ME340B – Elasticity of Microscopic Structures – Stanford University – Winter 2004

Lecture Note 2. Eshelby's Inclusion I

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Contents

1 Inclusion and eigenstain

Consider a homogeneous linear elastic solid with volume V and surface area S, with elastic constant C_{ijkl}, as shown in Fig. 1. Let a sub-volume V_0 with surface area S_0 undergo a uniform permanent (inelastic) deformation, such as a martensitic phase transformation. The material inside V_0 is called an *inclusion* and the material outside is called the *matrix*. If we remove V_0 from its surrounding matrix, it should assume a uniform strain e^*_{ij} and will experience zero stress. e^*_{ij} is called the *eigenstrain*, meaning the strain under zero stress. Notice that both the inclusion and the matrix have the same elastic constants. The *eigenstress* is defined as $\sigma^*_{ij} \equiv C_{ijkl} e^*_{kl}$.

In reality, the inclusion is surrounded by the matrix. Therefore, it is not able to reach the state of eigenstrain and zero stress. Instead, both the inclusion and the matrix will deform and experience an elastic stress field. The Eshelby's transformed inclusion problem is to solve the stress, strain and displacement fields both in the inclusion and in the matrix.

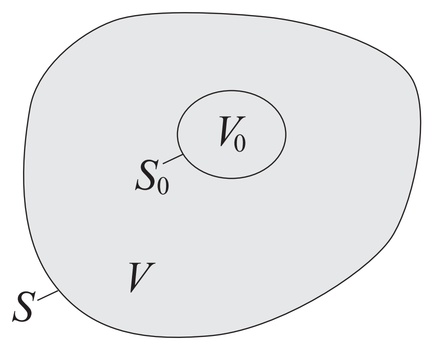


Figure 1: A linear elastic solid with volume V and surface S. A subvolume V_0 and surface S_0 undergoes a permanent (inelastic) deformation. The material inside V_0 is called an *inclusion* and the material outside is called the *matrix*.

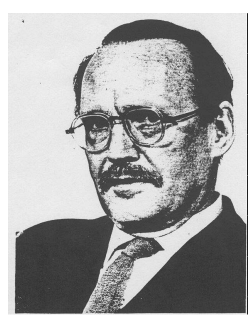
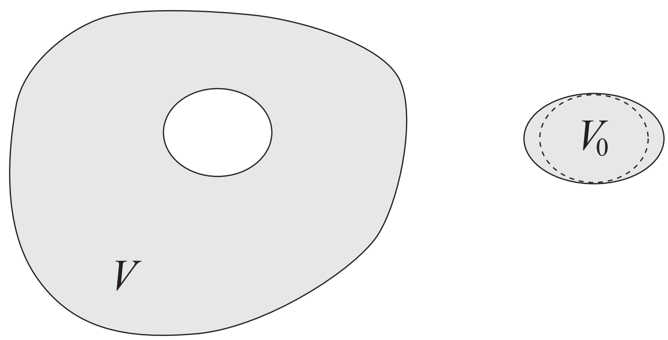


Figure 2: John Douglas Eshelby (1916-1981, United Kingdom).

2 Green's function and Eshelby's tensor $\mathcal{S}_{ijkl}$

Eshelby showed that the problem stated above can be solved elegantly by the superposition principle of linear elasticity and using the Green's function [1]. Eshelby used the following 4 steps of a "virtual" experiment to construct the desired solution.

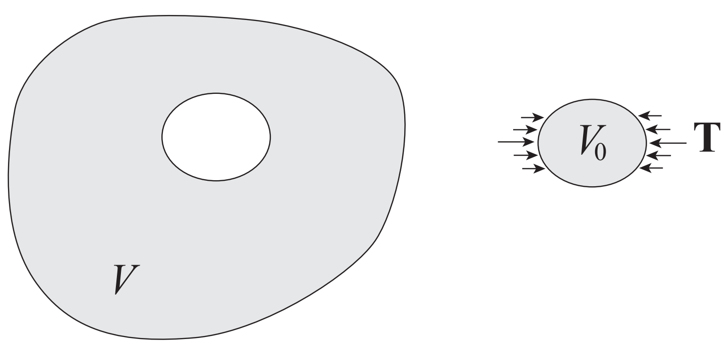
Step 1. Remove the inclusion from the matrix.



