Potential Field of a Uniformly Charged Ellipsoid

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Contents

1 Problem Statement

The purpose of this document is to discuss the derivation of a very useful result, which states that **the potential field of a uniformly charged ellipsoid is a quadratic function inside the ellipsoid** [1]. Specifically, consider an ellipsoid with uniform charge density ρ inside the following region,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

Let V_0 specify the volume of the ellipsoid. The potential field is defined as

$$\phi(\mathbf{x}) \equiv \int_{V_0} \frac{\rho}{|\mathbf{x} - \mathbf{x}'|}\, dV(\mathbf{x}') \tag{2}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{x}' = (x', y', z')$.

If point $\mathbf{x}$ is inside the ellipsoid [2],

$$\phi(x, y, z) = \pi\, abc\, \rho \int_0^\infty \left[1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{c^2 + s}\right] \frac{ds}{\sqrt{\varphi(s)}} \tag{3}$$

where

$$\varphi(s) \equiv (a^2 + s)(b^2 + s)(c^2 + s) \tag{4}$$

If point $\mathbf{x}$ is outside the ellipsoid [1],

$$\phi(x, y, z) = \pi\, abc\, \rho \int_\lambda^\infty \left[1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{c^2 + s}\right] \frac{ds}{\sqrt{\varphi(s)}} \tag{5}$$

where λ is the greatest root of the equation $f(s) = 0$, where

$$f(s) \equiv \frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} + \frac{z^2}{c^2 + s} - 1 \tag{6}$$

The physical significance of function $f(s)$ is that, for each $s > 0$, the equation $f(s) = 0$ defines an ellipsoid (larger than the original ellipsoid), which is an *isosurface* of the potential field $\phi(\mathbf{x})$ generated by the original ellipsoid (with uniform charge density). Therefore, all the value of $s \in [0, \infty)$ corresponds to a family of ellipsoids, called the *confocal family*; the potential field is a constant on each ellipsoid in the family.

Physical significance: As a consequence of the above result, the second spatial derivatives of the potential field is a uniform second order tensor inside the ellipsoid. This is closely analogous to *Eshelby's Theorem*, which states that the stress field inside an ellipsoidal inclusion is uniform [3]. Landau and Lifshitz also used the above result to show that normal stress distribution inside the contact area between two smooth elastic media (Hertzian contact problem) has the ellipsoidal shape [2].

2 Orthogonal Curvilinear Coordinates

The proof of this result is best discussed using *ellipsoidal coordinates*, which is an *orthogonal curvilinear coordinate* system. In this section, we first summarize the major results concerning orthogonal curvilinear coordinates. We will then introduce the *elliptic coordinates* (2D) and *ellipsoidal coordinates* (3D) in the following sections.

Let the Cartesian coordinates be specified by $(x_1, x_2, x_3) = (x, y, z)$. An arbitrary differential length in space ds is specified by $(ds)^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$. Now consider a general orthogonal curvilinear coordinate system, (q_1, q_2, q_3), which are related to the Cartesian coordinates by

$$q_m = q_m(x_1, x_2, x_3), \quad x_m = x_m(q_1, q_2, q_3) \tag{7}$$

An arbitrary differential length in space can be expressed by

$$(ds)^2 = (h_1\, dq_1)^2 + (h_2\, dq_2)^2 + (h_3\, dq_3)^2 \tag{8}$$

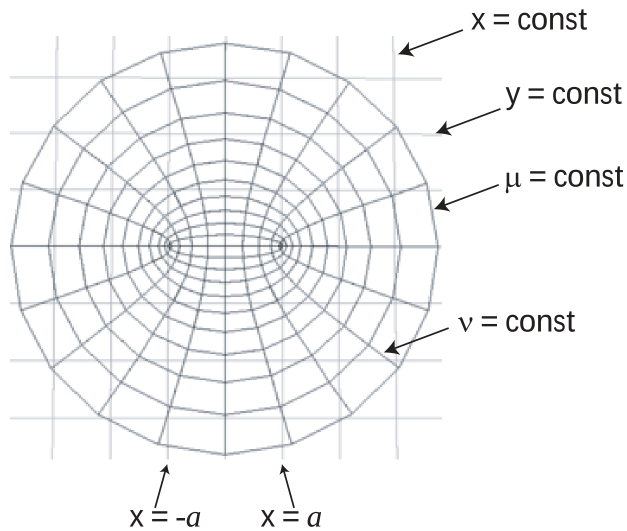

Figure 1: Contour lines in Cartesian and elliptic coordinates [5].

where h_i are called *scale factors* and have the following expressions [4].

$$\begin{aligned}(h_1)^2 &= \frac{\partial x_k}{\partial q_1}\frac{\partial x_k}{\partial q_1}\\ (h_2)^2 &= \frac{\partial x_k}{\partial q_2}\frac{\partial x_k}{\partial q_2}\\ (h_3)^2 &= \frac{\partial x_k}{\partial q_3}\frac{\partial x_k}{\partial q_3}\end{aligned} \tag{9}$$

The index notation is used here and k is a dummy index that is summed from 1 to 3. Let $\mathbf{e}_k$ be the basis vectors of the Cartesian coordinates (x_1, x_2, x_3) and let $\hat{\mathbf{e}}_k$ be the basis vectors of the curvilinear coordinates (q_1, q_2, q_3). The *gradient* of a scalar field $\phi(q_1, q_2, q_3)$ is,

$$\nabla\phi = \hat{\mathbf{e}}_1\frac{1}{h_1}\frac{\partial\phi}{\partial q_1} + \hat{\mathbf{e}}_2\frac{1}{h_2}\frac{\partial\phi}{\partial q_2} + \hat{\mathbf{e}}_3\frac{1}{h_3}\frac{\partial\phi}{\partial q_3} \tag{10}$$

