S}^E_{1222} & 2\mathcal{S}^E_{1233} & 2\mathcal{S}^E_{1223} & 2\mathcal{S}^E_{1213} & 2\mathcal{S}^E_{1212} \end{pmatrix}$$

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} \end{pmatrix} =$$

$$\begin{pmatrix} P_{1111} & P_{1122} & P_{1133} & 2P_{1123} & 2P_{1113} & 2P_{1112} \\ P_{2211} & P_{2222} & P_{2233} & 2P_{2223} & 2P_{2213} & 2P_{2212} \\ P_{3311} & P_{3322} & P_{3333} & 2P_{3323} & 2P_{3313} & 2P_{3312} \\ 2P_{2311} & 2P_{2322} & 2P_{2333} & 4P_{2323} & 4P_{2313} & 4P_{2312} \\ 2P_{1311} & 2P_{1322} & 2P_{1333} & 4P_{1323} & 4P_{1313} & 4P_{1312} \\ 2P_{1211} & 2P_{1222} & 2P_{1233} & 4P_{1223} & 4P_{1213} & 4P_{1212} \end{pmatrix}$$

These representations ensure that the relationship between the "constrained" strain and eigenstrain, $e^c_{ij} = \mathcal{S}^E_{ijmn} e^*_{mn} = P_{ijkl} C_{klmn} e^*_{mn}$, can be written in the matrix form as $e^c_I = \mathcal{S}^E_{IM} e^*_M = P_{IK} C_{KM} e^*_M$, where I, K, M are indices from 1 to 6.

B Energy In Medium Surrounding Inclusion

Here we show that the strain energy stored in the elastic medium (i.e. the matrix) surrounding the ellipsoidal inclusion (with arbitrary eigenstrain $\mathbf{e}^*$) can be written as

$$E^{\mathrm{M}} = \frac{1}{2} (\mathbf{e}^*)^{\mathrm{T}} (\mathcal{S}^E)^{\mathrm{T}} \mathbf{C} (\mathbf{I} - \mathcal{S}^E) \mathbf{e}^* \, V_0 \tag{32}$$

Eshelby showed that the total strain energy of the infinite elastic medium containing a transformed inclusion can be written as,

$$E^{\mathrm{tot}} = -\frac{1}{2} \, \sigma^{\mathrm{I}}_{ij} \, e^*_{ij} \, V_0 \tag{33}$$

where σ^{I}_{ij} is the final stress state inside the inclusion, which can be written as,

$$\sigma^{\mathrm{I}}_{ij} = C_{ijkl}(e^c_{kl} - e^*_{kl}) = C_{ijkl}(\mathcal{S}^E_{klmn} e^*_{mn} - e^*_{kl}) \tag{34}$$

Therefore, the total elastic energy contained in both matrix and inclusion can be written as,

$$E^{\mathrm{tot}} = \frac{1}{2}\, e^*_{ij}\, C_{ijkl}(e^*_{kl} - \mathcal{S}^E_{klmn} e^*_{mn})\, V_0 \tag{35}$$

In Voigt notation,

$$E^{\mathrm{tot}} = \frac{1}{2}\, e^*_I\, C_{IK}(e^*_K - \mathcal{S}^E_{KN} e^*_N)\, V_0 = \frac{1}{2}\, (\mathbf{e}^*)^{\mathrm{T}}\, \mathbf{C}\, (\mathbf{I} - \mathcal{S}^E)\, \mathbf{e}^*\, V_0 \tag{36}$$

Because the elastic energy E^{tot} and volume of the inclusion V_0 must be positive for arbitrary eigenstrain $\mathbf{e}^*$, matrix $\mathbf{C}\,(\mathbf{I} - \mathcal{S}^E)$ is PD. We used this property in Section 3.2.

The elastic strain energy E^{I} stored inside the inclusion can be obtained from the stress σ^I_{ij} and elastic strain $e^{\mathrm{I,el}}_{ij} = e^c_{ij} - e^*_{ij}$ inside the inclusion.

$$\begin{aligned} E^{\mathrm{I}} &= \frac{1}{2}\, \sigma^{\mathrm{I}}_{ij}\, e^{\mathrm{I,el}}_{ij}\, V_0 \\ &= \frac{1}{2}\, (e^c_{ij} - e^*_{ij})\, C_{ijkl}(e^c_{kl} - e^*_{kl})\, V_0 \\ &= \frac{1}{2}\, (e^*_{ij} - e^c_{ij})\, C_{ijkl}(e^*_{kl} - e^c_{kl})\, V_0 \\ &= \frac{1}{2}\, (\mathbf{e}^*)^{\mathrm{T}}\, (\mathbf{I} - \mathcal{S}^E)^{\mathrm{T}}\, \mathbf{C}\, (\mathbf{I} - \mathcal{S}^E)\, \mathbf{e}^*\, V_0 \end{aligned} \tag{37}$$

Therefore, the elastic energy stored in the matrix is,

$$\begin{aligned} E^{\mathrm{M}} &= E^{\mathrm{tot}} - E^{\mathrm{I}} \\ &= \frac{1}{2}\, e^c_{ij}\, C_{ijkl}(e^*_{kl} - e^c_{kl})\, V_0 \\ &= \frac{1}{2}\, (\mathbf{e}^*)^{\mathrm{T}}\, (\mathcal{S}^E)^{\mathrm{T}}\, \mathbf{C}\, (\mathbf{I} - \mathcal{S}^E)\, \mathbf{e}^*\, V_0 \end{aligned} \tag{38}$$

Because E^{M} must be positive, matrix $(\mathcal{S}^E)^{\mathrm{T}}\, \mathbf{C}\, (\mathbf{I} - \mathcal{S}^E)$ is PD.