Apply no force to the inclusion, nor to the matrix. The strain, stress and displacement fields in the matrix and the inclusion are,

matrix	inclusion
$e_{ij} = 0$	$e_{ij} = e^*_{ij}$
$\sigma_{ij} = 0$	$\sigma_{ij} = 0$
$u_i = 0$	$u_i = e^*_{ij} x_j$

Step 2. Apply surface traction to S_0 in order to make the inclusion return to its original shape

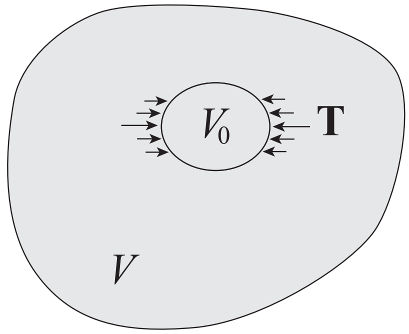


The elastic strain of the inclusion should exactly cancel the eigenstrain, i.e. $e^{\text{el}}_{ij} = -e^*_{ij}$. The strain, stress and displacement fields in the matrix and the inclusion are,

matrix	inclusion
$e_{ij} = 0$	$e_{ij} = e^{\text{el}}_{ij} + e^*_{ij} = 0$
$\sigma_{ij} = 0$	$\sigma_{ij} = C_{ijkl} e^{\text{el}}_{ij} = -C_{ijkl} e^*_{ij} = -\sigma^*_{ij}$
$u_i = 0$	$u_i = 0$

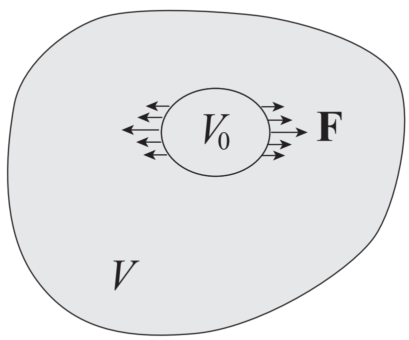
The traction force on S_0 is $T_j = \sigma_{ij} n_i = -\sigma^*_{ij} n_i$.

Step 3. Put the inclusion back to the matrix.



The same force $\mathbf{T}$ is applied to the internal surface S_0. There is no change in the deformation fields in either the inclusion or the matrix from step 2.

Step 4. Now remove the traction $\mathbf{T}$. This returns us to the original inclusion problem as shown in Fig. 1. The change from step 3 to step 4 is equivalent to applying a cancelling body force $\mathbf{F} = -\mathbf{T}$ to the internal surface S_0 of the elastic body.



Let $u_i^{\rm c}(\mathbf{x})$ be the displacement field in response to body force F_j on S_0. $u_i^{\rm c}(\mathbf{x})$ is called the *constrained* displacement field. It can be easily expressed in terms of the Green's function of the elastic body, (notice that $F_j = -T_j = \sigma^*_{jk} n_k$)

$$u_i^{\rm c}(\mathbf{x}) = \int_{S_0} F_j(\mathbf{x}') G_{ij}(\mathbf{x}, \mathbf{x}') dS(\mathbf{x}') = \int_{S_0} \sigma^*_{jk} n_k(\mathbf{x}') G_{ij}(\mathbf{x}, \mathbf{x}') dS(\mathbf{x}') \tag{1}$$

The displacement gradient, strain, and stress of the constrained field are

$$u_{i,j}^{\rm c}(\mathbf{x}) = \int_{S_0} \sigma^*_{lk} n_k(\mathbf{x}') G_{il,j}(\mathbf{x}, \mathbf{x}')\, {\rm d}S(\mathbf{x}') \tag{2}$$

$$e_{ij}^{\rm c}(\mathbf{x}) = \frac{1}{2}(u_{i,j}^{\rm c} + u_{i,j}^{\rm c}) = \frac{1}{2} \int_{S_0} \sigma^*_{lk} n_k(\mathbf{x}') \left[G_{il,j}(\mathbf{x}, \mathbf{x}') + G_{jl,i}(\mathbf{x}, \mathbf{x}') \right] {\rm d}S(\mathbf{x}') \tag{3}$$

$$\sigma_{ij}^{\rm c}(\mathbf{x}) = C_{ijkl} e_{kl}^{\rm c}(\mathbf{x}) \tag{4}$$