The *Laplacian* of a scalar field $\phi(q_1, q_2, q_3)$ is,

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right] \tag{11}$$

In 2-dimension, the *Laplacian* of a scalar field $\phi(q_1, q_2)$ reduces to the following.

$$\nabla^2\phi = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right)\right] \tag{12}$$

3 Elliptic Coordinates

The most common definition of elliptic coordinate (μ, ν) is [5],

$$\begin{aligned}x &= a\cosh\mu\,\cos\nu\\ y &= a\sinh\mu\,\sin\nu\end{aligned}$$

With this definition, we can show that

$$
\begin{aligned}
\frac{x^2}{a^2\cosh^2\mu} + \frac{x^2}{a^2\sinh^2\mu} &= \cos^2\nu + \sin^2\nu = 1 \\
\frac{x^2}{a^2\cos^2\nu} - \frac{x^2}{a^2\sin^2\nu} &= \cosh^2\mu - \sinh^2\mu = 1
\end{aligned}
\tag{13}
$$

Therefore, the contour lines of $\mu =$ const form a set of ellipses, and the contour lines of $\nu =$ const form a set of hyperbolas, as shown in Fig. 1. This figure also shows that in the limit of $a \to 0$, or when the distance from the origin is much greater than a, the elliptic coordinates becomes very close to polar coordinates.

The Jacobian matrix between Cartesian coordinates (x, y) and elliptic coordinates (μ, ν) is

$$
J \equiv \begin{bmatrix} \frac{\partial x}{\partial \mu} & \frac{\partial x}{\partial \nu} \\ \frac{\partial x}{\partial \mu} & \frac{\partial x}{\partial \nu} \end{bmatrix} = \begin{bmatrix} a\sinh\mu\cos\nu & -a\cosh\mu\sin\nu \\ a\cosh\mu\sin\nu & a\sinh\mu\cos\nu \end{bmatrix}
\tag{14}
$$

The elliptic coordinates is an orthogonal coordinate system because the two columns of matrix J are orthogonal to each other. The *scale factors* are

$$
\begin{aligned}
h_\mu &= \sqrt{\left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2} \\
h_\nu &= \sqrt{\left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2}
\end{aligned}
\tag{15}
$$

It is easy to show that

$$
h_\mu = h_\nu = a\sqrt{\sinh^2\mu + \sin^2\nu} = \sqrt{\det(J)}
\tag{16}
$$

Therefore, the Laplacian of a scalar field ϕ is

$$
\begin{aligned}
\nabla^2\phi &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x, y) \\
&= \frac{1}{a^2(\sinh^2\mu + \sin^2\nu)}\left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2}\right)\phi(\mu, \nu)
\end{aligned}
\tag{17}
$$

4 Alternative Definition of Elliptic Coordinates

An alternative definition of elliptic coordinates makes it more natural to generalize the concept to ellipsoidal coordinates in 3D. Consider an ellipse

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\tag{18}
$$

We will assume $a > b$ without loss of generality. Now consider a family of curves defined by

$$
f(s) \equiv \frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} - 1 = 0
\tag{19}
$$

When $s > -b^2$, it defines an ellipse. When $-b^2 > s > -a^2$, it defines a hyperbola.

For a given (x, y), let (μ, ν) be the two largest roots of the equation $f(s) = 0$. There is a one-to-one correspondence between (x, y) and (μ, ν), if we assume $x > 0$, $y > 0$. Specifically,

$$\begin{aligned} x^2 &= \frac{(a^2+\mu)(a^2+\nu)}{a^2-b^2} \\ y^2 &= \frac{(b^2+\mu)(b^2+\nu)}{b^2-a^2} \end{aligned} \tag{20}$$

This relationship can be verified by plugging it into the definition of $f(s)$,

$$\begin{aligned} f(\mu) &= \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} - 1 = \frac{a^2+\nu}{a^2-b^2} + \frac{b^2+\nu}{b^2-a^2} - 1 = 0 \\ f(\nu) &= \frac{x^2}{a^2+\nu} + \frac{y^2}{b^2+\nu} - 1 = \frac{a^2+\mu}{a^2-b^2} + \frac{b^2+\mu}{b^2-a^2} - 1 = 0 \end{aligned} \tag{21}$$

The four components of the Jacobian matrix can be obtained.

$$\begin{aligned} \frac{\partial x}{\partial \mu} &= \frac{1}{2}\left[\frac{a^2+\nu}{(a^2+\mu)(a^2-b^2)}\right]^{1/2} \qquad \frac{\partial x}{\partial \nu} = \frac{1}{2}\left[\frac{a^2+\mu}{(a^2+\nu)(a^2-b^2)}\right]^{1/2} \\ \frac{\partial y}{\partial \mu} &= \frac{1}{2}\left[\frac{b^2+\nu}{(b^2+\mu)(b^2-a^2)}\right]^{1/2} \qquad \frac{\partial y}{\partial \nu} = \frac{1}{2}\left[\frac{b^2+\mu}{(b^2+\nu)(b^2-a^2)}\right]^{1/2} \end{aligned} \tag{22}$$

The (μ, ν) coordinate system is orthogonal because

$$\frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \nu} = 0 \tag{23}$$

Define function $\varphi(s) \equiv (a^2+s)(b^2+s)$. The *scale factors* can be expressed as

$$\begin{aligned} h_\mu &= \frac{1}{2}\sqrt{\frac{\mu-\nu}{\varphi(\mu)}} \\ h_\nu &= \frac{1}{2}\sqrt{\frac{\nu-\mu}{\varphi(\nu)}} \end{aligned} \tag{24}$$