C Proving the Sum of Eigenvalues of $\mathcal{S}^E$ Equals 3

It has been shown by Milgrom and Shtrikman) [7] that the trace of the Eshelby tensor is 3, provided the eigenstrain is constant, and that their proof holds even for non-ellipsoidal inclusions. For non-ellipsoidal inclusions, the Eshelby tensor will vary with position inside the inclusion, but the trace still remains to be 3. For convenience of the reader, here we provide a short proof of the same result for the more specific case considered here, i.e. for the case of ellipsoidal inclusions.)

The trace of matrix $\mathcal{S}^E$ can be written in the index notation as,

$$\mathrm{Tr}(\mathcal{S}^E) = \mathcal{S}^E_{ipip} \tag{39}$$

Using Eq. (8) and Eq. (9), we have

$$\mathcal{S}^E_{ijmn} \quad = \quad \frac{abc}{8\pi} C_{klmn} \int_0^{\pi} \int_0^{2\pi} \left[z_i z_l \, (zz)^{-1}_{jk} + z_j z_l \, (zz)^{-1}_{ik} \right] \frac{\sin \Phi}{\beta^3} \, \mathrm{d}\Theta \mathrm{d}\Phi \tag{40}$$

where

$$\beta = \sqrt{(a^2 \cos^2 \Theta + b^2 \sin^2 \Theta) \sin^2 \Phi + c^2 \cos^2 \Phi} \tag{41}$$

So that

$$\begin{aligned}
\mathcal{S}^E_{ipip} &= \frac{abc}{8\pi} C_{klip} \int_0^{\pi} \int_0^{2\pi} \left[z_i z_l \, (zz)^{-1}_{pk} + z_p z_l \, (zz)^{-1}_{ik} \right] \frac{\sin \Phi}{\beta^3} \, \mathrm{d}\Theta \mathrm{d}\Phi \\
&= \frac{abc}{8\pi} \int_0^{\pi} \int_0^{2\pi} \left[(zz)_{pk} \, (zz)^{-1}_{pk} + (zz)_{ik} \, (zz)^{-1}_{ik} \right] \frac{\sin \Phi}{\beta^3} \, \mathrm{d}\Theta \mathrm{d}\Phi \\
&= \frac{3\, abc}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\sin \Phi}{\beta^3} \, \mathrm{d}\Theta \mathrm{d}\Phi \\
&= \frac{3\, abc}{4\pi} \int_{-1}^{1} \int_0^{2\pi} \frac{\mathrm{d}(\cos \Phi)}{[(a^2 \cos^2 \Theta + b^2 \sin^2 \Theta) \sin^2 \Phi + c^2 \cos^2 \Phi]^{3/2}} \, \mathrm{d}\Theta
\end{aligned} \tag{42}$$

Let $\mu \equiv \cos \Phi$, and $f \equiv \sqrt{a^2 \cos^2 \Theta + b^2 \sin^2 \Theta}$. Because

$$\int_{-1}^{1} \frac{\mathrm{d}\mu}{[f^2(1-\mu^2) + c^2 \mu^2]} = \frac{2}{f^2\, c} \tag{43}$$

we have

$$\mathcal{S}^E_{ipip} = \frac{3\, abc}{4\, \pi} \frac{2}{c} \int_0^{2\,\pi} \frac{\mathrm{d}\Theta}{a^2 \cos^2 \Theta + b^2 \sin^2 \Theta} \tag{44}$$

Using the expression that [19]

$$J(a,b) \equiv \int_0^{\pi/2} \frac{d\Theta}{a^2\cos^2\Theta + b^2\sin^2\Theta} = \frac{\pi}{2ab} \tag{45}$$

we have

$$\mathcal{S}^E_{ipip} = \frac{3\,abc}{4\,\pi}\,\frac{4\,\pi}{abc} = 3 \tag{46}$$

This completes the proof that the sum of all eigenvalues of $\mathcal{S}^E$ always equals 3. At present, a clear physical interpretation of this mathematical property is still lacking. A qualitative interpretation of this result is that given an arbitrary elastic medium and an arbitrary ellipsoidal shape of the inclusion, if the eigenstrain is chosen randomly (as a 6-dimensional vector in Voigt notation), then on the average the constrained strain (i.e. the actual strain of the inclusion when embedded in the matrix) is half of the eigenstrain.