In terms of the constrained field, the strain, stress and displacement fields in the matrix and the inclusion are,

matrix	inclusion
$e_{ij} = e^{\mathrm{c}}_{ij}$	$e_{ij} = e^{\mathrm{c}}_{ij}$
$\sigma_{ij} = \sigma^{\mathrm{c}}_{ij}$	$\sigma_{ij} = \sigma^{\mathrm{c}}_{ij} - \sigma^*_{ij} = C_{ijkl}(e^{\mathrm{c}}_{kl} - e^*_{kl})$
$u_i = u^{\mathrm{c}}_i$	$u_i = u^{\mathrm{c}}_i$

To obtain explicit expressions for the stresses and strains everywhere, the constrained field must be determined both inside and outside the inclusion. We can define a fourth order tensor $\mathcal{S}_{ijkl}$ that relates the constrained strain inside the inclusion to its eigenstrain,

$$e^c_{ij} = \mathcal{S}_{ijkl} e^*_{kl} \tag{5}$$

$\mathcal{S}_{ijkl}$ is often referred to as Eshelby's tensor. Because it relates two symmetric strain tensors, the Eshelby's tensor satisfies minor symmetries,

$$\mathcal{S}_{ijkl} = \mathcal{S}_{jikl} = \mathcal{S}_{ijlk} \tag{6}$$

But it does not satisfy the major symmetry, i.e. $\mathcal{S}_{ijkl} \neq \mathcal{S}_{klij}$. In the following sections, we derive the explicit expressions of Eshelby's tensor in an infinite elastic medium ($V \to \infty$). In principle, Eshelby's tensor is a function of space, i.e. $\mathcal{S}_{ijkl}(\mathbf{x})$. However, an amazing result obtained by Eshelby is that,

> **For an *ellipsoidal* inclusion in a homogeneous infinite matrix, the Eshelby tensor $\mathcal{S}_{ijkl}$ is a *constant* tensor. Hence the stress-strain fields inside the inclusion are *uniform*.**

3 Auxiliary tensor $\mathcal{D}_{ijkl}$

For convenience, let us define another tensor $\mathcal{D}_{ijkl}$ that relates the constrained displacement gradients to the eigenstress inside the inclusion [3],

$$u^c_{i,l}(\mathbf{x}) = -\sigma^*_{kj} \mathcal{D}_{ijkl}(\mathbf{x}) \tag{7}$$

Obviously, tensor $\mathcal{D}_{ijkl}$ is related to Eshelby's tensor,

$$\begin{aligned}
\mathcal{S}_{ijmn} e^*_{mn} &= e^c_{ij} && (8)\\
&= \frac{1}{2}(u^c_{i,j} + u^c_{j,i}) && (9)\\
&= -\frac{1}{2}(\sigma^*_{lk}\mathcal{D}_{iklj} + \sigma^*_{lk}\mathcal{D}_{jkli}) && (10)\\
&= -\frac{1}{2}\sigma^*_{lk}(\mathcal{D}_{iklj} + \mathcal{D}_{jkli}) && (11)\\
&= -\frac{1}{2}C_{lkmn} e^*_{mn}(\mathcal{D}_{iklj} + \mathcal{D}_{jkli}) && (12)
\end{aligned}$$

Therefore,

$$\mathcal{S}_{ijmn}(\mathbf{x}) = -\frac{1}{2}C_{lkmn}(\mathcal{D}_{iklj}(\mathbf{x}) + \mathcal{D}_{jkli}(\mathbf{x})) \tag{13}$$