Therefore, the Laplacian of a scalar field $\phi(\mu, \nu)$ is

$$\begin{aligned} \nabla^2\phi &= \frac{1}{h_\mu h_\nu}\left[\frac{\partial}{\partial \mu}\left(\frac{h_\nu}{h_\mu}\frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{h_\mu}{h_\nu}\frac{\partial \phi}{\partial \nu}\right)\right] \\ &= \frac{4\sqrt{\varphi(\mu)\varphi(\nu)}}{\mu-\nu}\left[\frac{\partial}{\partial \mu}\left(\sqrt{\frac{\varphi(\mu)}{\varphi(\nu)}}\frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\sqrt{\frac{\varphi(\mu)}{\varphi(\nu)}}\frac{\partial \phi}{\partial \nu}\right)\right] \\ &= \frac{4}{\mu-\nu}\left[\sqrt{\varphi(\mu)}\frac{\partial}{\partial \mu}\left(\sqrt{\varphi(\mu)}\frac{\partial \phi}{\partial \mu}\right) + \sqrt{\varphi(\nu)}\frac{\partial}{\partial \nu}\left(\sqrt{\varphi(\nu)}\frac{\partial \phi}{\partial \nu}\right)\right] \end{aligned} \tag{25}$$

For derivation details see `elliptic_coord.m`.

5 Ellipsoidal Coordinates

Generalizing the elliptic coordinates defined above, we obtain the ellipsoidal coordinates [6]. Consider an ellipsoid, Consider an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{26}$$

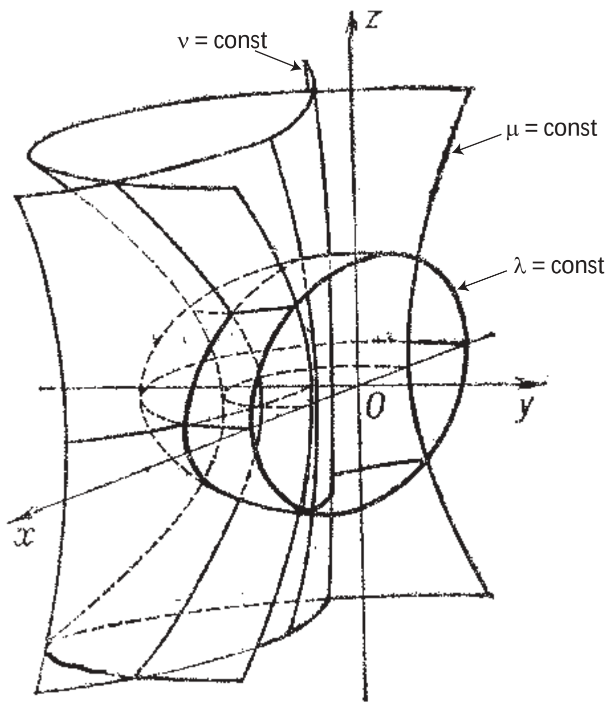

Figure 2: Isosurfaces of ellipsoidal coordinates [7].

We will assume $a > b > c$ without loss of generality. Now consider a family of curves defined by

$$f(s) \equiv \frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s} - 1 = 0 \tag{27}$$

For $\lambda > -c^2$, $f(\lambda) = 0$ defines an ellipsoid. When $-c^2 > \mu > -b^2$, $f(\mu) = 0$ defines a one-sheet hyperbola. When $-b^2 > \nu > -a^2$, $f(\nu) = 0$ defines a two-sheet hyperbola, as shown in Fig. 2.

For a given (x, y, z), let (λ, μ, ν) be the three largest roots of equation $f(s) = 0$. There is a one-to-one correspondence between (x, y, z) and (λ, μ, ν), if we assume $x > 0$, $y > 0$, $z > 0$.

$$\begin{aligned} x^2 &= \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} \\ y^2 &= \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-a^2)(b^2-c^2)} && (28) \\ z^2 &= \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)} && (29) \end{aligned}$$

The following limit applies,

$$\lambda > -c^2 > \mu > -b^2 > \nu > -a^2 \tag{30}$$

The nine components of the Jacobian matrix can be obtained.

$$
\begin{aligned}
\frac{\partial x}{\partial \lambda} &= \frac{1}{2}\left[\frac{(a^2+\mu)(a^2+\nu)}{(a^2+\lambda)(a^2-b^2)(a^2-c^2)}\right]^{1/2} \\
\frac{\partial x}{\partial \mu} &= \frac{1}{2}\left[\frac{(a^2+\lambda)(a^2+\nu)}{(a^2+\mu)(a^2-b^2)(a^2-c^2)}\right]^{1/2} \\
\frac{\partial x}{\partial \nu} &= \frac{1}{2}\left[\frac{(a^2+\lambda)(a^2+\mu)}{(a^2+\nu)(a^2-b^2)(a^2-c^2)}\right]^{1/2} \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial y}{\partial \lambda} &= \frac{1}{2}\left[\frac{(b^2+\mu)(b^2+\nu)}{(b^2+\lambda)(b^2-a^2)(b^2-c^2)}\right]^{1/2} \\
\frac{\partial y}{\partial \mu} &= \frac{1}{2}\left[\frac{(b^2+\lambda)(b^2+\nu)}{(b^2+\mu)(b^2-a^2)(b^2-c^2)}\right]^{1/2} \\
\frac{\partial y}{\partial \nu} &= \frac{1}{2}\left[\frac{(b^2+\lambda)(b^2+\mu)}{(b^2+\nu)(b^2-a^2)(b^2-c^2)}\right]^{1/2} \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial \lambda} &= \frac{1}{2}\left[\frac{(c^2+\mu)(c^2+\nu)}{(c^2+\lambda)(c^2-a^2)(c^2-b^2)}\right]^{1/2} \\
\frac{\partial z}{\partial \mu} &= \frac{1}{2}\left[\frac{(c^2+\lambda)(c^2+\nu)}{(c^2+\mu)(c^2-a^2)(c^2-b^2)}\right]^{1/2} \\
\frac{\partial z}{\partial \nu} &= \frac{1}{2}\left[\frac{(c^2+\lambda)(c^2+\mu)}{(c^2+\nu)(c^2-a^2)(c^2-b^2)}\right]^{1/2} \qquad (33)
\end{aligned}
$$