Rewrite Eq. (7) as $u^c_{i,j}(\mathbf{x}) = -\sigma^*_{kl}\mathcal{D}_{ilkj}(\mathbf{x})$ and compare it with From Eq. (2), we obtain,

$$\mathcal{D}_{ilkj}(\mathbf{x}) = -\int_{S_0} n_k(\mathbf{x}')G_{il,j}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \tag{14}$$

or equivalently,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\int_{S_0} G_{ij,l}(\mathbf{x}-\mathbf{x}')n_k(\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \tag{15}$$

Notice that we have used the fact that $G_{ij}(\mathbf{x},\mathbf{x}') = G_{ij}(\mathbf{x}-\mathbf{x}')$ for an infinite homogeneous medium. Applying Gauss's Theorem, we obtain

$$\begin{aligned}\mathcal{D}_{ijkl}(\mathbf{x}) &= -\int_{V_0} \frac{\partial}{\partial x'_k} G_{ij,l}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\ &= \int_{V_0} \frac{\partial}{\partial x_k} G_{ij,l}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')\end{aligned}$$

Therefore,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = \int_{V_0} G_{ij,kl}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \tag{16}$$

Recall that the Green's function for an anisotropic medium is,

$$G_{ij}(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]\frac{(zz)^{-1}_{ij}}{k^2}\,\mathrm{d}\mathbf{k} \tag{17}$$

where $\mathbf{z} = \mathbf{k}/k$. Substituting this into Eq. (16), we get

$$\begin{aligned}\mathcal{D}_{ijkl}(\mathbf{x}) &= \int_{V_0} \frac{\partial^2}{\partial x_k \partial x_l}\left[\frac{1}{(2\pi)^3}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]\frac{(zz)^{-1}_{ij}}{k^2}\,\mathrm{d}\mathbf{k}\right]\mathrm{d}V(\mathbf{x}') \\ &= \frac{1}{(2\pi)^3}\int_{V_0}\int_{-\infty}^{\infty}\left[\mathrm{d}\mathbf{k}(-ik_k)(-ik_l)\exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]\frac{(zz)^{-1}_{ij}}{k^2}\right]\mathrm{d}V(\mathbf{x}') \\ &= -\frac{1}{(2\pi)^3}\int_{V_0}\int \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right](zz)^{-1}_{ij} z_k\, z_l\,\mathrm{d}\mathbf{k}\,\mathrm{d}V(\mathbf{x}') \qquad (18)\end{aligned}$$

Because the integration over the inclusion volume V_0 only depends on $\mathbf{x}'$ but not on $\mathbf{x}$, it is helpful to rearrange integrals as,

$$\begin{aligned}\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\,\exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k\, z_l \int_{V_0}\exp(i\mathbf{k}\cdot\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\ &= -\frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\,\exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k\, z_l\, Q(\mathbf{k}) \qquad (19)\end{aligned}$$

where

$$Q(\mathbf{k}) \equiv \int_{V_0}\exp(i\mathbf{k}\cdot\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \tag{20}$$

Therefore, for an infinite homogeneous medium, the auxiliary tensor $\mathcal{D}_{ijkl}$ also satisfies minor symmetries,

$$\mathcal{D}_{ijkl} = \mathcal{D}_{jikl} = \mathcal{D}_{ijlk} \tag{21}$$

But it does not satisfy the major symmetry, i.e. $\mathcal{D}_{ijkl} \neq \mathcal{D}_{klij}$ (similar to Eshelby's tensor $\mathcal{S}_{ijkl}$).

4 Ellipsoidal inclusion

Now let us restrict our attention to inclusions that are ellipsoidal in shape. The goal is to prove that $\mathcal{D}_{ijkl}(\mathbf{x})$ is a constant inside an ellipsoidal inclusion. The volume V_0 occupied by the inclusion can be expressed as,

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 + \left(\frac{z'}{c}\right)^2 \leq 1 \tag{22}$$

where a, b, c specify the size of the ellipsoid. Define new variables,

$$\begin{aligned} X' &\equiv \frac{x'}{a} && (23)\\ Y' &\equiv \frac{y'}{b} && (24)\\ Z' &\equiv \frac{z'}{c} && (25)\\ \mathbf{R} &\equiv X'\mathbf{e}_1 + Y'\mathbf{e}_2 + Z'\mathbf{e}_3 && (26)\\ R &\equiv |\mathbf{R}| && (27) \end{aligned}$$