The (λ, μ, ν) coordinate system is orthogonal because

$$
\begin{aligned}
\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \mu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \mu} &= 0 \\
\frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial \mu}\frac{\partial z}{\partial \nu} &= 0 \\
\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \nu} &= 0 \qquad (34)
\end{aligned}
$$

Define function $\varphi(s) \equiv (a^2+s)(b^2+s)(c^2+s)$. The *scale factors* can be expressed as

$$
\begin{aligned}
h_\lambda &= \frac{1}{2}\sqrt{\frac{(\lambda-\mu)(\lambda-\nu)}{\varphi(\lambda)}} \\
h_\mu &= \frac{1}{2}\sqrt{\frac{(\mu-\lambda)(\mu-\nu)}{\varphi(\mu)}} \\
h_\nu &= \frac{1}{2}\sqrt{\frac{(\nu-\lambda)(\nu-\mu)}{\varphi(\nu)}} \qquad (35)
\end{aligned}
$$

The Laplacian of a scalar field $\phi(\lambda, \mu, \nu)$ is,

$$
\begin{aligned}
\nabla^2\phi &= \frac{1}{h_\lambda h_\mu h_\nu}\left[\frac{\partial}{\partial \lambda}\left(\frac{h_\mu h_\nu}{h_\lambda}\frac{\partial \phi}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{h_\nu h_\lambda}{h_\mu}\frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{h_\lambda h_\mu}{h_\nu}\frac{\partial \phi}{\partial \nu}\right)\right] \\
&= \frac{4\sqrt{\varphi(\lambda)}}{(\lambda-\mu)(\lambda-\nu)}\frac{\partial}{\partial \lambda}\left(\sqrt{\varphi(\lambda)}\,\frac{\partial \phi}{\partial \lambda}\right) + \frac{4\sqrt{\varphi(\mu)}}{(\mu-\lambda)(\mu-\nu)}\frac{\partial}{\partial \mu}\left(\sqrt{\varphi(\mu)}\,\frac{\partial \phi}{\partial \mu}\right) \\
&\quad + \frac{4\sqrt{\varphi(\nu)}}{(\nu-\lambda)(\nu-\mu)}\frac{\partial}{\partial \nu}\left(\sqrt{\varphi(\nu)}\,\frac{\partial \phi}{\partial \nu}\right) \qquad (36)
\end{aligned}
$$

For derivation details see `ellipsoidal_coord.m`.

6 Ellipsoidal Conductor

The discussion from here on follows Kellogg's book closely [1], with some variables renamed to follow the notation here. Suppose we would like to solve for the potential function in space produced by an ellipsoidal conductor that contains surface charges [1]. For a perfect conductor, the potential on its surface (as well as the interior) is a constant. Therefore, we are trying to solve the Poisson's equation,

$$\nabla^2 \phi(\mathbf{x}) = 0 \tag{37}$$

subject to the boundary condition that $\phi(\mathbf{x}) = \phi_0$ when point $\mathbf{x}$ is on the ellipsoidal surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{38}$$

and that $\phi(\mathbf{x}) = 0$ as $|\mathbf{x}| \to \infty$.

Introducing the ellipsoidal coordinates (λ, μ, ν) as defined in the previous section. The surface of the (original) ellipsoid is simply the isosurface of $\lambda = 0$. The limit of $|\mathbf{x}| \to \infty$ corresponds to the limit of $\lambda \to \infty$. Therefore, when ϕ is expressed in term of the ellipsoidal coordinates, i.e. $\phi(\lambda, \mu, \nu)$, the boundary condition is very simple,

$$\begin{aligned} \phi(\lambda = 0, \mu, \nu) &= \phi_0 \\ \phi(\lambda \to \infty, \mu, \nu) &= 0 \end{aligned} \tag{39}$$

Notice that the Laplacian in the Possion's equation ($\nabla^2 \phi = 0$) in the elliptical coordinates is defined in Eq. (36). A natural trial solution is a function $\phi(\lambda)$ that only depends on λ, but not on μ or ν. In this case,

$$\nabla^2 \phi = \frac{4\sqrt{\varphi(\lambda)}}{(\lambda - \mu)(\lambda - \nu)} \frac{\partial}{\partial \lambda} \left(\sqrt{\varphi(\lambda)} \frac{\partial \phi}{\partial \lambda} \right) = 0 \tag{40}$$

$$\begin{aligned} \sqrt{\varphi(\lambda)} \frac{\partial \phi}{\partial \lambda} &= -\frac{E}{2} \\ \frac{\partial \phi}{\partial \lambda} &= -\frac{E}{2\sqrt{\varphi(\lambda)}} \\ \phi(\lambda) &= \int_\lambda^\infty \frac{E\, ds}{2\sqrt{\varphi(s)}} \end{aligned} \tag{41}$$

where E is a constant. The potential field inside the conductor is a constant and equals to the potential on the surface ($\lambda = 0$), which is,

$$\phi_0 = \int_0^\infty \frac{E\, ds}{2\sqrt{\varphi(s)}} \tag{42}$$

The surface charge of the conductor $\sigma(\mathbf{x})$ can be obtained from the following relationship.

$$\frac{\partial \phi(\mathbf{x})}{\partial n_+} = -4\pi\sigma(\mathbf{x}) \tag{43}$$

where $\frac{\partial}{\partial n_+}$ is the gradient along the surface normal.

$$\begin{aligned}
\sigma &= -\frac{1}{4\pi}\frac{\partial\phi(\mathbf{x})}{\partial n_+} \\
&= -\frac{1}{4\pi}\left(\frac{1}{h_\lambda}\frac{\partial\phi(\lambda)}{\partial\lambda}\right)_{\lambda=0}
\end{aligned} \tag{44}$$

Notice that at $\lambda = 0$, $h_\lambda = \sqrt{\mu\nu/\varphi(\lambda)}/2$. Therefore,

$$\sigma = \frac{1}{4\pi}\left(\frac{2\sqrt{\varphi(\lambda)}}{\sqrt{\mu\nu}}\frac{E}{2\varphi(\lambda)}\right) = \frac{E}{4\pi\sqrt{\mu\nu}} \tag{45}$$

On the surface of the ellipsoid, $\lambda = 0$,

$$\mu\nu = a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) \tag{46}$$

The equation of the plane tangent to ellipsoid at point (x, y, z) is

$$(X-x)\frac{x}{a^2}+(Y-y)\frac{y}{b^2}+(Z-z)\frac{z}{c^2} = 0 \tag{47}$$

The surface normal of the tangent plane is

$$\mathbf{n} = \left(\frac{x}{a^2}\,,\ \frac{y}{b^2}\,,\ \frac{z}{c^2}\right)\Big/\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}} \tag{48}$$

The shortest distance from the origin to this plane is

$$p = \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} = \frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} = \frac{abc}{\sqrt{\mu\nu}} \tag{49}$$

Therefore, the surface charge density is

$$\sigma = \frac{E}{4\pi\,abc}\,p = \frac{E}{4\pi\,abc}\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} \tag{50}$$

This result is related to the problem of a uniformly charged ellipsoid. As will be shown in the following section, the above expression of σ is exactly the amount of charge contained in a thin shell between two similar ellipsoids, in the limit of shell thickness going to zero.

7 Ellipsoidal Shell

Consider a set of similar ellipsoids,

$$\frac{x^2}{(au)^2}+\frac{y^2}{(bu)^2}+\frac{z^2}{(cu)^2} = 1\ , \quad \text{equivalently} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = u^2 \tag{51}$$

whose semi-axes are au, bu and cu. They are simply the original ellipsoid scaled by a factor u in all three directions. Notice that this family of ellipsoids are different from the family of ellipsoids defined by $f(\lambda) = 0$ (whose shapes are not similar to each other). For $0 < u < 1$, these ellipsoids are smaller than the original ellipsoid (while for $0 < \lambda < \infty$ the ellipsoids defined by $f(\lambda) = 0$ are all larger than the original ellipsoid).

Consider an ellipsoidal shell contained between two ellipsoids defined by u_1 and $u_2 = u_1 + \Delta u$. Let the volume density of the charge distribution inside the shell to be a constant ρ. In the limit of $\Delta u \to 0$, the shell reduces to a surface with a surface charge density σ. Obviously, the surface charge density is proportional to the local thickness of the shell, Δh, i.e.,

$$\sigma = \rho\,\Delta h \tag{52}$$

Let (ux, uy, uz) be a point on the surface of the ellipsoid defined by u. Let $p(x, y, z, u)$ be the shortest distance from the origin to the plane tangent to the ellipsoid at point (ux, uy, uz). Then,

$$\begin{aligned} p(x,y,z,u) &= \frac{u}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} \\ \Delta h(x,y,z) &= \frac{\Delta u}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} \\ \sigma &= \frac{\rho\;\Delta u}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} \end{aligned} \tag{53}$$

Compare this with Eq. (50), we can conclude that the surface charge of the shell is the same as the equilibrium surface charge of a ellipsoidal conductor. The correspondence is made complete if we set $u_1 = 1$, $u_2 = 1 + \Delta u$, and

$$\begin{aligned} \frac{E}{4\pi abc} &= \rho\Delta u \\ E &= 4\pi\, abc\, \rho\Delta u \end{aligned} \tag{54}$$

This means that the potential field produced by this ellipsoidal shell ($u_1 = 1$, $u_2 = 1 + \Delta u$) is

$$\phi(\mathbf{x}) = \phi(\lambda) = 2\pi\, abc\, \rho\, \Delta u \int_\lambda^\infty \frac{ds}{\sqrt{\varphi(s)}} \tag{55}$$

The potential field inside the ellipsoidal shell is a constant and equals to the potential on the surface ($\lambda = 0$), which is,

$$\phi_0 = 2\pi\, abc\, \rho\, \Delta u \int_0^\infty \frac{ds}{\sqrt{\varphi(s)}} \tag{56}$$

In summary, Eq. (55) describes the potential generated by an ellipsoidal shell with uniform density ρ, whose boundary is the original ellipsoid and a similar ellipsoid scaled by a factor $(1+\Delta u)$. This result can be generalized to an ellipsoidal shell between $u_1 = u$ and $u_2 = u + \Delta u$ for arbitrary u. The potential field at a point $\mathbf{x} = (x, y, z)$ outside this shell is,

$$\phi(\mathbf{x}) = 2\pi\, abc\, \rho\, u^2\, \Delta u \int_{\lambda(u)}^\infty \frac{ds}{\sqrt{\varphi(u,s)}} \tag{57}$$

where $\lambda(u)$ is the greatest root of equation $f(u, s) = 0$ for given (x, y, z) and u. $f(u, s)$ and $\varphi(u, s)$ are generalization of the previously defined functions $f(s)$ and $\varphi(s)$.

$$f(u,s) \equiv \frac{x^2}{a^2u^2+\lambda} + \frac{y^2}{b^2u^2+\lambda} + \frac{z^2}{c^2u^2+\lambda} - 1 \tag{58}$$

$$\varphi(u,s) \equiv (a^2u^2+s)(b^2u^2+s)(c^2u^2+s) \tag{59}$$

The factor u^2 in Eq. (57) accounts for the fact that surface area of the scaled shell and hence its total charge content is u^2 times those of the original shell ($u = 1$).