Then the integration over V_0 becomes an integration over a unit sphere in the space of $\mathbf{R}$,

$$\int_{V_0} \mathrm{d}V(\mathbf{x}') \quad \Rightarrow \quad abc \int_{|\mathbf{R}|\leq 1} \mathrm{d}\mathbf{R} \tag{28}$$

Also define new variables in Fourier space,

$$\begin{aligned} \lambda_x &\equiv ak_x && (29)\\ \lambda_y &\equiv bk_y && (30)\\ \lambda_z &\equiv ck_z && (31)\\ \boldsymbol{\lambda} &\equiv \lambda_x\mathbf{e}_1 + \lambda_y\mathbf{e}_2 + \lambda_z\mathbf{e}_3 && (32)\\ \lambda &\equiv |\boldsymbol{\lambda}| = \sqrt{a^2k_x^2 + b^2k_y^2 + c^2k_z^2} && (33) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{k}\cdot\mathbf{x}' &= \boldsymbol{\lambda}\cdot\mathbf{R} && (34)\\ Q(\mathbf{k}) &\equiv \int_{V_0} \exp(i\mathbf{k}\cdot\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \\ &= abc\int_{|\mathbf{R}|\leq 1} \exp(i\boldsymbol{\lambda}\cdot\mathbf{R})\,\mathrm{d}\mathbf{R} && (35) \end{aligned}$$

In polar coordinates,

$$
\begin{aligned}
Q(\mathbf{k}) &= abc \int_0^1 \int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi \exp(i\lambda R\cos\phi)\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}R && (36)\\
&= 2\pi abc \int_0^1 \mathrm{d}R\, R^2 \int_{-1}^{1} \mathrm{d}s\, \exp(i\lambda R s) && (37)\\
&= 2\pi abc \int_0^1 R^2 \left[\frac{2\sin(\lambda R)}{\lambda R}\right] \mathrm{d}R && (38)\\
&= 4\pi \frac{abc}{\lambda} \int_0^1 R\sin\lambda R\, \mathrm{d}R && (39)\\
&= 4\pi \frac{abc}{\lambda^3} (\sin\lambda - \lambda\cos\lambda) && (40)
\end{aligned}
$$

Substituting this result into Eq. (19), we have

$$
\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{1}{8\pi^3} \int_{-\infty}^{\infty} \mathrm{d}\mathbf{k}\, \exp(-i\mathbf{k}\cdot\mathbf{x})(zz)^{-1}_{ij} z_k\, z_l\, \frac{4\pi}{\lambda^3} abc(\sin\lambda - \lambda\cos\lambda)\\
&= -\frac{abc}{2\pi^2} \int_{-\infty}^{\infty} (zz)^{-1}_{ij} z_k\, z_l\, \exp(-i\mathbf{k}\cdot\mathbf{x})\, \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\, \mathrm{d}\mathbf{k} && (41)
\end{aligned}
$$

Again we go to polar coordinates. Define new variables Φ, Θ, γ through,

$$
\begin{aligned}
k_x &= k\sin\Phi\cos\Theta && (42)\\
k_y &= k\sin\Phi\sin\Theta && (43)\\
k_z &= k\cos\Phi && (44)\\
\gamma &\equiv (\mathbf{k}\cdot\mathbf{x})/k = x\sin\Phi\cos\Theta + y\sin\phi\sin\Theta + z\cos\Phi && (45)\\
\beta &\equiv \lambda/k = \sqrt{(a^2\cos^2\Theta + b^2\sin^2\Theta)\sin^2\Phi + c^2\cos^2\Phi} && (46)
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{D}_{ijkl}(\mathbf{x}) &= -\frac{abc}{2\pi^2} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} k^2 (zz)^{-1}_{ij} z_k\, z_l\, \exp(-ik\gamma) \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3} \sin\Phi\, \mathrm{d}\Theta\, \mathrm{d}\Phi\, \mathrm{d}k\\
&= -\frac{abc}{2\pi^2} \int_0^{\pi} \int_0^{2\pi} (zz)^{-1}_{ij} z_k\, z_l\, \kappa(\gamma) \sin\Phi\, \mathrm{d}\Theta\, \mathrm{d}\Phi && (47)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa(\gamma) &\equiv \int_0^{\infty} \mathrm{d}k\, k^2 \exp(-ik\gamma) \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\\
&= \int_0^{\infty} \mathrm{d}k\, k^2 \exp(-ik\gamma) \frac{\sin k\beta - k\beta\cos k\beta}{k^3\beta^3}\\
&= \frac{1}{\beta^3} \int_0^{\infty} \mathrm{d}k \exp(-ik\gamma) \left[\frac{\sin k\beta}{k} - \beta\cos k\beta\right] && (48)
\end{aligned}
$$