8 Uniformly Charged Ellipsoid

A uniformly charged ellipsoid can be considered as a collection of many layers of ellipsoid shells considered above. For a point $\mathbf{x}$ outside the ellipsoid, its potential value should be an integral of u from 0 to 1,

$$U_{\mathrm{e}}(\mathbf{x}) = 2\pi\, abc\, \rho \int_0^1 u^2 \int_{\lambda(u)}^{\infty} \frac{ds}{\sqrt{\varphi(u,s)}}\, du \tag{60}$$

Define new variables $v \equiv \lambda(u)/u^2$ and $t \equiv s/u^2$. Then $\phi(u,s) = u^2\phi(t)$.

$$U_{\mathrm{e}}(\mathbf{x}) = 2\pi\, abc\, \rho \int_0^1 u \int_{v}^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du \tag{61}$$

Perform integration by parts on $\int du$,

$$\int_0^1 u \int_v^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du = \left[\frac{u^2}{2}\int_v^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\right]_0^1 + \frac{1}{2}\int_0^1 u^2 \frac{1}{\sqrt{\varphi(v)}}\frac{dv}{du}\, du \tag{62}$$

Notice that v is the greatest root of equation,

$$\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} = u^2 \tag{63}$$

For $u = 1$, $v = \lambda$, while for $u \to 0$, $v \to \infty$. Hence

$$\left[\frac{u^2}{2}\int_v^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\right]_0^1 = \frac{1}{2}\int_{\lambda}^{\infty} \frac{dt}{\sqrt{\varphi(t)}} \tag{64}$$

$$\begin{aligned} \frac{1}{2}\int_0^1 u^2 \frac{1}{\sqrt{\varphi(v)}}\frac{dv}{du}\, du &= \frac{1}{2}\int_{\infty}^{\lambda} u^2 \frac{1}{\sqrt{\varphi(v)}}\, dv \\ &= -\frac{1}{2}\int_{\lambda}^{\infty} \left(\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v}\right) \frac{1}{\sqrt{\varphi(v)}}\, dv \end{aligned} \tag{65}$$

Therefore, the potential outside the ellipsoid is

$$U_{\mathrm{e}}(\mathbf{x}) = \pi\, abc\, \rho \int_{\lambda}^{\infty} \left(1 - \frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v}\right) \frac{1}{\sqrt{\varphi(v)}}\, dv \tag{66}$$

where λ is the greatest root of equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \tag{67}$$

as previously defined.

Let us now consider the potential field at a point $\mathbf{x} = (x, y, z)$ inside the ellipsoid. Here we have to cut the ellipsoid into two parts. Point $\mathbf{x}$ is on the outside of part 1, but is on the inside of part 2. Let u_0 correspond to the ellipsoid that pass through point $\mathbf{x}$, i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = u_0^2 \tag{68}$$

Therefore, part 1 contains the ellipsoidal shells with $0 < u < u_0$ and part 2 contains the ellipsoidal shells with $u_0 < u < 1$. The potential at point $\mathbf{x}$ from an ellipsoidal shell in part 1 has the same expression as the above. But potential at point $\mathbf{x}$ from an ellipsoidal shell in part 2 equals to the potential value at the shell surface (because point $\mathbf{x}$ is inside the shell). Therefore, the total potential at an interior point $\mathbf{x}$ is,

$$U_{\rm i}(\mathbf{x}) = 2\pi\, abc\, \rho \left[\int_0^{u_0} u \int_v^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du + \int_{u_0}^{1} u \int_0^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du \right] \tag{69}$$

Perform integration by parts on the first integral,

$$\int_0^{u_0} u \int_v^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du = \left[\frac{u^2}{2} \int_v^{\infty} \frac{dt}{\sqrt{\varphi(t)}} \right]_0^{u_0} + \frac{1}{2} \int_0^{u_0} u^2 \frac{1}{\sqrt{\varphi(v)}} \frac{dv}{du}\, du \tag{70}$$

Notice that when $u = 0$, $v = \infty$, and when $u = u_0$, $v = 0$. Hence

$$\begin{aligned} \int_0^{u_0} u \int_v^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du &= \frac{u_0^2}{2} \int_0^{\infty} \frac{dt}{\sqrt{\varphi(t)}} - \frac{1}{2} \int_0^{\infty} u^2 \frac{1}{\sqrt{\varphi(v)}}\, dv \\ &= \frac{u_0^2}{2} \int_0^{\infty} \frac{dt}{\sqrt{\varphi(t)}} - \frac{1}{2} \int_0^{\infty} \left(\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} \right) \frac{1}{\sqrt{\varphi(v)}}\, dv \end{aligned}$$

The second integral in Eq. (69) can be carried out because the inner integral is a constant.

$$\int_{u_0}^{1} u \int_0^{\infty} \frac{dt}{\sqrt{\varphi(t)}}\, du = \frac{1-u_0^2}{2} \int_0^{\infty} \frac{dt}{\sqrt{\varphi(t)}} \tag{71}$$

Therefore, the total potential at an interior point $\mathbf{x}$ is,

$$U_{\rm i}(\mathbf{x}) = \pi\, abc\, \rho \int_0^{\infty} \left(1 - \frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} \right) \frac{1}{\sqrt{\varphi(v)}}\, dv \tag{72}$$

Notice that the only difference between Eq. (66) and Eq. (72) is in the lower limit of integration (λ versus 0). Because the range of integration is constant, the potential field inside the ellipsoid, $U_{\rm i}(\mathbf{x})$, is simply a quadratic function of space.