Notice that the dependence of $\mathcal{D}_{ijkl}$ on $\mathbf{x}$ is through $\gamma = (\mathbf{k}\cdot\mathbf{x})/k$ in $\kappa(\gamma)$. To evaluate $\kappa(\gamma)$, notice that the term in the square bracket is an even function of k. Because $\mathcal{D}_{ijkl}$ is real,

$\kappa(\gamma)$ must be real as well. Therefore, we can rewrite the integral as,

$$\kappa(\gamma) = \frac{1}{2\beta^3}\int_{-\infty}^{\infty} \mathrm{d}k\, \exp(-ik\gamma)\left[\frac{\sin k\beta}{k} - \beta\cos k\beta\right] \tag{49}$$

Notice that

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}k\, \exp(-ik\gamma)\cos k\beta &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-ik\gamma}(\mathrm{e}^{ik\beta} + \mathrm{e}^{-ik\beta}) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k\, \left[\mathrm{e}^{-ik(\gamma-\beta)} + \mathrm{e}^{-ik(\gamma+\beta)}\right] \\
&= \pi\left[\delta(\beta-\gamma) + \delta(\beta+\gamma)\right] && (50) \\
\frac{d}{d\beta}\int_{-\infty}^{\infty} \mathrm{d}k\, \exp(-ik\gamma)\frac{\sin k\beta}{k} &= \int_{-\infty}^{\infty} \mathrm{d}k\, \exp(-ik\gamma)\cos k\beta && (51) \\
\int_{-\infty}^{\infty} \mathrm{d}k\, \exp(-ik\gamma)\frac{\sin k\beta}{k} &= \pi\left[h(\beta-\gamma) + h(\beta+\gamma)\right] && (52)
\end{aligned}$$

where

$$h(\alpha) = \begin{cases} -\frac{1}{2} & \text{if } \alpha < 0 \\ 0 & \text{if } \alpha = 0 \\ \frac{1}{2} & \text{if } \alpha > 0 \end{cases} \tag{53}$$

Therefore, if $\beta \pm \gamma > 0$ then the $\kappa(\gamma)$ reduces to

$$\begin{aligned}
\kappa(\gamma) &= \frac{\pi}{2\beta^3}\left[h(\beta-\gamma) + h(\beta+\gamma) - \beta\delta(\beta-\gamma) - \beta\delta(\beta+\gamma)\right] \\
&= \frac{\pi}{2\beta^3} && (54)
\end{aligned}$$

In other words, $\kappa(\gamma)$ becomes a constant if $\beta \pm \gamma > 0$. In this case, $\mathcal{D}_{ijkl}(\mathbf{x})$ reduces to a surface integral that is independent of $\mathbf{x}$,

$$\mathcal{D}_{ijkl}(\mathbf{x}) = -\frac{abc}{2\pi^2}\int_0^{\pi}\int_0^{2\pi} (zz)^{-1}_{ij}\, z_k\, z_l\, \frac{\pi}{2\beta^3}\, \sin\Phi\, \mathrm{d}\Theta\, \mathrm{d}\Phi \tag{55}$$

We will now show that if $\mathbf{x}$ is within the ellipsoid, then $\beta \pm \gamma > 0$. This will then prove that $\mathcal{D}_{ijkl}$ and $\mathcal{S}_{ijkl}$ are constants within the ellipsoidal inclusion. To see why this is the case, consider vector $\boldsymbol{\rho}$ such that,

$$\boldsymbol{\rho} = \frac{x}{a}\mathbf{e_1} + \frac{y}{b}\mathbf{e_2} + \frac{z}{c}\mathbf{e_3} \tag{56}$$