9 Special Cases

9.1 Uniformly Charged Sphere

The situation of $a = b = c$ describes a sphere with radius a. When the sphere has a uniform charge density ρ, the potential distribution inside the sphere is,

$$\phi(x, y, z) = \pi\, a^3\, \rho\, (A - B\, x^2 - B\, y^2 - B\, z^2) = \pi\, a^3\, \rho\, (A - B\, r^2) \tag{73}$$

where

$$A = \int_0^{\infty} \frac{ds}{(a^2+s)^{3/2}} = \frac{2}{a} \tag{74}$$

$$B = \int_0^{\infty} \frac{ds}{(a^2+s)^{5/2}} = \frac{2}{3a^3} \tag{75}$$

Therefore

$$\phi(r) = \frac{2\pi\rho}{3}(3a^2 - r^2) \tag{76}$$

The potential value on the sphere surface is

$$\phi(r=a) = \frac{2\pi\rho}{3}(3a^2 - a^2) = \frac{4\pi\rho}{3}a^2 \tag{77}$$

Notice that $Q = 4\pi\, a^3\rho/3$ is the total charge contained in the sphere. Hence

$$\phi(r=a) = \frac{Q}{a} \tag{78}$$

This is equivalent to the potential produced by a point charge at origin. This confirms Newton's Theorem, which states that the potential field outside a uniformly charged sphere is equivalent to that produced by a point charge located at the center of the sphere. The potential field outside the sphere ($r > a$) is

$$\phi(r) = \frac{Q}{r} \tag{79}$$

9.2 Charged Metal Disc

Consider the case of $a = b$ and $c \to 0$. In this limit, the region inside the ellipsoid reduces to a circle of radius a in the x-y plane ($z = 0$). The potential distribution inside this area is,

$$\phi(x,y) \;=\; \iiint \frac{\rho\, dx'\, dy'\, dz'}{\sqrt{(x-x')^2 + (y-y')^2 + z'^2}}$$

Because the integration limit for z' is $\pm\, c\sqrt{1 - (x'/a)^2 - (y'/a)^2}$,

$$\phi(x,y) \;=\; 2\rho\, c \iint_{x'^2+y'^2 \le a^2} \frac{dx'\, dy'}{\sqrt{(x-x')^2 + (y-y')^2}} \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \tag{80}$$

Using the results obtained above, the potential must be a quadratic function inside the area of radius a,

$$\begin{aligned}
\phi(x,y) &= \pi\, a^2\, c\, \rho\, (A - B\, x^2 - B\, y^2) \\
A &= \int_0^\infty \frac{ds}{(a^2+s)\sqrt{s}} = \frac{\pi}{a} && (81) \\
B &= \int_0^\infty \frac{ds}{(a^2+s)^2\sqrt{s}} = \frac{\pi}{2a^3} && (82)
\end{aligned}$$

In other words, we have obtained the following relationship,

$$\begin{aligned}
\iint_{x'^2+y'^2 \le a^2} \frac{dx'\, dy'}{\sqrt{(x-x')^2 + (y-y')^2}} \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \;&=\; \frac{\pi}{2}\, a^2\, (A - B\, x^2 - B\, y^2) \\
&=\; \frac{\pi^2}{4\, a}(2a^2 - r^2) && (83)
\end{aligned}$$

where $r = \sqrt{x^2 + y^2}$. The potential value is maximum at the circumference of the circle, $r = a$,

$$\phi(r=a) = 2\rho\, c\, \frac{\pi^2}{4\, a}\, a^2 = \frac{\pi\, Q}{3\, a} \tag{84}$$

where $Q = \frac{3}{4}\pi\, a^2\, c\, \rho$ is the total charge in the ellipsoid.

The results obtained here can be used to model a very thin circular metal (conductor) plate with radius a. The total charge in the metal plate is Q, whereas the equilibrium charge distribution on the surface is

$$\sigma(x, y) = \frac{3\, Q}{2\pi\, a^2} \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \tag{85}$$

The potential on the metal surface is a constant

$$\phi_0 = \frac{\pi\, Q}{3\, a} \tag{86}$$

10 Application to Hertz Contact Problem

For simplicity, let us consider a rigid sphere of radius R indenting an elastic half space. The discussion here follows closely that in Landau and Lifshitz [2]. Choose the coordinate system such that the elastic half space occupies the $z \leq 0$ domain and the $z = 0$ plane is the surface of the half space. The Boussinesq solution tells us about the surface displacement of the elastic half space in response to a point force of magnitude F acting at the origin in the $-z$ direction.

$$u_z = -\frac{F(1-\nu^2)}{\pi\, E}\, \frac{1}{r} \tag{87}$$

where $E = 2\mu(1+\nu)$ is the Young's modulus of the elastic half space.

The shape of the indentor can be described by function

$$u_0(x, y) = \frac{x^2}{2R} + \frac{y^2}{2R} \tag{88}$$

Let d be the indentation depth. Hence inside the region of contact (S), the surface displacement of the half space is

$$u_z(x, y) = -d + u_0(x, y) = -d + \frac{x^2}{2R} + \frac{y^2}{2R} \tag{89}$$

Let $p_z(x, y)$ be the normal stress on the surface of the half space. Inside the region of contact, $p_z(x, y) < 0$; outside the region of contact, $p_z(x, y) = 0$. The total indenting force F is,

$$F = \iint_S -p_z(x, y)\, dx\, dy \tag{90}$$

Using the Boussinesq solution, the surface stress $p_z(x, y)$ and the surface displacement $u_z(x, y)$ are related by,

$$u_z(x, y) = \frac{1-\nu^2}{\pi\, E} \iint_S \frac{p_z(x', y')}{\sqrt{(x-x')^2 + (y-y')^2}}\, dx'\, dy' \tag{91}$$

Therefore, our task is to find a function $p_z(x, y)$ that satisfies the following condition,

$$\iint_S \frac{p_z(x', y')}{\sqrt{(x-x')^2 + (y-y')^2}}\, dx'\, dy' = \frac{\pi\, E}{1-\nu^2} \left(-d + \frac{x^2}{2R} + \frac{y^2}{2R}\right) \tag{92}$$

By symmetry, we expect the contact area to be a circle, and let a be its radius. Motivated by Eq. (83), we postulate the following form for $p_z(x,y)$,

$$p_z(x,y) = -p_0 \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \tag{93}$$