If $\mathbf{x}$ lies within the ellipsoid, then

$$\rho \equiv |\boldsymbol{\rho}| = \sqrt{(x/a)^2 + (y/b)^2 + (z/c)^2} < 1 \tag{57}$$

At the same time,

$$\begin{aligned}
\gamma &\equiv (\mathbf{k}\cdot\mathbf{x})/k = (\boldsymbol{\lambda}\cdot\boldsymbol{\rho})/k && (58) \\
\beta &\equiv \lambda/k && (59)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
|\gamma| &= |\boldsymbol{\lambda}\cdot\boldsymbol{\rho}|/k \le \lambda\rho/k < \lambda/k = \beta \\
\beta \pm \gamma &> 0
\end{aligned}
\tag{60}
$$

Therefore, when $\mathbf{x}$ lies within the ellipsoid, the $\mathcal{D}_{ijkl}$ tensor can be calculated by simply performing a surface integral over a unit sphere,

$$
\mathcal{D}_{ijkl} = -\frac{abc}{4\pi}\int_0^{\pi}\int_0^{2\pi}(zz)^{-1}_{ij}\,z_k\,z_l\,\frac{\sin\Phi}{\beta^3}\,\mathrm{d}\Theta\,\mathrm{d}\Phi \tag{61}
$$

When $\mathbf{x}$ lies outside the ellipsoid, $\mathcal{D}_{ijkl}$ shall depend on $\mathbf{x}$, and can be calculated directly from the Green's function,

$$
\mathcal{D}_{ijkl}(\mathbf{x}) = -\int_S G_{ij,l}(\mathbf{x}-\mathbf{x}')n_k(\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') \tag{62}
$$

Once D_{ijkl} is obtained, Eshelby's tensor S_{ijkl} can be found by Eq. (13).

5 Eshelby's tensor in isotropic medium

The derivation of the Eshelby tensor in isotropic materials can be found in [1] (for isotropy) and [2]. For isotropic medium, the Eshelby's tensor can be expressed in terms of elliptic integrals. Assuming that $a > b > c$ and that the semi axis a aligns with the coordinate x (and similarly b with y and c with z) then

$$
\begin{aligned}
\mathcal{S}_{1111} &= \frac{3}{8\pi(1-\nu)}a^2 I_{11} + \frac{1-2\nu}{8\pi(1-\nu)}I_1 \\
\mathcal{S}_{1122} &= \frac{1}{8\pi(1-\nu)}b^2 I_{12} + \frac{1-2\nu}{8\pi(1-\nu)}I_1 \\
\mathcal{S}_{1133} &= \frac{1}{8\pi(1-\nu)}c^2 I_{13} + \frac{1-2\nu}{8\pi(1-\nu)}I_1 \\
\mathcal{S}_{1212} &= \frac{a^2+b^2}{16\pi(1-\nu)}I_{12} + \frac{1-2\nu}{16\pi(1-\nu)}(I_1+I_2) \\
\mathcal{S}_{1112} &= \mathcal{S}_{1223} = \mathcal{S}_{1232} = 0
\end{aligned}
$$

The rest of the nonzero terms can be found by cyclic permutation of the above formulas. Notice that we should also let $a \to b \to c$ together with $1 \to 2 \to 3$. The I terms are defined in terms of standard elliptic integrals,

$$
\begin{aligned}
I_1 &= \frac{4\pi abc}{(a^2-b^2)(a^2-c^2)^{1/2}}\left[F(\theta,k) - E(\theta,k)\right] \\
I_3 &= \frac{4\pi abc}{(b^2-c^2)(a^2-c^2)^{1/2}}\left[\frac{b(a^2-c^2)^{1/2}}{ac} - E(\theta,k)\right]
\end{aligned}
$$

where

$$\theta = \arcsin\sqrt{\frac{a^2-c^2}{a^2}}$$
$$k = \sqrt{\frac{a^2-b^2}{a^2-c^2}}$$