The constant p_0 is related with F by

$$\begin{aligned} F &= p_0 \iint_S \sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2}\, dx\, dy \\ &= p_0 \frac{2\,\pi\, a^2}{3} \\ p_0 &= \frac{3\,F}{2\,\pi\, a^2} \end{aligned} \tag{94}$$

Plug the expression of $p_z(x,y)$ into Eq. (92), and using Eq. (83), we have

$$\begin{aligned} -p_0 \frac{\pi^2}{4\,a} (2a^2 - r^2) &= \frac{\pi\, E}{1-\nu^2} \left(-d + \frac{r^2}{2R}\right) \\ -\frac{3\,F}{2\,\pi\, a^2} \frac{\pi^2}{4\,a} (2a^2 - r^2) &= \frac{\pi\, E}{1-\nu^2} \left(-d + \frac{r^2}{2R}\right) \\ \frac{3\,(1-\nu^2)\,F}{8\,E\,a^3} (-2a^2 + r^2) &= -d + \frac{r^2}{2R} \end{aligned} \tag{95}$$

Therefore,

$$\begin{aligned} \frac{1}{R} &= \frac{3\,(1-\nu^2)\,F}{4\,E\,a^3} \\ a &= \left[\frac{3\,(1-\nu^2)\,F\,R}{4\,E}\right]^{1/3} \end{aligned} \tag{96}$$

The indentation depth is

$$d = \frac{3\,(1-\nu^2)\,F}{8\,E\,a^3}\, 2a^2 = \frac{3\,(1-\nu^2)\,F}{4\,E\,a} = \left[\frac{3\,(1-\nu^2)}{4\,E}\right]^{2/3} \frac{F^{2/3}}{R^{1/3}} \tag{97}$$

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A Matlab Files for Analytic Derivation

```
% File: elliptic_coord.m
% Purpose: analytic derivation of the properties of elliptic coordinates

syms x y a b mu nu

x = sqrt( (a^2+mu)*(a^2+nu)/(a^2-b^2) );
y = sqrt( (b^2+mu)*(b^2+nu)/(b^2-a^2) );

dxdmu = simplify(diff(x,mu));
dxdnu = simplify(diff(x,nu));
dydmu = simplify(diff(y,mu));
dydnu = simplify(diff(y,nu));

disp('dxdmu='); pretty(dxdmu);
disp('dxdnu='); pretty(dxdnu);
disp('dydmu='); pretty(dydmu);
disp('dydnu='); pretty(dydnu);

%check orthogonality
simplify(dxdmu*dxdnu + dydmu*dydnu)

h_mu = simplify( sqrt( (dxdmu)^2 + (dydmu)^2 ) );
h_nu = simplify( sqrt( (dxdnu)^2 + (dydnu)^2 ) );

disp('h_mu='); pretty(h_mu);
disp('h_nu='); pretty(h_nu);
```

```
% File: ellipsoidal_coord.m
% Purpose: analytic derivation of the properties of ellipsoidal coordinates

syms x y z a b c lm mu nu

x = sqrt( (a^2+lm)*(a^2+mu)*(a^2+nu)/(a^2-b^2)/(a^2-c^2) );
y = sqrt( (b^2+lm)*(b^2+mu)*(b^2+nu)/(b^2-a^2)/(b^2-c^2) );
z = sqrt( (c^2+lm)*(c^2+mu)*(c^2+nu)/(c^2-a^2)/(c^2-b^2) );

dxdlm = simplify(diff(x,lm));
dxdmu = simplify(diff(x,mu));
dxdnu = simplify(diff(x,nu));
dydlm = simplify(diff(y,lm));
dydmu = simplify(diff(y,mu));
dydnu = simplify(diff(y,nu));
dzdlm = simplify(diff(z,lm));
dzdmu = simplify(diff(z,mu));
dzdnu = simplify(diff(z,nu));

disp('dxdlm='); pretty(dxdlm);
disp('dxdmu='); pretty(dxdmu);
disp('dxdnu='); pretty(dxdnu);
disp('dydlm='); pretty(dydlm);
disp('dydmu='); pretty(dydmu);
disp('dydnu='); pretty(dydnu);
disp('dzdlm='); pretty(dzdlm);
disp('dzdmu='); pretty(dzdmu);
disp('dzdnu='); pretty(dzdnu);

%check orthogonality
[ simplify(dxdlm*dxdmu + dydlm*dydmu + dzdlm*dzdmu)
  simplify(dxdmu*dxdnu + dydmu*dydnu + dzdmu*dzdnu)
  simplify(dxdlm*dxdnu + dydlm*dydnu + dzdlm*dzdnu)
]

h_lm = simplify( sqrt( (dxdlm)^2 + (dydlm)^2 + (dzdlm)^2 ) );
h_mu = simplify( sqrt( (dxdmu)^2 + (dydmu)^2 + (dzdmu)^2 ) );
h_nu = simplify( sqrt( (dxdnu)^2 + (dydnu)^2 + (dzdnu)^2 ) );

disp('h_lm='); pretty(h_lm);
disp('h_mu='); pretty(h_mu);
disp('h_nu='); pretty(h_nu);
```

References

[1] O. D. Kellogg, *Foundations of Potential Theory*, (Dover, New York, 1953).

[2] L. D. Landau and E. M. Lifshitz, *Mechanics*, (Permagon, New York, 1959).

[3] C. R. Weinberger, W. Cai, D. M. Barnett, Micromechanics lecture notes, `http://micro.stanford.edu/~me340b`

[4] M. H. Sadd, *Elasticity: Theory, Applications and Numerics*, (Elsevier, New York, 2005).

[5] `http://en.wikipedia.org/wiki/Elliptic_coordinates`

[6] `http://en.wikipedia.org/wiki/Ellipsoidal_coordinates`

[7] `http://eom.springer.de/E/e035420.htm`