and

$$I_1 + I_2 + I_3 = 4\pi$$
$$3I_{11} + I_{12} + I_{13} = \frac{4\pi}{a^2}$$
$$3a^2 I_{11} + b^2 I_{12} + c^2 I_{13} = 3I_1$$
$$I_{12} = \frac{I_2 - I_1}{a^2 - b^2}$$

and the standard elliptic integrals are defined as

$$F(\theta,k) = \int_0^\theta \frac{\mathrm{d}w}{(1-k^2\sin^2 w)^{1/2}} \tag{63}$$
$$E(\theta,k) = \int_0^\theta (1-k^2\sin^2 w)^{1/2}\,\mathrm{d}w \tag{64}$$

For a spherical inclusion ($a = b = c$), Eshelby's tensor has the following compact expression,

$$\mathcal{S}_{ijkl} = \frac{5\nu-1}{15(1-\nu)}\delta_{ij}\delta_{kl} + \frac{4-5\nu}{15(1-\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{65}$$

Notice that tensor itself does not depend on the radius of the sphere!
For an elliptic cylinder ($c \to \infty$)

$$\begin{aligned}
\mathcal{S}_{1111} &= \frac{1}{2(1-\nu)}\left[\frac{b^2+2ab}{(a+b)^2} + (1-2\nu)\frac{b}{a+b}\right] \\
\mathcal{S}_{2222} &= \frac{1}{2(1-\nu)}\left[\frac{a^2+2ab}{(a+b)^2} + (1-2\nu)\frac{a}{a+b}\right] \\
\mathcal{S}_{3333} &= 0 \\
\mathcal{S}_{1122} &= \frac{1}{2(1-\nu)}\left[\frac{b^2}{(a+b)^2} - (1-2\nu)\frac{b}{a+b}\right] \\
\mathcal{S}_{2233} &= \frac{1}{2(1-\nu)}\frac{2\nu a}{a+b} \\
\mathcal{S}_{2211} &= \frac{1}{2(1-\nu)}\left[\frac{a^2}{(a+b)^2} - (1-2\nu)\frac{a}{a+b}\right] \\
\mathcal{S}_{3311} &= \mathcal{S}_{3322} = 0 \\
\mathcal{S}_{1212} &= \frac{1}{2(1-\nu)}\left[\frac{a^2+b^2}{2(a+b)^2} + \frac{(1-2\nu)}{2}\right] \\
\mathcal{S}_{1133} &= \frac{1}{2(1-\nu)}\frac{2\nu b}{a+b} \\
\mathcal{S}_{2323} &= \frac{a}{2(a+b)} \qquad \mathcal{S}_{3131} = \frac{b}{2(a+b)}
\end{aligned}$$

Another important geometry is the flat ellipsoid ($a > b \gg c$). The I integrals in this limiting case reduce to

$$\begin{aligned}
I_1 &= 4\pi(F(k) - E(k))\frac{bc}{a^2 - b^2} \\
I_2 &= 4\pi\left(E(k)\frac{c}{b} - (F(k) - E(k))\,\frac{bc}{a^2 - b^2}\right) \\
I_3 &= 4\pi\left(1 - E(k)\frac{c}{b}\right) \\
I_{12} &= 4\pi\left[E(k)\frac{c}{b} - 2(F(k) - E(k))\frac{bc}{a^2 - b^2}\right]/(a^2 - b^2) \\
I_{23} &= 4\pi\left[1 - 2E(k)\frac{c}{b} + (F(k) - E(k))\frac{bc}{a^2 - b^2}\right]/b^2 \\
I_{31} &= 4\pi\left[1 - E(k)\frac{c}{b} - (F(k) - E(k))\frac{bc}{a^2 - b^2}\right]/a^2 \\
I_{33} &= \frac{4\pi}{3c^2}
\end{aligned}$$

where $E(k)$ and $F(k)$ are complete elliptic integrals defined as

$$F(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}w}{(1 - k^2 \sin^2 w)^{1/2}} \tag{66}$$

$$E(k) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 w)^{1/2}\, \mathrm{d}w \tag{67}$$

Eshelby's tensor for various other shapes can be found in [2] and [4].

References

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